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NINTH SUMMER MEETING OF THE ASSOCIATION

The ninth summer meeting of the Mathematical Association of America was held at Cornell University, Ithaca, N.Y., on Tuesday and Wednesday, September 8-9, 1925, in conjunction with the summer meeting and colloquium of the American Mathematical Society. The total attendance at the two sessions of the Association was 175, including the following 129 members:

C. R. Adams, Brown University.
R. P. Agnew, Cornell University.
O. P. Akers, Allegheny College.
W. E. Anderson, Miami University.
R. C. Archibald, Brown University.
Clara L. Bacon, Goucher College.
Grace M. Bareis, Ohio State University.
Ethelwynn R. Beckwith, Western Reserve University.
B. R. Beisel, Cornell University.
E. T. Bell, University of Washington.
A. A. Bennett, Lehigh University.
T. L. Bennett, University of Illinois.
G. D. Birkhoff, Harvard University.
G. A. Bliss, University of Chicago.
S. L. Boothroyd, Cornell University.
H. S. Brown, Hamilton College.
Margaret Buchanan, West Virginia University.
W. G. Bullard, Syracuse University.
R. W. Burgess, Western Electric Co., New York.
W. H. Bussey, University of Minnesota.
A. D. Campbell, University of Arkansas.
I. S. Carroll, Syracuse University.
E. H. Carus, La Salle, Illinois.
W. B. Carver, Cornell University.
A. B. Coble, University of Illinois.
Julia T. Colpitts, Iowa State College.
J. L. Coolidge, Harvard University.
Lennie P. Copeland, Wellesley College.
N. A. Court, University of Oklahoma.
E. F. Cox, Cornell University.
F. F. Decker, Syracuse University.
W. W. Denton, University of Michigan.
H. A. DoBell, Colgate University.
L. W. Dowling, University of Wisconsin.
Arnold Dresden, University of Wisconsin.
Otto Dunkel, Washington University.
J. A. Eiesland, West Virginia University.
L. P. Eisenhart, Princeton University.
G. C. Evans, Rice Institute.
G. W. Evans, Charlestown High School, Boston, Mass.
H. S. Everett, Bucknell University.
Fay Farnum, Cornell University.
Peter Field, University of Michigan.
W. B. Ford, University of Michigan.
Tomlinson Fort, Hunter College.
T. C. Fry, Bell Tel. Laboratories, New York.
M. G. Gaba, University of Nebraska.
A. S. Gale, University of Rochester.
W. V. N. Garretson, Rutgers College.
D. C. Gillespie, Cornell University.
R. E. Gilman, Brown University.
O. E. Glenn, University of Pennsylvania.
J. W. Glover, University of Michigan.
W. C. Graustein, Harvard University.
C. E. Harrington, University of Buffalo.
E. R. Hedrick, University of California, So. Branch.
M. E. Hekimian, Cornell University.
T. H. Hildebrandt, University of Michigan.
Einar Hille, Princeton University.
T. H. Hollcroft, Wells College.
Anna M. Howe, Tulane University.
W. A. Hurwitz, Cornell University.
M. H. Ingraham, University of Wisconsin.
Dunham Jackson, University of Minnesota.
B. W. Jones, Western Reserve University.
O. D. Kellogg, Harvard University.
B. F. Kimball, Cornell University.
H. W. Kuhn, Ohio State University.
W. D. Lambert, Coast and Geodetic Survey, Washington.
Solomon Lefschetz, Princeton University.
C. A. Lindemann, Bucknell University.
Richard Long, University of Rochester.

R. S. LUBBEN, University of Texas.
W. T. MACCREADIE, Cornell University.
C. C. MACDUFFEE, Ohio State University.
J. V. MCKELVEY, Iowa State College.
MARTHA M. MCKELVEY, Iowa State College.
HELEN A. MERRILL, Wellesley College.
A. D. MICHAL, Rice Institute.
G. A. MILLER, University of Illinois.
NORMAN MILLER, Queens University.
E. C. MOLINA, Amer. Tel. & Tel. Co., New York.
EUGENIE M. MORENUS, Sweet Briar College.
C. L. E. MOORE, Mass. Inst. of Technology.
C. N. MOORE, University of Cincinnati.
RICHARD MORRIS, Rutgers College.
D. S. MORSE, Union College.
MARSTON MORSE, Brown University.
F. H. MURRAY, Dalhousie University.
J. J. NASSAU, Case School of Applied Science.
G. D. OLDS, Amherst College.
H. L. OLSON, Michigan State College.
HELEN B. OWENS, Ithaca, N. Y.
F. W. OWENS, Cornell University.
MARGARET C. PACKER, Hood College.
L. R. PERKINS, Middlebury College.
H. PORITSKY, Cornell University.
ARTHUR RANUM, Cornell University.
S. E. RASOR, Ohio State University.
C. E. RHODES, Heidelberg University.
R. G. D. RICHARDSON, Brown University.
HORTENSE RICKARD, Ohio State University.
H. L. RIETZ, University of Iowa.
G. M. ROBISON, Duke University.
E. D. ROE, JR., Syracuse University.
J. ROSENBAUM, Milford, Conn.
GEORGE SAUTÉ, Brown University.
G. T. SELLEW, Knox College.
H. C. SHAUB, Cornell University.
W. A. SHEWHART, Bell Tel. Laboratories, New York.
J. A. SHOHAT, University of Michigan.
H. E. SLAUGHT, University of Chicago.
L. L. SMAIL, University of Texas.
CLARA E. SMITH, Wellesley College.
G. W. SNEDECOR, Iowa State College.
VIRGIL SNYDER, Cornell University.
A. HELEN TAPPAN, Iowa State College.
MARIAN W. TORREY, Goucher College.
H. S. VANDIVER, University of Texas.
ROXANA H. VIVIAN, Wellesley College.
W. J. WALLIS, High Schools, Washington, D. C.
C. W. WATKEYS, University of Rochester.
H. E. WEBB, Central High School, Newark, N. J.
J. H. M. WEDDERBURN, Princeton University.
LOUIS WEISNER, University of Rochester.
W. D. A. WESTFALL, University of Missouri.
C. O. WILLIAMSON, College of Wooster.
ELIZABETH W. WILSON, High Schools, Washington, D. C.
C. H. YEATON, Oberlin College.

The arrangements for the various meetings of the week and for the convenience, comfort, and pleasure of the members and their families and friends seemed to be carried out to perfection under the guidance of Professor W. A. Hurwitz, chairman of the committee. The dormitory accommodations in Sage College were convenient, comfortable, and greatly appreciated, and the meeting rooms in the Baker Laboratory were well suited to their purpose.

The provision for entertainment included a reception in Sage College on Tuesday evening where President Farrand on behalf of Cornell University welcomed in felicitous words the visiting mathematicians and responses were made by the presidents of the two organizations, Professors G. D. Birkhoff and J. L. Coolidge; an excursion on Thursday afternoon to beautiful Enfield Glen under the direction of Professor and Mrs. F. W. Owens; a delightful afternoon tea on Friday at the Ithaca Country Club with Mrs. J. H. Tanner, Mrs. Virgil Snyder and other faculty wives as hostesses; a visit to the observatory on Friday evening, and an organ recital by Professor W. C. Ballard of the Electrical Engineering department. All of these arrangements were greatly enjoyed and formal resolutions of appreciation were tendered to the committee and all who rendered assistance.

The joint dinner on Wednesday evening was attended by 174, including members of the two organizations and their families and friends. Professor H. E. Slaught presided and short talks were given by Professors Tomlinson Fort, A. A. Bennett, E. T. Bell, W. C. Graustein, Lennie P. Copeland, and W. B. Ford. An honored guest of the evening was Dr. Edward H. Carus, son of Mrs. Mary Hegeler Carus whose gift to the Association made possible the Carus Mathematical Monographs.

The two sessions of the Association for the reading of papers were held on Tuesday and Wednesday mornings. The program was arranged by Professors J. W. Bradshaw, University of Michigan, Mary E. Wells, Vassar College, and W. H. Roever, chairman, Washington University. The following papers were read at the two sessions:

1. "Elementary geometry and its foundations," by Mr. H. E. WEBB, Central High School, Newark, N. J.

2. Discussion of Mr. Webb's paper by Professor NATHAN ALTSHILLER-COURT, University of Oklahoma.

3. "The mathematical basis of art," by Professor G. D. BIRKHOFF, Harvard University.

4. "Certain applications of differential and integral calculus to actuarial science," by the retiring president, Professor H. L. RIETZ, University of Iowa.

5. "The College Entrance Examination Board from behind the scenes," by Professor VIRGIL SNYDER, Cornell University.

6. "Outlines of fields of research: projective geometry," by Professor L. W. DOWLING, University of Wisconsin.

7. "Mathematical problems which arise in a research laboratory," by Dr. T. C. FRY, Western Electric Company, New York, N.Y.

Professor J. L. Coolidge, president of the Association, presided at both sessions. Following are brief abstracts of the papers, numbered as in the foregoing list:

(1) Mr. Webb contended that elementary geometry as presented in American text books suffers from too meticulous a demand for nicety of form and at the same time from serious errors in the logical development of the subject. These have arisen from the effort to simplify Euclid's *Elements* without impairing his method of procedure. Recent advances in the field of synthetic projective geometry and in the foundations of geometry have served to correct the few errors in the Euclidean text and the many in those of later date, but have given the subject a form which is beyond the powers of a beginner. The remedy proposed is an arbitrary list of improved propositions which would include besides the familiar assumptions of elementary plane geometry many propositions which are now proved but are quite obvious, and also certain propositions which are now by custom assumed tacitly, but which are readily

understood and actually demanded for complete demonstration. In preparing such a list, regard should be had for the development of synthetic projective geometry, one purpose of which would then be to establish logically either as independent assumptions or as demonstrated theorems the propositions thus laid down. A typical list was submitted.

(2) In opening the discussion of Mr. Webb's paper, Professor Altshiller-Court stated that there are several fundamental reasons why the traditional course in elementary geometry should be changed. We know now vastly more mathematics, and in particular much more elementary geometry than was known in the time of Euclid. The *Elements* of Euclid was written for mature men of high intellectual development while we are teaching the subject to boys and girls of a tender age. The more intuitional theorems ought to be assumed without proof. Other theorems, like those concerning areas and volumes, should be stated in arithmetic and their proofs given in the calculus. The axioms should be stated as explicitly as possible, and no vicious circles in the demonstrations should be tolerated. However, in view of the limited mathematical experience of the beginner, it would be a harmful exaggeration to put into the proofs of elementary geometry all the circumspection that might be required of them by a trained mathematician who is familiar, say, with the elliptic and hyperbolic geometries.

(3) In his lecture on "The mathematical basis of art," Professor Birkhoff discussed the possibility of quantitatively estimating the comparative aesthetic value of simple works of art of a prescribed type. He expressed the view that such a measure might be obtained as the ratio O/C, where O is the order and C is the complexity of the work of art. Definite methods of analysis were presented in the case of tiles, vases, and paintings, and illustrations were given by means of lantern slides.

(4) The address of Professor Rietz appears in full in this number.

(5) Professor Snyder explained informally the method of procedure observed by the College Entrance Examination Board in the annual examinations. The subject matter in each subject is minutely defined by the syllabus prepared by various agencies appointed by organizations other than the Board; the question papers are submitted to a committee of revision to insure that each question is within the proper definition; every answer having a rating of not over sixty is not only re-read, but after having been re-read and the two ratings minutely compared, every such paper is then again read by a special committee and all of the ratings are adjusted to a common basis. Signed complaints, either from candidates or from others, are carefully considered. In fact every effort is made to make the tests as fair and as indicative of the candidate's proper standing as can possibly be made.

(6) The address of Professor Dowling appeared in the MONTHLY for December, 1925.

(7) Dr. Fry outlined a number of problems which grow out of the activities of the Bell Telephone System, and which illustrate the wide range of mathematical subject matter that finds a use in industrial work. Among them were: The use of *a posteriori* probability in interpreting factory inspection data; the use of *a priori* probability in the design of telephone exchanges; the use of function theory, integral equations, operational calculus, and generalized Fourier integrals in the solution of transmission problems; continued fraction expansion of functions and divergent series in the design of networks with preassigned impedances; analysis situs in the classification of impedance nets; and multiform solutions of differential equations in diffraction problems.

Considerable discussion arose in connection with Mr. Webb's paper, participated in by Professors Eisenhart, Jackson, Slaught and others. A resolution to have a special committee consider and report upon the whole question of the teaching of geometry in the secondary school was referred to the Trustees for final action. At a later meeting of the Trustees, it was voted to postpone for the present any official action of the Association with reference to reform in the teaching of elementary geometry, it being felt that the report of the National Committee and the subsequent report of the College Entrance Examination Board should be allowed to operate unhampered for a reasonable period before any further action by the Association is taken.

Much regret was expressed on all sides over the enforced absence of the secretary, Professor W. D. Cairns, it being the first meeting of the Association which he has missed since its organization nine years ago.

MEETINGS OF THE BOARD OF TRUSTEES

The Trustees of the Association held two meetings at Ithaca. President Coolidge presided and ten members were present at each session. The following are the chief items of business transacted, though informal discussion of many items of interest to the Association occupied serious attention during the two sessions.

The following thirty persons and one institution were elected to membership on applications duly certified:

To Individual Membership

C. W. ANDREWS, A.M. (Harvard); LL.D. (Northwestern). Librarian, The John Crerar Library, Chicago, Ill.

E. P. BLACKBURN, A.B. (Oakland City Coll.) Wadesville, Ind.

STANLEY BOLKS, M.S. (Iowa State). Instr., Purdue Univ., W. LaFayette, Ind.

MARY E. CLARKE, M.S. (Kentucky). Teacher, Senior High School, Lexington, Ky.

E. F. COX, Ph.D. (Cornell). Math. and Physics, West Virginia Coll. Inst., Institute, W. Va.

D. A. FLANDERS, A.B. (Haverford). Prof., Texas Tech. Coll., Lubbock, Texas.

H. A. FOSTER, A.B. (Baylor). Instr., Physics, A. and M. Coll. of Texas, College Station, Texas.

A. H. FOX, A.B. (Western Reserve). Instr., Oberlin College, Oberlin, Ohio.

GEORGE HARTNELL. Observer in Charge, Magnetic Observ., Cheltenham, Md.

T. W. HATCHER, M.E. (Va. Polytech. Inst.); M.S. (Iowa State Coll.) Asst. Prof., Virginia Polytech. Inst., Blacksburg, Va.

M. E. HEKIMIAN, C.E., A.M. (Cornell). Hotel Bevan, Larchmont, N. Y.

V. B. HINSCH, M.E. (School of Mines). Asso. Prof., School of Mines and Metallurgy, Rolla, Mo.

W. R. HUTCHERSON, A.M. (Kentucky). Asso. Prof., Berea Coll., Berea, Ky.

FRED KADERLI. Teacher, High School, San Marcos, Texas.

KATHERINE D. KIENZLE, A.B. (Louisville). Teacher, Margaret Hall, Versailles, Ky.

S. B. LITTAUER, Ch.E. (Rensselaer). Teacher of Physics, Boys' High School, Brooklyn, N.Y.

C. A. MANEY, M.S. (Chicago). Prof., Transylvania Coll., Lexington, Ky.

E. W. MARTIN, M.S. (Illinois). Prof. and Head of Dept., Western State Coll., Gunnison, Colo.

G. T. NEWTON, A.B. (Texas). Head of Dept. of Math., High School, Cameron, Texas.

E. K. PAXTON, A.M. (Washington and Lee; Columbia). Asso. Prof., Washington and Lee Univ., Lexington, Va.

R. I. PEPPER, A.B. (Centre Coll.) Teacher, High School, Paris, Ky.

A. J. PYKE, B.A. (Toronto). Prof., Univ. of Saskatchewan, Saskatoon, Sask., Canada.

C. E. RHODES, A.B. (Cornell). Instr., Heidelberg Univ., Tiffin, Ohio.

LAURA RIESBECK, B.S. in Educ. (Ohio Univ.) 1001 Greenwood Ave., Zanesville, Ohio.

L. R. SALVOSA, A.B. (Philippines); Grad. U.S. Milit. Acad. Prof. Math. and Eng., Los Baños Coll., Laguna, P.I.

F. A. SCOTT, A.M. (Columbia). Prin., High School, Paris, Ky.

J. S. SEWELL, Grad. U.S. Milit. Acad. Pres., Ala. Marble Co., Birmingham, Ala.

W. A. SHEWHART, Ph.D. (California). Bell Telephone Labs., 463 West St., New York, N.Y.

V. A. TAN, A.M., C.E. (Cornell), Ph.D. (Chicago). Prof., Univ. of the Philippines, Manila, P.I.

L. D. WILLS, A.B. (Reed). Part-time Instr., Princeton Univ., Princeton, N.J.

To Institutional Membership

BUTLER COLLEGE, Indianapolis, Ind.

This makes a total of 146 individuals and seven institutions elected since the publication of the last Register in November, 1924. It was voted to publish a new Register.

Invitations for future meetings were received and accepted as follows: the summer meeting in 1926 at Ohio State University, Columbus, Ohio; the summer meeting in 1927 at the University of Wisconsin. The annual meeting in December, 1925 had already been fixed for Kansas City, Missouri, and the annual meeting in 1927 at the University of Pennsylvania, Philadelphia, Pennsylvania, both in affiliation with the American Association for the Advancement of Science. The Program Committee for the Kansas City meeting is L. W. Dowling, chairman; W. C. Brenke, Olive C. Hazlett, G. H. Light.

It was reported by the committee on Carus Monographs that 879 copies of the first monograph had been taken by members of the Association and that about 375 had been sold to the general public by the Open Court Publishing Company. The receipts from sales to the members, after payment of the honorarium to the authors, are deposited to the credit of the Carus Publishing Fund in the treasury of the Association. This fund now amounts to $2099.18.

The second monograph on *Functions of a Complex Variable* by Professor D. R. Curtiss will be ready for the printers early in November and should be ready for distribution by February, 1926.

Owing to the fact that President Coolidge will be out of the country at the time of the next summer meeting, it was voted that he should deliver his retiring address at the annual meeting in Kansas City. His address will be on "Robert Adrain and the beginnings of American mathematics."

Professor Archibald reported that the contract has been let for the publication of the *Rhind Papyrus* by Chancellor Chace, of Brown University, under the auspices of the Association. There will be 550 sets for sale at cost by the Association to its members. A committee consisting of Professors Archibald, Cairns and Slaught was appointed to administer the details of this great benefaction to the Association. Professor Archibald was asked to prepare an article for the MONTHLY giving a full description of this notable work by Chancellor Chace.

The Chauvenet prize for mathematical exposition, which was proposed by President Coolidge last spring, was formally adopted and the first award will be made at the annual meeting in Kansas City. A full description of the details of the prize has already appeared in the MONTHLY.

H. E. SLAUGHT, *Acting Secretary.*

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The seventeenth regular meeting of the Maryland-Virginia-District of Columbia Section was held in McDowell Hall, St. John's College, Annapolis, Md., on May 16, 1925. There were two sessions, Professor F. D. Murnaghan being the chairman at each session. The attendance was thirty-two including the following twenty-six members of the Association: O. S. Adams, Sarah Beall, W. W. Bigelow, G. A. Bingley, C. C. Bramble, J. A. Bullard, Paul Capron, G. R. Clements, A. Cohen, A. Dillingham, Jessie B. Edmondson, H. English, D. M. Garrison, Alice A. Grant, H. Gwinner, W. M. Hamilton, W. D. Lambert, E. S. Mayer, F. D. Murnaghan, C. A. Nelson, B. C. Patterson, R. E. Root, J. B. Scarborough, W. F. Shenton, J. Tyler, E. W. Woolard. Those in attendance were guests at the luncheon given by the Annapolis members, and were given an address of welcome by the President of St. John's College.

The report of the nominating committee, Mr. English and Mr. Bramble, for officers for the next year was unanimously adopted and was as follows: Chairman, W. D. LAMBERT; Secretary-treasurer, J. A. BULLARD; State representatives, C. A. NELSON and T. McN. SIMPSON, Jr. The next meeting will be in Washington, D. C., probably in December.

The following seven papers were read:

(1) "The compression of a gravitating sphere," by Mr. W. D. LAMBERT, U. S. Coast and Geodetic Survey.

(2) "Some formulae connected with great circle sailing," by Mr. W. M. HAMILTON, U. S. Naval Observatory.

(3) "The accuracy of Simpson's rule" by Professor J. B. SCARBOROUGH, U. S. Naval Academy.

(4) "The identification of stars" by Miss SARAH BEALL, U. S. Coast and Geodetic Survey.

(5) "Some mathematical problems of meteorology" by Mr. E. W. WOOLARD, U. S. Weather Bureau.

(6) "The Neuberg cubic," by Mr. B. C. PATTERSON, Johns Hopkins University.

(7) "Some cissoidal curves," Dr. C. A. NELSON, Johns Hopkins University.

There were lively discussions of all papers, especially of that on the accuracy of Simpson's rule. Abstracts of some of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. Mr. Lambert's paper on the compression of a gravitating sphere brought out the difference between a gravitating and a non-gravitating fluid sphere, when uniform pressure is applied to the outer surface. This problem, which can be treated by comparatively elementary means, illustrates some of the difficulties encountered in the problem of the elastic deformation of a gravitating body as large as the earth, such a body being already in a state of internal stress under its own gravitation. If the modulus of compressibility of the earth is such as may be deduced from Laplace's law of density, a pressure of one atmosphere will shorten the radius by 2.91 meters if gravitation be considered, but only by 1.95 meters if it is neglected. The radial contraction in the more general case of any radially symmetrical distribution of density and compressibility may be found by solving a differential equation of the second order.

The gravitational energy evoked by such a radial contraction far exceeds the energy due to ordinary elastic strain. In the case considered, the former is 0.787×10^{33} ergs, in the latter 1.26×10^{27} ergs, a ratio of 600,000 : 1. If this gravitational energy were turned into heat, the resulting expansion would tend to diminish the radial shortening, which accordingly would attain the calculated value given above only when the heat had been radiated away, a process which, at the observed rate of radiation from the earth, would require 70,000 years. There would thus arise a distinction having some analogies to that between isothermal and adiabatic compression. The differential equation for the radial shortening in the "adiabatic" case contains additional terms depending on the specific heat and the coefficient of thermal expansion.

4. Miss Beall discussed the use of the astronomical globe and the stereographic projection for star identification and star spotting. The astronomical globe has been used by the Coast and Geodetic Survey since 1917 with great success. The stereographic projection for the use of the navigator is a great advance over star identification tables. Stars may be quickly identified, or observing lists may be prepared enabling the observer to get a star in the telescope of the sextant before it can be seen with the naked eye. This insures an observation while the horizon is well defined. The number of stars which may be used is extended and the use of the stars is made more attractive.

5. Given an urn in which there are n balls, of which a_1 are marked with the number x_1, a_2 with the number x_2, and so on to x_s. If we draw a ball at random, note the number, return the ball to the urn, and draw another, and so on, we obtain a sequence of numbers in random order. Suppose we take the absolute difference between each number in the series and the next following number; what, in the long run, will be the arithmetic mean of these differences?

In this paper, a formula for this mean value is derived, and its accuracy verified by experimental tests. The problem arose in connection with the statistical treatment of meteorological data; and the solution is designed to afford a means of testing time series to ascertain whether or not the variable quantity is taking on its values at random, such information frequently being of importance in deciding whether or not certain statistical formulae may be validly applied to the given data. The work is an extension of that of Goutereau and Maillet, who, in 1906, showed that in a random series drawn from a Gaussian distribution the ratio of the above mean to the mean deviation is equal to $\sqrt{2}$; mean daily temperatures, *e. g.*, show a Gaussian distribution, but the ratio mentioned is in general only about half $\sqrt{2}$, since the mean temperatures of consecutive days are not independent of one another. This paper has been published in full in the *Monthly Weather Review*, March, 1925.

7. Dr. Nelson put the Zahradruk construction for a rational plane cubic in projective form and gave a geometric characterization of the generating line and conic.

HARRY ENGLISH, *Secretary-Treasurer.*

ON CERTAIN APPLICATIONS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS IN ACTUARIAL SCIENCE[1]

By H. L. RIETZ, University of Iowa

1. Introduction. The position which the Mathematical Association of America should take in the development of a broader and deeper interest in

[1] Presidential address delivered before the Mathematical Association of America, September 9, 1925.

applied mathematics has been emphasized by both Professor Hedrick and Professor Huntington in their retiring presidential addresses. In line with the general character of their suggestions, I shall attempt to bring to your attention certain applications of the calculus in actuarial science.

We may no doubt take it for granted as a general proposition that teachers of mathematics are looking over many fields for a wider range of applications with which to enrich their courses of instruction. This fact might perhaps be regarded as a sufficient reason for presenting a paper dealing with applications even from a field in which relatively few members of the Association have had a special interest. However, the speaker was moved mainly in the preparation of the present paper by the following somewhat more specific consideration: the teaching of certain very elementary parts of life insurance mathematics has grown so rapidly in recent years in connection with courses in the mathematics of finance that it seems not unlikely that a fairly large number of teachers would be interested in a simple exposition of the connection of the calculus with certain actuarial problems. It is with a feeling that you may be thus interested that I venture to take up your time by bringing to your attention applications from a field that is considerably removed from the most common source of applications—the physical sciences.

The time at my disposal will permit me to deal only with a certain class of very direct applications. It will not permit me to dwell upon the somewhat indirect applications of the calculus such as are involved in the development of interpolation, graduation, and quadrature formulas which are applied in the determination of mortality rates from statistical data, or in the approximate summation of the series which arise in actuarial science. To be more specific about the contents of the present paper my purpose in this address will be limited to bringing to your attention a few of the applications of the calculus to the theory and practice of life contingencies by assuming a continuous function to represent the law of mortality. Such applications have led to great improvements in actuarial theory which center around the ideas of continuous flow of population, continuous annuities, and assurances payable at the moment of death.

2. The representation of the law of mortality by a continuous function. On the basis of certain observations of the Halley tables (constructed in 1693) DeMoivre in his *Annuities on Lives*, published in 1725, put forth his well-known hypothesis that the decrement d_x per year in the number living l_x at age x is a constant for all values of x. This hypothesis is, of course, equivalent to an assumption that the number living l_x is given by a linear function of x within the age interval in question. It is doubtful whether much significance was attached by DeMoivre to the continuity of the function. In 1765, J. H.

Lambert[1] proposed a continuous function in the form of a polynomial of the fifth degree to represent the number living. Other formulas involving a combination of integral and exponential functions were proposed by Lambert and Duvillard.[2] These functions were simply in the nature of empirical formulas and it is again doubtful whether special use was made of the continuity properties of the function employed except for interpolation. But in 1825, B. Gompertz[3] not only introduced the continuous function $l_x = kg^{c^x}$ on a priori grounds to represent the number living for a certain range of ages, but he made use of the property of continuity of the function in defining what is sometimes called the instantaneous rate of mortality. He called this function the intensity of mortality and it is now commonly called the force of mortality by English and American actuaries. This step really gave the derivative of the function l_x and the corresponding force of mortality a place of fundamental importance in actuarial theory. The practical importance of Makeham's[4] first modification of the law of Gompertz, particularly in joint life insurance, has undoubtedly tended to increase the significance attached to the force of mortality and to the use of continuous functions in the treatment of life contingencies. Gauss, G. F. Hardy, Lazarus, Thiele, Dormey, Laurent, Wittsein, Quiquet, and other mathematicians and actuaries have proposed various formulas in the nature of continuous functions to represent mortality functions within certain age intervals. For an interesting discussion of such formulas, let me refer you to a recent paper by James S. Elston[5] on a "Survey of mathematical formulas that have been used to express a law of mortality." Most of these functions contain too many parameters to be convenient for practical purposes. However, the Makeham and the Wittstein[6] formulas contain a relatively small number of parameters and the latter may be regarded as of special interest in relation to the present paper because the number living l_x is expressed by means of an integral, and the maximal and minimal values of the probability q_x that a person aged x will die within a year are easily found by the use of the derivative dq_x/dx.

3. Certain elementary mortality functions involving derivatives and integrals. As a setting for our later remarks it seems desirable to introduce at this point a few very elementary mortality functions expressed in terms of

[1] *Beiträge, Zum Gebrauche der Mathematik und deren Anwendung* (1765).

[2] *Recherches sur les rentes, les emprunts et les remboursements* (1788).

[3] *Phil. Trans.* London, 115 (1825), pp. 513-583.

[4] *Journal of the Institute of Actuaries*, vol. 8, (1860), pp. 301-310, vol. 13, (1867), pp. 325-358; vol. 28, (1890), pp. 152-159; pp. 185-192; pp. 316-331.

[5] *Record of American Institute of Actuaries*, vol. 12, (1923), pp. 66-95. See also Quiquet, "Représentation algèbrique des tables de survie"; *Bulletin de L'Institute des Actuaries Français*, vol. III, no. 14, (1893).

[6] *Loc. cit.*, p. 82.

derivatives and integrals. First of all, assuming that the continuous function l_x has a derivative with regard to x at a given point, we define

$$\mu_x = -\frac{1}{l_x} \cdot \frac{dl_x}{dx} = -\frac{d \log_e l_x}{dx} \tag{1}$$

as the force of mortality at age x. In other words, the force of mortality at any age is the rate of change of the number living per person living at the given age.

Then the probability that[1] (x) will die in the age interval between $x+t$ and $x+t+dt$ is given to within infinitesimals of higher order by

$$-\frac{1}{l_x} \cdot \frac{dl_{x+t}}{dt} dt = \mu_{x+t} \frac{l_{x+t}}{l_x} dt = {}_t p_x \mu_{x+t} dt \ , \tag{2}$$

where ${}_t p_x$ is the probability that (x) will live t years. From (2) we have for the probability ${}_{n|m}Q_x$ that (x) will die between ages $x+n$ and $x+n+m$

$${}_{n|m}Q_x = -\frac{1}{l_x} \int_n^{n+m} \frac{dl_{x+t}}{dt} dt = \frac{l_{x+n} - l_{x+n+m}}{l_x} \ . \tag{3}$$

Given

$$\mu_{x+t} = -\frac{d \log_e l_{x+t}}{dt} \ ,$$

we have

$$-\int_0^t \mu_{x+t} dt = \log_e \frac{l_{x+t}}{l_x} = \log_e {}_t p_x \quad \text{or} \quad {}_t p_x = e^{-\int_0^t \mu_{x+t} dt} \ . \tag{4}$$

The force of mortality of joint lives, say of (x), (y), (z) at a time t from the present date may be defined by

$$\begin{aligned} \mu_{x+t:y+t:z+t} &= -\frac{1}{l_{x+t}\, l_{y+t}\, l_{z+t}} \ \frac{d\,(l_{x+t}\, l_{y+t}\, l_{z+t})}{dt} = -\frac{d}{dt} \log(l_{x+t} l_{y+t} l_{z+t}) \\ &= -\frac{1}{l_{x+t}} \cdot \frac{dl_{x+t}}{dt} - \frac{1}{l_{y+t}} \cdot \frac{dl_{y+t}}{dt} - \frac{1}{l_{z+t}} \cdot \frac{dl_{z+t}}{dt} = \mu_{x+t} + \mu_{y+t} + \mu_{z+t} \ . \end{aligned} \tag{5}$$

That is, the force of mortality for any number of joint lives is equal to the sum of the forces of mortality of the separate lives. The number living between ages x and $x+1$ at any moment is given by

$$L_x = \int_0^1 l_{x+t} dt \ . \tag{6}$$

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[1] The symbol (x) is used as an abbreviation for "a person aged x."

Then

$$\frac{dL_x}{dx} = \int_0^1 \frac{dl_{x+t}}{dt}\, dt = l_{x+1} - l_x = -d_x\,. \tag{7}$$

The central death rate m_x is defined by

$$m_x = \frac{d_x}{L_x} = -\frac{1}{L_x}\;\frac{dL_x}{dx}\,. \tag{8}$$

The complete expectation of life is defined by

$$\mathring{e}_x = \int_0^{\omega} {}_tp_x dt\;, \tag{9}$$

where ω is used to denote the upper bound of age and is sometimes written as ∞.

We might easily continue the list of mortality functions represented by means of derivatives and integrals, but those given will perhaps indicate the general nature of this representation, and serve our purposes in the present paper.

4. **Continuous annuities.** The payments under an annuity of annual rent 1 may be assumed to become more and more frequent so that the interval between two successive payments approaches zero as a limiting value. In the limiting situation, the annuity is called a continuous annuity.

The introduction of continuous functions in a systematic manner for the development of the theory of annuities and assurance should be attributed to W. S. B. Woolhouse,[1] in a paper published in 1869 "On an improved theory of annuities and assurances." The most elementary theory of life annuities is ordinarily based on annual payments. In the introduction to this paper, Woolhouse remarks that "it cannot but appear to be remarkable that the investigation of so elementary a value as that of an annuity, when diverted from its usual course and made payable half-yearly or quarterly, should not have been determined with greater accuracy in works on annuities and assurances. This may in some measure be ascribed to a sort of vague impression that a rough approximation might be considered sufficient for the cases which may be expected to arise in the course of actua' business. But, looking at the formidable character of some of the investigations, there can be but little doubt that a more potent reason is to be traced to the mathematical difficulty encountered in previous efforts to arrive at anything like an accurate formula, which would at the same time be sufficiently simple for the ordinary routine

[1] *Journal of the Institute of Actuaries*, vol. 15, (1869), pp. 95-195.

of office calculation." According to the principles laid down in the improved theory of Woolhouse, the number of lives, instead of being subjected to successive yearly decrements are considered to be decreasing continuously. As advantages of the new method, Woolhouse points out that it is with this method no longer necessary to be dependent on the gratuitous assumption that deaths which take place in any year are uniformly distributed throughout the year, and that the new method is not only more accurate in principle, and in all respects philosophically consistent, but that the various formulas are ordinarily simple and commodious for calculation.

The present value of the continuous life annuity of annual rent 1 for (x) is given by

$$\bar{a}_x = \frac{1}{l_x} \int_0^\omega l_{x+t} v^t dt = \int_0^\omega {}_t p_x \, e^{-\delta t} \, dt \, , \tag{10}$$

where $v = 1/1+i$ is the discount factor, i being the interest rate, $\delta = \log_e (1+i)$ is the force of interest, and ω is the upper bound of age. Similarly, for a joint life annuity for n persons (x_1), (x_2), . . . (x_n), we have

$$\bar{a}_{x_1 x_2 \, . \, . x_n} = \int_0^\omega {}_t p_{x_1 x_2 \, . \, . x_n} e^{-\delta t} dt \, . \tag{11}$$

Different methods of evaluating this integral would be used according as the mortality table does or does not follow Makeham's first modification of the law of Gompertz.

A. When the mortality table follows Makeham's law. If the mortality table on which the annuity is based is given by Makeham's first modification of the law of Gompertz, that is, if

$$l_{x+t} = k s^{x+t} g^{c^{x+t}} \, ,$$

then

$$\bar{a}_x = \int_0^\omega s^t e^{-\delta t} g^{c^x (c^t - 1)} dt. \tag{12}$$

Let[1]

$$e^{-\delta} s = e^{-\alpha}, \qquad g^{c^x} = e^{-q}$$

Then (12) becomes

$$\bar{a}_x = \int_0^\omega e^{-\alpha t} e^{-q(c^t - 1)} dt \, . \tag{13}$$

[1] Cf. Insolera, *Corso di Matematica Finanziaria* (1923), p. 400.

Next, let $qc^t = y$, and $\alpha/\log c = p_1$ then (13) becomes

$$\bar{a}_x = \frac{e^q q^{p_1}}{\log c} \int_q^\infty y^{-p_1-1} e^{-y} \, dy \,. \tag{14}$$

Integration of (14) by parts gives

$$\bar{a}_x = \frac{1}{p_1 \log c} \left[1 - e^q q^{p_1} \int_q^\infty y^{-p_1} e^{-y} \, dy \right] . \tag{15}$$

In (15),

$$\int_q^\infty y^{-p_1} e^{-y} \, dy = \int_0^\infty y^{-p_1} e^{-y} \, dy - \int_0^q y^{-p_1} e^{-y} \, dy \,.$$

$$= \Gamma(1-p_1) - \int_0^q {}^{-p_1} e^{-y} \, ydy \,. \tag{16}$$

From (15) and (16), we have

$$\bar{a}_x = \frac{1}{p_1 \log c} \left[1 - e^q q^{p_1} \left\{ \Gamma(1-p_1) - \int_0^q y^{-p_1} e^{-y} \, dy \right\} \right] . \tag{17}$$

Hence, we have $\bar{a}_x$ expressed in terms of a gamma-function and an incomplete gamma-function.

But for ordinary rates of interest, p_1 is likely to be a number between 0 and 1, and the Pearson tables[1] of the incomplete gamma-function would be directly applicable only when the exponent of y is positive. By further integration by parts, we obtain instead of (17),

$$\bar{a}_x = \frac{1}{\alpha} \left[1 - e^q q^{p_1} \Gamma(1-p_1) + \frac{q}{1-p_1} + \frac{e^q q^{p_1}}{1-p_1} \int_0^q y^{1-p_1} e^y \, dy \right] . \tag{18}$$

By the use of tables of gamma-functions and of incomplete gamma-functions, we now readily compute the continuous life annuity $\bar{a}_x$. The method is capable of immediate extension to joint life annuities. One of my recent graduate students—Mr. J. Van S. Longenecker of the Equitable Life Insurance Company of Iowa—tested this method of finding both single and joint life annuities by the use of the incomplete gamma-functions in (18) against the usual method of summation formulas such as the Woolhouse, Lubbock, Hardy and Weddle formulas. The test consisted in actually computing annuities for ages well distributed, and for several of the best known tables of mortality. Mr. Longenecker arrived at the conclusion from this fairly extensive study

[1] Karl Pearson, *Tables of the Incomplete Gamma-function*, 1922.

that both from the standpoints of accuracy and facility in calculation, the method of the incomplete gamma-functions is preferable to the other methods. So far as I have been able to learn the use of the incomplete gamma-functions to compute annuities is new, but Emory McClintock made use of the ordinary gamma-function for this purpose as early as 1874. In 1873 W. M. Makeham[1] expressed the continuous annuity $\bar{a}_x$, based on a mortality table given by his first modification of the law of Gompertz, in a form which involved no difficulties except such as are involved in an integral of the form

$$\frac{1}{\log} \quad \frac{1}{10^{-10^z} e^{-nz}} \int_z^\infty 10^{-10^z} e^{-nz} dz.$$

He then prepared a double entry table of

$$\log \frac{1}{10^{-10^z} e^{-nz}} \int_z^\infty 10^{-10^z} e^{-nz} dz$$

with respect to the arguments n and z for the calculation of continuous annuities. In 1874, Emory McClintock[2] published a rival method of calculation of the continuous annuities for which he held he could obtain the same results as by Makeham's method and without any other table than that of the ordinary gamma-function. His method involved the expansion of an integral of the form

$$\int_v^\infty e^{-v} v^r dv$$

in a series the first term of which involved the gamma-function. The slow convergence of the series makes this method of little value in many cases of joint life assurance and even in some cases for an annuity on a single life.

On the other hand, by the use of the Pearson tables of the incomplete gamma-function it seems fairly well established that we have now a simpler and more satisfactory method than has previously been known for the calculation of annuities based on a Makehamized table and at a rate of interest for which auxiliary tables such as commutation columns have not been calculated.

B. When the mortality table does not follow Makeham's law. The general theory of continuous annuities dating from the fundamental paper of Woolhouse is not dependent upon the use of a Makehamized mortality table. Thus, we write for the continuous annuity as in (10),

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[1] *Journal of the Institute of Actuaries*, vol. 17, (1873), pp. 306-327.

[2] *Journal of the Institute of Actuaries*, vol. 18, (1874), pp. 242-251.

$$\bar{a}_x=\frac{1}{l_x}\int_0^\omega v^t l_{x+t}dt\ . \tag{19}$$

From (19), by making $v^{x+t}l_{x+t}=D_{x+t}$, we have

$$\bar{a}_x=\frac{1}{D_x}\int_0^\omega D_{x+t}dt\ . \tag{20}$$

If the object is to express the continuous annuity $\bar{a}_x$ in terms of an ordinary life annuity a_x and certain correctional terms, we may ordinarily simply use the Euler-Maclaurin formula and obtain from (20)

$$\bar{a}_x=\frac{1}{D_x}\left\{\sum_{t=1}^{t=\omega}D_{x+t}+\tfrac{1}{2}D_x+\frac{1}{12}\frac{dD_x}{dx}+\cdots\cdot\right\}$$

$$=a_x+\frac{1}{2}-\frac{1}{12}(\mu_x+\delta)\ \text{approx.} \tag{21}$$

On the other hand, suppose the present value of the ordinary annuity is not available, and that we wish to obtain such value on the basis of a mortality table for which auxiliary functions are not available. To do this we proceed to the valuation of an integral such as (19), by the use of some quadrature formula. Actuaries have found one of G. F. Hardy's summation formulas,[1] the famous formula (39a), particularly convenient for such annuity calculations. By such approximate valuation of the integral in (19), we have a convenient method of finding the present value of an ordinary life annuity based on a mortality table or rate of interest for which auxiliary tables are not available. This method of finding the value of annuities is found particularly useful in the case of joint life and survivorship annuities.

5. Assurance payable at the moment of death. It is common practice to define the net premium for life assurance on the hypothesis that the claim is to be paid at the end of the policy year in which death occurs; but as a matter of business practice, the claim is paid as nearly the instant of death as is possible. Thus, a theory of assurance payable at the instant of death conforms much more closely to actual practice than a theory based on the commonly accepted hypothesis that claims are paid at the ends of the policy years.

Since the probability that (x) will die at an age between $x+t$ and $x+t+dt$ is to within infinitesimals of higher order

$$-\ \frac{1}{l_x}\cdot\frac{dl_{x+t}}{dt}dt\ ,$$

[1] *Text-book of the Institute of Actuaries*, (1902), p. 488.

it follows that the value of a whole life assurance of 1 payable at the moment of death of (x) is

$$\bar{A}_x = -\frac{1}{l_x}\int_0^\omega v^t \frac{dl_{x+t}}{dt}\,dt \tag{22}$$

$$= 1 - \delta \bar{a}_x\,, \tag{23}$$

as may be shown by integration by parts where $\bar{a}_x$ is the present value of a continuous life annuity of annual rent 1.

Thus, the single premiums for a temporary or for a whole life assurance are given very simply in terms of the corresponding continuous annuities.

6. Contingent probabilities of life. A comparison of the new book (1922) of the Institute of Actuaries on *Life Contingencies* by E. F. Spurgeon with the classic text-book by George King shows particularly the improvement in the presentation of contingent probabilities by the more extensive use of the differential and integral calculus. The older treatment involved cumbersome approximations dependent on the assumption of the uniform distribution of deaths throughout the year, and corrections for errors growing out of this assumption. These approximations have been eliminated, to a considerable extent, and a more elegant treatment substituted by the use of the calculus. The application of the calculus in this connection can perhaps be seen with sufficient clearness for our purposes by considering a few examples involving two and three lives.

A. Two lives. Let us find the probability that (x) will die in the $(n+1)$th year from the present time, (y) being alive at the moment of death of (x).

The probability that (x) will die at an *age* between $x+t$ and $x+t+dt$, (y) being alive at age $y+t$ is, to within infinitesimals of higher order,

$$-{}_tp_y \frac{dl_{x+t}}{l_x} = {}_tp_x\ \mu_{x+t}\ {}_tp_y\ dt = {}_tp_{xy}\ \mu_{x+t}\ dt\ , \tag{24}$$

where ${}_tp_{xy}$ is the probability that both (x) and (y) will live t years.

This expression integrated from $t=n$ to $t=n+1$ is the probability ${}_{n|}q^1_{xy}$ that (x) will die before (y) in the $(n+1)$th year. That is,

$${}_{n|}q_{\overset{1}{x}y} = \int_n^{n+1} {}_tp_{xy}\ \mu_{x+t}\ dt\ . \tag{25}$$

Similarly, the probability that (x) will die before (y) within n years is

$${}_{|n}Q_{\overset{1}{x}y} = \int_0^n {}_tp_{xy}\ \mu_{x+t}\, dt\ . \tag{26}$$

B. Three lives. The probability Q^1_{xyz} that of three lives (x), (y), and (z), (x) will die first is

$$Q_{\overset{1}{x}yz}=\int_0^{\omega} {}_tp_{xyz}\,\mu_{x+t}\,dt\ , \tag{27}$$

where ${}_tp_{xyz}$ is the probability that (x), (y), and (z) will each live t years.

As another example, the probability that (x) will die third of three lives (x), (y), and (z) is

$$\begin{aligned} Q_{\overset{3}{x}yz}&=\int_0^{\omega} (1-{}_tp_y)(1-{}_tp_z)\,{}_tp_x\mu_{x+t}dt \\ &=\int_0^{\omega} (1-{}_tp_y-{}_tp_z+{}_tp_{yz})\,{}_tp_x\mu_{x+t}dt=1-Q_{\overset{1}{x}y}-Q_{\overset{1}{x}z}+Q_{\overset{1}{x}yz} \end{aligned} \tag{28}$$

where $\overset{1}{x}$ indicates that (x) is to die first.

The numerical valuation of the integrals giving such contingent probabilities of life is usually accomplished by approximate integration, although this would not necessarily be the case if the mortality table follows a formula such as that of Makeham.

7. **Joint life assurances.** The present value of 1 payable at a time between t and $t+dt$ years from the present if the joint existence of a set of lives, say (x), (y), and (z) fails at that time is, to within infinitesimals of higher order,

$$-\frac{v^t}{l_xl_yl_z}\ \frac{d(l_{x+t}l_{y+t}l_{z+t})}{dt}dt=v^t\,{}_tp_{xyz}\,\mu_{x+t\,:\,y+t\,:\,z+t}\,dt \tag{29}$$

and the value of a joint life assurance of 1 payable at the moment of the first death among these lives,

$$\bar{A}_{xyz}=\int_0^{\omega} v^t\,{}_tp_{xyz}\,\mu_{x+t\,:\,y+t\,:\,z+t}\,dt \tag{30}$$

$$=\int_0^{\omega} v^t\,{}_tp_{xyz}\,(\mu_{x+t}+\mu_{y+t}+\mu_{z+t})\,dt \tag{31}$$

by formula (5).

It may be observed that the integral (30) giving the continuous joint life assurance may be formed from the integral (11) for the corresponding continuous joint life annuity by multiplying the integrand in (11) by the force of mortality of the set of lives involved.

8. Contingent life assurances. In its simplest form, a contingent life assurance involving only two lives provides for the payment of 1 on the death of (x) known as the "failing" life provided (y) known as the "counter" life should survive him.

The value 1 payable at time t from the present if (x) die between ages $x+t-dt$ and $x+t$, (y) surviving him, is to within infinitesimals of higher order

$$v^t {}_tp_{xy}\, \mu_{x+t}\, dt\ ,$$

and the total value of such an assurance for the whole of life, payable at the moment of death of (x), provided (y) survive him is

$$\bar{A}_{\overset{1}{x}y} = \int_0^\omega v^t {}_tp_{xy}\, \mu_{x+t}\, dt\ , \tag{32}$$

where ω is the upper bound of age. It may be observed that this integral for a contingent assurance payable on the death of (x) may be obtained from the integral for a continuous joint life annuity to (x) and (y) by multiplying the integrand by the force of mortality of the "failing" life at age $x+t$.

Contingent life assurance may be payable in some other way than on the first death, and these cases also lend themselves to treatment by the means of definite integrals. For example, the value $\bar{A}_{\overset{2}{x}y}$ of an insurance of 1 payable at the moment of death of (x) if he die after (y) is

$$\bar{A}_{\overset{2}{x}y} = \int_0^\omega v^t(1-{}_tp_y)\,{}_tp_x\mu_{x+t}dt = \bar{A}_x - \bar{A}_{\overset{1}{x}y} \tag{33}$$

which is equal to an assurance of 1 payable on the death of (x) diminished by an assurance payable on the death of (x) if he die before (y).

9. Reversionary annuities. A continuous reversionary annuity $\bar{a}_{y|x}$ to (x) after (y) is an annuity to begin on the death of (y) and to continue thereafter during the life of (x). Such an annuity may be expressed as an integral in the following form:

$$\bar{a}_{y|x} = \int_0^\omega v^t(1-{}_tp_y)\,{}_tp_x dt \tag{34}$$

$$= \bar{a}_x - \bar{a}_{xy}\ . \tag{35}$$

The increased tendency to apply the calculus in dealing with reversionary annuities is shown by the fact that in the text-book of the Institute of Actuaries last revised in 1902, the integral sign appears only on the last two pages of the fourteen pages devoted to reversionary annuities whereas in the corresponding recent book of the Institute by Spurgeon published in 1922, the integral sign

appears on the first page and the calculus plays an important role throughout the entire chapter of twenty pages.

10. Compound survivorship annunities. In the ordinary reversionary annuity, payable to (x) after the deaths of (y) and (z), no specification is made as to the order of deaths of (y) and (z).

But assume an annuity payment to (x) after the failure of the joint lives (y) and (z) by the death of (z). Then we have a condition of compound survivorship. If both (x) and (y) are living at the moment of death of (z), (x) enters into possession of a continuous life annuity. Denote such an annuity by $\bar{a}_{y\overset{1}{z}|x}$. Then

$$\bar{a}_{y\overset{1}{z}|x} = \int_0^\omega v^t \ {}_tp_{xyz}\mu_{z+t}\bar{a}_{x+t}dt \,. \tag{36}$$

This function $\bar{a}_{y\overset{1}{z}|x}$ has attracted much attention relative to its commercial importance. The actuary is very generally seeking for formulas amenable to calculation. Ordinarily he would resort to approximate summation when numerical results are needed from such a formula as (36), but in 1895, F. E. Colenso published a paper[1] which included the reduction of (36) to a form suitable for direct numerical calculation when the mortality table follows Makeham's first modification of the law of Gompertz.

We have commented on the relatively small commercial importance of compound survivorship annuities. With the extension of the inheritance taxes which may be levied severely on successive interests we may find increased applications of compound survivorship annuities. That is to say, such taxation may result in proposals that insurance companies go much further with survivorship annuities than has hitherto been the rule.

11. On the relation between assurance functions based on two different mortality tables. When we take account of the laborious calculations involved in the preparation of appropriate commutation columns, valuation functions, pure endowment functions, forborne immediate annuity functions, and other actuarial functions which accompany certain mortality tables which are much used in assurance practice, we appreciate the importance of any contribution to the solution of the problem of expressing assurance functions based on a mortality table for which the numerical values of the auxiliary actuarial functions are not available in terms of available functions of a standard mortality table.

Among the applications of the calculus to be considered in the present paper, it seems fitting to direct attention to a method of calculation of assurance func-

[1] *Jour. Institute of Actuaries*, vol. 31, pp. 337-56.

tions based on any mortality table that follows Makeham's law, and at an arbitrary rate of interest, by means of available tables based on a standard Makehamized mortality table at various rates of interest.

In 1904, the eminent Danish actuary, Dr. J. P. Gram, published a paper[1] dealing with this problem.

A brief exposition of the method developed by Gram was given by Christian Jensen in a paper[2] in the *Transactions of the Actuarial Society of America*, in 1908.

Given the force of mortality $\mu_x=\alpha+\beta e^{\gamma x}$ of a standard table and

$$\mu_x'=\alpha'+\beta' e^{\gamma' x}$$

of a special table on which we wish to base our calculations. The method of obtaining the assurance functions of the special table rests on showing that

$$\mu_x'=\rho+\mu_{\lambda x+\epsilon}, \tag{37}$$

where

$$\rho=\alpha'-\alpha\ , \qquad \epsilon=\frac{\log_e\beta'-\log_e\beta}{\gamma}\ , \qquad \lambda=\frac{\gamma'}{\gamma}\ .$$

We observe from (37) that we may use a force of mortality from a standard table transformed by (*a*) multiplication of the age by a constant, (*b*) addition of a constant to the thus transformed age, (*c*) addition of a constant ρ to such force of mortality.

It may now be readily shown that a continuous temporary life annuity $\bar{a}'_{x\overline{n}|}$ based on our special mortality table is given in terms of a corresponding continuous annuity $\bar{a}_{z\overline{m}|}$ on a standard table by

$$\bar{a}'_{x\overline{n}|}=\frac{1}{\lambda}\bar{a}_{z\overline{m}|}\ ,$$

where

$$z=\lambda x+\xi\ , \qquad \xi=\frac{\log_{10}\dfrac{\beta'}{\lambda\beta}}{\gamma\log_{10}{}^{e}}\ , \qquad m=\lambda n\ , \quad \text{and } \delta+\alpha=\frac{\alpha'+\delta}{\lambda}\ ,$$

where δ and δ' are forces of interest.

The relation $\alpha+\delta=(\alpha'+\delta')/\lambda$ gives the required value of δ since the remaining symbols are assumed to be known. If we have annuity tables based on our standard mortality table following Makeham's first modification of the law of Gompertz, and for a sufficient number of values of the interest rate for interpolation, then the annuity values for an assigned force of interest δ can be found by interpolation. The important conclusion in this connection is that

[1] *Skandinavisk Aktuarietidsskrift* (1904).

[2] Vol. 10, pp. 503-8.

we can calculate the annuity values based on our special mortality table following Makeham's law and an arbitrary rate of interest by means of interpolation for age and rate of interest and multiplication by a constant. The calculus is obviously involved throughout Gram's method which is based on forces of mortality.

12. The use of the calculus in the problem of the flow of population. In the applications of the calculus to which attention has thus far been directed in the present paper, we have assumed a law of mortality given by a continuous function l_x of the age. This assumption implies, on the numerically practical side, that we are basing our developments on a given table of mortality. For pedagogical reasons, it seems desirable to assume a table of mortality as a starting point in the teaching of actuarial theory. But it is fairly obvious that the more logical order would be to develop first the theory and methods of construction of mortality tables from statistical data. We find in recent methods of construction of such tables that very important use is made of the infinitesimal calculus in the theory of the flow of population which underlies the most elegant methods for the construction of life tables.

The development of the theory of the flow of population to which I refer centers around the names of Knapp, Zeuner, Lexis, Perozzo, and Blaschke. The work is most readily available to the American student through the excellent exposition of the theory given by James W. Glover under the title "Mathematical theory of construction of life tables" in his *United States Life Tables* (1921). Professor Glover deserves very much credit for the careful and useful work he has done in the exposition of the theory and its application to U. S. census data. In this theory, the aggregates of those living to the same age and of those living at the same time are represented by definite integrals. The function giving the number of deaths between ages x and $x+\xi$ satisfies a linear differential equation of the first order and degree with ξ as the independent variable. The solution of this differential equation gives the value of the number of deaths between ages x and $x+\xi$ in terms of the number E exposed and the force of mortality. This solution combined with the fundamental formula $q_x = 1 - e^{-\int_0^1 \mu_{x+\xi} d\xi}$ for the probability of death within a year enables us to compute tables of probabilities of death from statistical aggregates.

In conclusion, I think we may say that it is fairly obvious from the developments of actuarial theory in the past thirty years that the applications of the infinitesimal calculus in this field have tended to increase during this period both in the development of a more satisfactory theory for the construction of mortality tables and in the theory of life contingencies based on a given mortality table.

THE ARITHMETIC OF JEHAN ADAM, 1475 A.D.

By LYNN THORNDIKE, Columbia University

In a manuscript of the fifteenth century in the Bibliothèque Sainte Geneviève, Paris,[1] is an arithmetic in French. Both its text and author appear to have passed hitherto unnoted in works upon the history of mathematics. Yet it seems to contain an earlier example of numeration carried as far as trillions than has hitherto been recorded.[2] It was composed in the year 1475. As the introductory epistle at the beginning of the manuscript states, the author is Jehan Adam, who was secretary to Nicolle Tilhart, who in his turn was notary, secretary, and auditor of accounts to Louis XI, king of France from 1461 to 1483. That monarch more than once figures in the examples given in our text. Thus under Addition (fol. 9v) we are told, "le Roy donne a mons^{r} Lepaunatier IIII^{m}V^{c}XXIII (i. e. 4523), a mons^{r} Leschaconn II^{m}III^{c}XLI (2341), a mons^{r} Lesciuer III^{m}II^{c}XXIII (3223), a mons^{r} le maître doustel XIII^{c}XLII (1342). . ." ("The king gives Monsieur Lepaunatier 4523, M. Leschaconn 2341, M. Lesciver 3223, Master Doustel 1342"). Or in the first example of halving (*Mediacion*, at fol. 5r) we read, "The king gives the half of 51607 livres to Monsieur de Hourluum, that is, 25803 livres, ten sous."[3]

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[1] Ste. Geneviève MS Français 3143.

Since this article went to press Professor David Eugene Smith has called my attention to an item in the 1898 sales catalogue of the library of Prince Boncompagni which indicates that he had somehow come into possession of a recent copy or facsimile of the work of Jehan Adam. This copy was apparently made from our Sainte Geneviève MS., but the sales catalogue states that Boncompagni had asked Marre to search for the original MS. in the Bibliothèque Nationale and other libraries of Paris, and that he had failed to locate it. It is nevertheless duly listed in Ch. Kohler's *Catalogue des manuscrits de la Bibliothèque de Ste. Geneviève*, published in 1893. The following is the notice in the *Catalogo della Biblioteca Boncompagni*, Parte Prima, Roma, 1898, p. 94: "499. Adam Iehan. Traicté d'arismetique pour la pratique par gectouers (SIC) faite et compillé a Paris en lan mil 475, della Biobl. di 79 carte membranacee, salvo le prime due e le due ultime che sono cartacee. Scritto nel sec. xix. Facsimile, con grande accuratezza eseguito, con iniziali e figure a oro e colori e titoli rubricati. Il Princ. Boncompagni, avendo pregato il Sig. Marre di ricercare se l'originale del detto trattato di Jehan Adam si trovasse nella biblioteca Nazionale od in altra di Parigi, egli gli fece sapere che ogni ricerca in proposito riusci infruttuosa."

[2] The earliest example hitherto known appears to have been in the *Triparty* of Nicolas Chuquet, a bachelor of medicine in Paris, dated 1484 at Lyons.

See Aristide Marre, "Notice sur Nicolas Chuquet et son Triparty en la science des nombres" in Boncompagni's *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche*, Tomo XIII, 1880, pp. 555-592, followed at p. 593 *et seq.* by Chuquet's text. I owe this reference and other helpful suggestions to the kindness of Professor David Eugene Smith. Chuquet carried the nomenclature on beyond trillion and quadrillion to *nonyllion*, as the form is given in the printed text, or *nouyllion* (or, *novyllion*) as I read it in the facsimile of the manuscript (Paris, Bibliothèque Nationale, MS Français 1346; once Codex Colbert 2170, Regius 7483) in Professor Smith's library. Neither Chuquet nor his modern editor mentions the arithmetic of our Jehan Adam which is earlier by nine years.

[3] fol. 5r, "Mediacion est assavoir combien monte la moictie dun nombir proposi Exemple Le Roy donne la moitie de li^{m}vi^{c}vii l. (i. e. the sign for livres) a mons^{r} de hourluum cest xxv^{m} huic cens troys livres dix sols . . . ."

Our arithmetic is of the abacus type, reckoning by *jetons* or counters.[1] This word is commonly spelled *gectoners* in our manuscript,[2] but the form *greton* also occurs at least once.[3] Jehan Adam, however, although he still usually employs the Roman numerals, was acquainted with the Hindu-Arabic numerals. At fol. 9r he writes them out as far as 100, then by hundreds up to one thousand,[4] "2000, 8000,[5] etc. Et ainsi jusques infini en assemblent les nombres digitz articulz."

Our arithmetic is the only, or at least the chief, treatise in the manuscript, which is a small, illuminated codex with seventy-five leaves of text in all. The writing is in a fairly large hand, so that there is not very much text per page. The spelling of the old French is at times difficult to decipher. Illuminated figures to illustrate graphically the sums and calculations of the text also occupy some of the space. None occurs, however, between fol. 33r and fol. 60r inclusive. The manuscript has been incorrectly bound so that the leaf now numbered fol. 5r-v and devoted to the topics of Halving and Doubling should be transposed with the leaf now numbered fol. 13r-v, which continues the opening chapter of the text proper begun on fol. 4v. At fol. 63r this main body of text proper ends after three lines at the top of the page. Most of the page is then left blank until, near the bottom of the page, comes the rubric, "Conclusion et fin de ce present traicte en continuant lespitre escripte au commencement dicelluy." That is to say, the author here resumes the epistle with which he opened his work at fols. 1-4. His devout conclusion terminates at fol. 66r with the words, " . . . . a laquelle nous puissons tous venir a la reste de nos jours. Amen." This *Explicit*, as it would seem, is then followed at fols. 66v-69v by a table of contents. The manuscript, however, does not end

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[1] See E. Littré, *Dictionnaire*, 1869; "Les jetons se réduisent à une échelle dont les puissances succes sives au lieu de se placer de droite à gauche comme dans l'arithmétique ordinaire se mettent du bas en haut chacune dans une ligne où il faut autant de jetons qu'il y a d'unités dans les coefficients." Littré cites Buffon, *Ess. arithm.*, (*Œuvres*, X, 781).

More recently the whole matter of these *jetons* or *jettons* has been the subject of elaborate treatment with handsome plates by Francis Pierrepont Barnard, *The Casting-Counter and the Counter-Board*, Oxford, Clarendon Press, 1916, 358 pp., lxiii Pl. For briefer accounts, with especial reference to early printed arithmetics and the history of mathematics see David Eugene Smith, *Computing Jetons*, The American Numismatic Society, New York, 1921; and the same author's recent *History of Mathematics*, 1924, vol. II, chapter 3.

[2] fol. 4r, "arismeticque pour la praticque par gectoners"; fol. 20v, Rubric, "Septiesme et dernier espece darismeticque par gectoners." Littré, *op. cit.*, cites De Laborde, *Emaux*, p. 328, for the form *gectons* in the fourteenth century, "Gectons de la chambre des comptes de Monseigneur le duc d'Orléans." *Gecton* also occurs among some fifty forms of the word in French which Professor Barnard lists at pp. 26-27 of his aforesaid work, but neither of our forms, *gectoners* or *greton*, is among them. Possibly in the two passages quoted *gectoners* may denote those who make use of the counters or *gectons*.

[3] fol. 7r, "Item noctes que le premier greton dembas vault ung."

[4] By a slip 600 is written twice.

[5] Our author employs the old characters, 8, ꟼ, and ʌ, for four, five, and seven respectively.

there, but, with a "regle pour faire une taille de Monnoye," and further text and headings, goes on to fol. 75r. Perhaps these apparent additions should be transposed back into the text at some point.

Jehan Adam divides arithmetic into nine parts: numeration, addition, subtraction, halving, duplication, multiplication, division, progression, and extraction of roots.[1] He states, however, that in this present treatise he will treat of only seven of these parts, leaving progressions and the extraction of square and cube roots to another treatise.[2] Whether it is in existence or was ever written by him, I do not know. He is aware, moreover, that arithmetic may be subdivided differently. Thus Master Bartholomew of Roumanis (or Romains?), professor of Holy Scripture, makes but five parts of arithmetic, namely, addition, subtraction, multiplication, division, and extraction of roots. And it is true that halving is simply dividing by two, and that doubling is simply multiplying by two, while progression may be explained as addition. Division is of two sorts, "*simple et miste.*"[3]

The most interesting and novel feature of Jehan Adam's work is the paragraph and the illuminated figure in his section on Numeration which he devotes to the twenty decimal numbers from one to ten trillions (*dix trimillions*).[4] These are represented in the figure by as many counters (*jetons*), balls, or circles, superimposed one above another as if on an abacus, and with only one ball or counter for each denomination. Instead of "billion" Jehan writes *bymillion*, and instead of "trillion," *trimillion*.[5] He quotes the Latin verses of Alexander of Villa Dei[6] which enumerate up to ten figures (1,000,000,000), but then he goes on by himself to the twentieth figure or *dix trimillions* (10,000,000,000,000,000,000). Explaining the illumination, he says:

"Also note that the first counter from the bottom stands for one, the second stands for ten, the third stands for one hundred, the fourth stands for one thousand, the fifth stands for ten thousand, the sixth

[1] fol. 13r, "Et est assavoir que en arismeticque sont ix especes Numeracion Addiction Substraction meditacion dupplacion Multiplicacion division progression Extraction de Radices.

[2] fol. 4r.

[3] fol. 21v. The treatment of simple division continues to fol. 28v; then follow mixed division and the rule of three. "En division mixte la regle de trois est la forme a laquelle toute subtille question se doit reduire tant per nombre houpt que autier (or, *aultre*)."

[4] The text occurs at fol. 7r; the illumination on fol. 7v.

[5] Chuquet, on the other hand, spells the words in question "byllion" and "tryllion."

[6] fol. 8v, "Unum prima, secunda decem, dat tercia centum,
Quarta dabit mille, millia quinta decem,
Centum mille sexta dat, septima millia mille,
Mille dat octava millesies decies,
Centesies nova dat millesies quoque mille,
Millesies mille millesies decima,
Sic per millarium centenum denariumque."

These lines do not occur, however, in the version of the *Carmen de Algorismo* of Alexander of Villa Dei published in J. O. Halliwell's *Rara Mathematica*, 1841, pp. 73-83.

stands for one hundred thousand, the seventh stands for a million, the eighth stands for ten millions, the ninth stands for one hundred millions, the tenth stands for one thousand millions, the eleventh for ten thousand millions, the twelfth for one hundred thousand millions, the thirteenth for a billion, the fourteenth for ten billions, the fifteenth for a hundred (thousand)[1] billions, the sixteenth for a thousand billions, the seventeenth for ten thousand billions, the eighteenth for a hundred thousand billions, the nineteenth stands for a trillion, the twentieth for ten trillions."

Jehan Adam makes one or two brief incursions into the history of arithmetic. He has several suggestions to make as to the etymology of the word.[2] Arithmetic he derives from the Greek, *Ares*, meaning virtue, and from the Greek word for number. But the art is also called Algorismus from the Arabic, meaning an introduction to numbers, and this word may come from Algos, the name of the Arabic inventor of the art, and *Richmos*, a Chaldean word for number, or perhaps from the Greek *al*, equivalent to the Latin *in*, and *gogos*, equivalent to *dictio*. In another passage besides "the noble philosopher, Algus," (Jehan seems to have no preference as between the Latin and Greek forms of proper names) "the inventor and first compiler" of arithmetic, are listed Aristotle, Plato, Pythagoras, Isidore, Boethius, Albert, Alexander of Villa Dei, Masters Bartholomew des Roumanis, John of Sacrobosco, Johannes de Lineriis, Jean de Meun,[3] and Jehan Loquemeron, as past masters of the art.[4]

The problems of our arithmetic range from such simple ones as to multiply 2321 by xxiiii,[5] or, "If twelve ells of cloth are worth 34 livres, how much will 26 ells be worth,"[6] to more complicated problems concerning the division of profits among merchants forming a company and contributing different amounts of capital,[7] or even investing different sums for different periods of time,[8] or "touching a serpentine concerning which the King wants to know

[1] "cent mil bymillions" is evidently a slip for "cent bymillions." The French text of the passage reads: "item noctes que le premier greton dembas vault ung, le second vault (. . . . here some words seem to be omitted) cent, le quart vult mille, le Ve vault dix M, le VIe vault cent M, le VIIe vault Milion, Le VIIIe vault dix Million, Le IXe vault cent Millions, Le Xe vault Mil Millions, Le XIe vault dix mil Millions, Le XIIe vault Cent mil Millions, Le XIIIe vault bymillion, Le XIIIIe vault dix bymillions, Le XVe vault cent (mil) bymillions, Le XVIe vault mil bymillions, Le XVIIe vault dix Mil bymillions, Le XVIIIe vault cent mil bymillions, Le XIXe vault trimillion, Le XXe vault dix trimillions."

[2] fol. 4v.

[3] One might expect mention of John of Meurs (Johannes de Muris) rather than of the second author of *The Romance of the Rose* in a list of arithmeticians, but Iehan de Mehung seems unmistakably to denote the latter.

[4] fols. 2v—3r, "Et depuis Aristote platon pitagoras ysodore Boisse Alebert Alixandre de Villedieu Maistres bartholomieux des Roumanis Iehan de sacro bosco Iehan de Ligneriis Iehan de Mehung et Iehan Loquemeron en ont si bien oy (?) souverament traicte que nulle Reprehencion ny doit estre faicte."

[5] fol. 16v.

[6] fol. 30v.

[7] fol. 33v

[8] fol. 42v

the quantity of each metal that will be in one of the broken pieces."[1] We also have "Rules" for over-charge and annulments (? *recindemens*),[2] for recovery of annulments,[3] for equal and mixed proportion,[4] for fractions[5]—as to divide a sum in ratios of 1/2, 1/3, and 1/4, or 1/3, 1/4, 1/5, and 1/6—and for alloying money.[6]

It is interesting to associate this arithmetic of Jehan Adam composed in 1475, with the important *Triparty* of Nicholas Chuquet,[7] finished in 1484. Both works were thus composed within a decade of each other, both were written in French and by authors intimately connected with Paris, and they are the first two works known to employ the terms, billion and trillion. There is another similarity and sign of close connection between them. The master Bartholomew, "des Romanis" or "des Roumanis," whom Jehan Adam mentioned at least twice, is also cited and a passage from his work criticized in the appendix to Chuquet's *Triparty*, where he is called "*maistre berthelemy de rõmans*, formerly of the Order of Friars Preachers at Valence and doctor in theology."[8] Similarly Jehan Adam spoke of him as "*professeur en la saincte escripture*." What was the debt of our two authors to this Bartholomew and can his own work be recovered?

THE EARLY CONTRIBUTIONS OF CARL SCHOY[9]

By DAVID EUGENE SMITH, Columbia University

The recent appointment of Dr. Carl Schoy as "Lehrauftrag für Geschichte der exacten Naturwissenschaften im Orient" in the University of Frankfort am Main, the work beginning on October first of the current academic year, is so significant in the study of the history of mathematics as to deserve more than a mere passing notice. The number of scholars who are proficient not only in mathematics and astronomy but also in the eastern languages has always been limited, even as it is today. Woepcke, who began as a Privatdozent at

[1] fol. 41r.

[2] fol. 36r, Rubric, "Regle de trop charge et Recindemens."

[3] fol. 38v, Rubric, "Aultre Regle et maniere de Repprendre les Recindemens."

[4] fols. 44v-45r.

[5] fol. 46r *et seq.*

[6] fol. 59v, Rubric, "Regle pour aloyer monnoye avecques au lire monnoye."

[7] See note 1a above.

[8] Aristide Marre, "Appendice au Triparty en la science des nombres de Nicolas Chuquet Parisien," in Boncompagni's *Bulletino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche*, Tomo XIV, (1881) pp. 415-16 and 442.

[9] See also in this connection a "Note on the mathematics of the Arabs" by Professor L. C. Karpinski published in the MONTHLY (1925), pp. 44-45.

Bonn was such as scholar, and so was Sédillot. The younger Hankel was unusually well qualified to undertake the interpretation of Arabic mathematics, but his early death prevented his work from progressing beyond the initial stages. Rosen, nearly a century ago, gave us a worthy translation of Al-Khowârizmî, and Braunmühl, at the close of the last century, contributed materially to our knowledge of Arabic trigonometry; but it was not until recently that men like Carra de Vaux and particularly the late Professor Suter brought modern scholarship to bear upon the problem of making the Arabic literature in mathematics better known to scholars of the present day. It was a century before a worthy successor to Colebrooke appeared in the person of the late Professor Rangacharya, and it is only now that scholars like Mikami and Père Vanhée are making known any considerable amount of the ancient mathematics of the Far East.

Within the last half dozen years a new name has been added to the list of scholars who have been or are able to interpret clearly the Mohammedan astronomy and mathematics, and his contributions have been such as to give confidence that he is destined to rank among the leaders in this field of science. Dr. Schoy was a student under Professor J. Chr. Seybold of Tübingen and was, in fact, his only one during the war period. With him he had a thorough training in the reading and interpretation of Arabic manuscripts, besides studying Persian and Turkish and devoting a considerable amount of time to mathematics and astronomy. From 1919 to 1921 he was Privatdozent at Bonn, his chief interests being in the field of mathematical geography and the history of geography and astronomy.

Since that time his output of work has been notable, including the following memoirs and volumes:

1920. "Das 20. Capitel der grossen Ḥākimitischen Tafeln des Ibn Jūnis: Ueber die Berechnung des Azimuts aus der Höhe und der Höhe aus dem Azimut." (*Annalen der Hydrographie und maritimen Meteorologie.*)

1920. "Abhandlung des Ḥasan ibn al-Ḥusain ibn al-Haiṯam: Ueber eine Methode, die Polhöhe mit grösster Genauigkeit zu bestimmen." (*De Zee.*)

1921. "Ueber eine arabische Methode die geographische Breite aus der Höhe im 1. Vertical (Höhe ohne Azimut) zu bestimmen." (*Annalen der Hydrographie*, etc.)

1922. "Abhandlungen des al-Hasan ibn al-Ḥasan ibn al-Haiṯam (Alhazen) über die Bestimmung der Richtung der Qibla." (*Zeitschrift der deutschen morgenländ. Gesellschaft.*)

1922. "Abhandlung von al-Faḍl ibn Hâtim an-Nairîzî: Ueber die Richtung der Qibla." (*Sitzungsberichte des bayer. Akad. der Wissenschaften.*)

1922. "Die Bestimmung der geographischen Breite eines Ortes durch Beobachtung der Meridianhöhe der Sonne oder mittels der Kenntnis 2^er^ anderer Sonnenhöhen und den zugehörigen Azimuten, nach dem arabischen Text der Ḥākimitischen Tafeln des Ibn Yūnus." (*Annalen der Hydrographie*, etc.)

1922. "Abhandlung über die Ziehung der Mittagslinie dem Buche über das Analemma entnommen, samt dem Beweis dazu, von Abū Sa'id aḍ-Darîr." (*Ibid.*)

1922. "Aus der astronomischen Geographie der Araber." Originalstudien nach al-Qânûn al-Mas'ûdî des Al-Bîrûnî. (*Isis.*)

1923. "Gnomonik der Araber." Berlin.
1923. "Beiträge zur arabischen Trigonometrie." (*Isis.*)
1923. "Ueber den Gnomonschatten und die Schattentafeln der arabischen Astronomie." Hannover.
1924. "Sonnenuhren der spätarabischen Astronomie." (*sis.*)
1924. "The geography of the Moslems of the Middle Ages." (*Geographical Review.*)
1925. "Die Bestimmung der geographischen Breite der Stadt Ġazna mittels Beobachtungen im Meridian, durch den arabischen Astronomen und Geographen al-Bîrûnî." (*Annalen der Hydrographie*, etc.)
1925. "Drei planimetrische Aufgaben des arabischen Mathematikers Abû'l-Jûd Muḥammed ibn al-Lîth." (*Isis.*)
1925. "Abhandlung des Schaichs Ibn 'Âlî al-Hasan ibn al-Hasan ibn al-Haitham: Ueber die Natur der Spuren (Flecken), die man auf der Oberfläche des Mondes sieht." Hannover.
1925. Die trigonometrischen Lehren des Muhammed ibn Aḥmad, Abû'l-Rîhân al-Bîrûnî.

Any such an array of monographs, each based upon a study of original manuscripts, would be impressive merely on the score of number, but in this case the product is even more so in respect to scholarship. To give, as in his article on the "Geography of the Moslems of the Middle Ages," the first translation of the method of Ibn Yûnis for finding longitude by observing solar eclipses; to show by means of a translation from Sibṭ al-Mâridînî that the Egyptians of the Middle Ages were the first to use a sundial directed towards the world pole; to be the first to give us a translation of Ibn 'Alî al-Ḥasan's work on the interpretation of the irregularities of the lunar surface; and to make more clearly known, by accurate translations, the nature of the umbra recta and umbra versa in the leading Arabic works on trigonometry,—these (to mention only a few instances) are gratifying promises of future contributions of great moment. We have by no means solved the problem of the value to be assigned to the Arab contributions to mathematics. Even such a seemingly simple question as that relating to the introduction of our numeral system (or its forerunner) into Bagdad, and somewhat later into Spain, is quite unanswered. We still have the origin of the ġobar numerals to settle, and the story of the other numeral forms in use in the Mediterranean area in the Middle Ages has yet to be written. The sources and the "prodromic symptoms" of the Arabic algebra have still to be searched with greater care; and it seems fair to assert that there is a larger field to be explored than has generally been imagined. The glosses on Euclid alone, as shown in a considerable number of unpublished manuscripts, might well repay the study of the pupils whom Dr. Schoy will gather about him in his new field of labor. He will lecture on the history of Arabic astronomy and trigonometry and will, no doubt, inspire a small group of students to enter the promising field of the study of Arabic manuscripts.

It is, of course, to be hoped that a scholar with such an equipment will soon be enabled to devote all his time to research in the best sense of this much-debased term, and to work with a limited number of sufficiently equipped

students. It would be a great contribution to scholarship in America if a chair could be offered to such a man in one of our own universities.

It may not be out of place to express appreciation of the typographical appearance of at least one of Dr. Schoy's latest works, the *Abhandlung des Schaichs Ibn 'Alî al-Ḥasan ibn al-Ḥasan ibn al-Haitham*, issued by the Orient-Buchhandlung Heinz Lafaire, at Hannover, in 1925. As a piece of printing of a scientific work it leaves little to be desired, and some of our American publishers might do well to examine it as a model of a monograph of its kind.[1]

NOTE ON INVOLUTIONS OF THE NTH ORDER

By E. F. ALLEN, University of Missouri

1. Introduction. In the first chapter of Newson's posthumous book on collineations,[2] the theory of linear fractional transformations is developed. The general form of this transformation is

$$x_1=\frac{ax+b}{cx+d}\ , \tag{1}$$

which is called the explicit form of the transformation. The following well-known properties of this transformation are derived:[3]

(*a*) If $a=d$ and $b=c=0$, then $x_1\equiv x$.

(b) If $a+d=0$, then (1) transforms x into x_1 and also x_1 into x. In this case the transformation is called an involution of the second order.

(*c*) The invariant points of the transformation are[4]

$$(A,A')=\frac{(a-d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2c}\ . \tag{2}$$

If the invariant points are distinct the transformation is said to be of type I, otherwise it belongs to type II.

(*d*) Transformations of type I may be written in the form[5]

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[1] The day after reading the proof of the above article word was received of the sudden death of Dr. Schoy, on December 6, 1925, as the result of an apoplectic stroke. He was forty-eight years of age and had not been in good health for some time, as his latest letter to me showed. His funeral took place at Frankfort am Main on December 9. Thus passes away a brilliant scholar and an indefatigable worker. Not until he was about forty years of age did he find his real life work, but in the few years that followed he made remarkable progress. D. E. S.

[2] H. B. Newson, *Theory of Collineations*. Press of the State Printer, Topeka, Kansas.

[3] *Ibid.*, p. 3.

[4] *Ibid.*, p. 9.

[5] Newson, p. 10.

$$\frac{x_1-A}{x_1-A'}\cdot\frac{x-A'}{x-A}=k\ , \tag{3}$$

where A and A' are the invariant points of the transformation and k, which is called the characteristic invariant, has the value

$$k=\frac{[(a+d)-\sqrt{(a+d)^2-4(ad-bc)}\,]^2}{4(ad-bc)}\ . \tag{4}$$

It is proved[1] that (3) is an involution of the second order if $k=-1$. Form (3) is called the implicit normal form of type I.

In this note we propose to find a value of k for which (3) is the implicit normal form of an involution of the nth order, and to show that the invariant points of all involutions of form (1), having real coefficients, are conjugate imaginaries, except for the second order involution which may have either real or imaginary invariant points. We shall also derive the explicit form of the involution of the nth order. The discussion shall be limited to transformations of type I, having real coefficients and a non-vanishing modulus.[2]

2. **Theorems.** We shall prove the following theorems:

THEOREM I. *The necessary and sufficient condition that the invariant points of a linear fractional transformation be conjugate imaginaries is that the characteristic invariant be a complex number, unless $k=-1$, when they may be real.*

THEOREM II. *The implicit form of a linear fractional transformation is an involution of the nth order if k is a primitive root of the equation $k^n-1=0$.*

THEOREM III. *The explicit form of an involution of the nth order is*

$$x_1=\frac{\left(\alpha\sin\dfrac{\pi l}{n}-\beta\cos\dfrac{\pi l}{n}\right)x-(\alpha^2+\beta^2)\sin\dfrac{\pi l}{n}}{\left(\sin\dfrac{\pi l}{n}\right)x-\left(\alpha\sin\dfrac{\pi l}{n}+\beta\cos\dfrac{\pi l}{n}\right)}\ ,$$

where α, β $(\beta\neq 0)$ are real numbers and the integer l is prime to n.

3. **Proofs of the theorems.** Theorem I[3] follows immediately from equations (2) and (4) which show that if k is a complex number the invariant points of the transformation must be conjugate imaginaries, and conversely, unless indeed $a+d=0$, when $k=-1$ in which case we have the well-known second order involution. Equation (2) shows that the invariant points of the second order involution may be either real or imaginary according to the value of the modulus $ad-bc$.

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[1] Newson, p. 22.

[2] The expression $ad—bc$ is called the modulus of the transformation.

[3] See Newson, p. 32, for part of this theorem.

To prove the second theorem we note that the implicit normal form (3) transforms the point x into the point x_1. Likewise

$$\frac{x_2-A}{x_2-A'}\cdot\frac{x_1-A'}{x_1-A}=k \tag{5}$$

transforms the point x_1 into the point x_2. By eliminating x_1 from equations (3) and (5) we obtain

$$\frac{x_2-A}{x_2-A'}\cdot\frac{x-A'}{x-A}=k^2\,, \tag{6}$$

which transforms x directly into x_2.

The necessary and sufficient condition that $x_2=x$ is that $k^2=1$. For if $k^2=1$, equation (6) reduces to $x_2=x$, since we have assumed that the invariant points A and A' are distinct. If $x_2=x$, equation (6) reduces to $k^2=1$. By an easy generalization it is seen that $x_n=x$ if $k^n=1$ and conversely. This proves the theorem.

In order to derive the equation of theorem III let us solve equation (3) for x_1. Thus

$$x_1=\frac{(kA'-A)\,x+AA'(1-k)}{(k-1)x+(A'-kA)}\,. \tag{7}$$

If this is the equation of an involution of the nth order, having imaginary invariant points, then, by theorems I and II, $k=\cos 2\pi l/n+i\sin 2\pi l/n$, $A=\alpha+\beta i$, and $A'=\alpha-\beta i$. When these values of k, A, and A' are substituted in equation (7) we obtain an equation which reduces to the equation of theorem III.

4. Geometric derivation of the formula. The equation of theorem III may be obtained geometrically as follows: Let $P(\alpha,\beta)$ be a point in the $xy-$ plane not on the x-axis. Join $Q(x, 0)$ to P by a straight line and draw another line from P, making an angle of $-l\pi/n$ radians with the line PQ. Call the coordinates of the point R, where this second line intersects the x-axis, $(x_1, 0)$. Then the required equation may be obtained by applying the formula for the angle between two lines.

5. Conclusions. The following conclusions may be drawn from the above discussion: The invariant points of all involutions of type I, having real coefficients, are conjugate imaginaries if the order of the involution is greater than the second. All involutions of order n having imaginary invariant points, can be generated by revolving n equally spaced rays about the point (α, β) $\beta\neq 0$ in the xy plane. The points of the involution are determined by the intersections of these rays with the x-axis. It is possible to construct an in-

volution of the nth order by ruler and compasses if the angle π/n can be constructed by the aid of those instruments. The product of two involutions of orders n and m respectively is an involution of order nm.

QUESTIONS AND DISCUSSIONS

EDITED BY TOMLINSON FORT, Hunter College, Park Ave. and 68th St., New York City.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. MINIMIZING THE ARITHMETIC IN SOLUTIONS BY SUCCESSIVE APPROXIMATIONS

By C. C. CAMP, University of Illinois

There are two extremely important considerations in solving equations by the method of successive approximations. One is the rapidity of the convergence to the root and the other is the amount of arithmetical labor required for each new approximation.

Algebra may be regarded as the art of avoiding tedious numerical calculations. On the practical side higher mathematics in general should be an extension of this idea of algebra. In this connection it is surprising how the arithmetic of certain problems in the mathematics of finance can be shortened.

As Professor Forsyth says in his article in the MONTHLY for March, 1925, the convergence of a sequence may be exasperatingly slow. It is believed that three examples will best illustrate a method by which the computing as well as the number of approximations may be diminished.

Taking the problem given, *i. e.*, to find the rate of interest of an annuity of \$1,000 per year for 20 years, whose present value is \$12,102, we have $a_{20}=12.102$. From Glover's *Tables* the value for five per cent is 12.462 and for five and one-half per cent 11.950. By ordinary interpolation we obtain $.05+360/1024$ or .0535 as the first approximation. The iterative method should start here. The use of Schrön's[1] table of logarithms of numbers from 1 to 108,000 is advised for this purpose since the error in any logarithm given in the latter part of the table can be reduced to $\frac{1}{4}$ in the eighth decimal. The 20th power of 1.0535 would have then a maximum error of 7 in the 8th digit, if extreme accuracy is desired. For the first interpolation a slide rule is more

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[1] This table of logarithms of numbers can now be had bound and is the handiest 7 (&8) place table available, none being more accurate.

convenient than a calculating machine or a multiplication table such as Crelle's or Peters'.

For a second illustration take the one presented in the MONTHLY (1925), pp. 272-287, by Professor L. R. Ford in his excellent article: Find the return on a five per cent bond due in twenty years bought at 90.

His form (c), p. 276, may be written

$$i = r - (A-1)a_n^{-1} = .05 + .1a_{20}^{-1} .$$

If we use $i_1 = .055$, then without interpolation from Glover's *Table*, p. 272, we have $i_2 = .0584$. A simple interpolation gives $i_3 = .05861$ wherein .0070 is the tabular difference for one per cent. Another simple interpolation gives $i_4 = .058621$. One here begins the regular method of successive approximations as given in form (c) and by the use of Schrön's table finds $i_5 = .05862111$. This value is in error less than unity in the eighth decimal.

What is emphatic here is the importance of reaching a close approximation as soon as possible. The same is true in using Newton's method. In problems of finding the investment rate, particularly in connection with building and loan associations, the equation can usually be simplified so that the tables may be used for the first few approximations. The numerical work is thus abridged tremendously over that required by methods presented in the best current text books on the mathematics of finance.

Take for example the problem of finding the investment rate from the viewpoint of the borrower in an association which matures its stock on the 79th payment by accepting the regular \$7 interest on a loan of \$1200 but only \$2.40 instead of the regular \$12 dues. To simplify the solution consider first the monthly rate. Then

$$19 + 19a_{78} - 9.60v^{78} = 1200 .$$

Choosing as an initial value $b_1 = 7/12$ per cent, we first approximate to the third term and solve for a_{78}, obtaining

$$a_{78} = 1187.09876/19 = 62.47888 .$$

One interpolation between 7/12 and 5/8 per cent gives $b_2 = .0058526$, all of the digits of which are correct. Hence $i = 7.254$ per cent.

Formula (B'), p. 126, in Professor Forsyth's article gives a very slowly converging sequence as may be seen by taking differentials of

$$b_2 = \frac{1-(1+b_1)^{-78}}{62.4784} , \quad \text{since } db_2 = .786\ db_1 \text{ (nearly).}$$

This method of using differentials for the approximate errors in testing the rapidity of the convergence of a sequence of approximations is quite general

and of wide application not only in the mathematics of finance but also in engineering problems and other fields including the solution of higher algebraic equations.

The method of attack in the case of a sequence which converges slowly will depend on the type of equation to be solved. The use of tables illustrated above is especially appropriate for problems in finance. The author is reserving for a future paper another method of speeding up the convergence in general.

II. Six Decimal Places from a Five-place Table

By L. S. Dederick, University of British Columbia

1. It has probably occurred to most users of mathematical tables that at certain points in certain tables a more accurate value than that given by a single entry may easily be inferred from the neighboring entries, *e. g.*, log cos $16'$ or log cos $17'$ in a five-place table.

In any part of a table where the second differences are relatively small considerable increase of accuracy can commonly be obtained by merely applying linear interpolation to several pairs of neighboring entries. The reader can easily verify that, aside from the effect of higher differences, the result of linear interpolation is subject to an error not greater than a half unit in the last place if the errors in the entries used are thus restricted. The original entry and the several interpolations therefore each give unit intervals within which the true value must lie. Hence the latter must lie in the part common to all these, an interval commonly much less than a unit.

For example, from a five-place table of common logarithms, $.653205 < \log 4.5 < .653215$. By interpolating between the values for 4.499 and 4.501 we get .653215, which shows that $.653210 < \log 4.5 < .653220$. Also from interpolation between log 4.499 and 4.502, we get $.6532117 < \log 4.5 < .6532217$. Combining these, we get that log 4.5 lies between .6532117 and .6532150, an interval just one-third the size of the original. One can readily verify that the effect of second differences is here negligible. We have thus materially increased the accuracy of the single entry by a trifling amount of labor.

If the process is to be carried further, a selection must be made of values to be used. This can be done by inspecting the tabular differences, seeking on the one hand pairs of entries whose tabular errors appear to be large and positive, and on the other those where they seem large and negative. In the particular example no further improvement in the lower limit presents itself. For the upper limit, if we use the entry for 4.505 combined in succession with those for 4.497, 4.494, 4.491, 4.488, and 4.485, we get successively decreasing upper limits, the last giving us .6532125. So if we could neglect second differences we could assert that $.6532117 < \log 4.5 < .6532125$, or that log 4.5 is

.653212 correct to six places. This is not far wrong but the effect of the second difference has become appreciable for this interval.

2. A much more general method, taking account of differences of all orders that may affect the result, is the following. Let the value of the function near the point investigated be given by the power series $f(x_0+h)=a_0+a_1h+a_2h^2+\cdots$. Then we may form the following differences:

$$\Delta_1(h)=f(x_0+h)-f(x_0-h)=2a_1h+2a_3h^3+2a_5h^5+\cdots,$$
$$\Delta_2(h)=f(x_0+h)-2f(x_0)+f(x_0-h)=2a_2h^2+2a_4h^4+2a_6h^6+\cdots,$$
$$\Delta_3(h)=\Delta_1(2h)-2\Delta_1(h)=12a_3h^3+60a_5h^5+\cdots,$$
$$\Delta_1'(h)=8\Delta_1(h)-\Delta_1(2h)=12a_1h-48a_5h^5+\cdots,$$
$$\Delta_4(h)=\Delta_2(2h)-4\Delta_2(h)=24a_4h^4+120a_6h^6+\cdots,$$

and as many more of the same sort as are needed. If they are formed from the tabulated values each is subject to accumulated tabular errors. These may amount to a maximum of 1 in Δ_1, 2 in Δ_2, 3 in Δ_3, 9 in Δ_1', 8 in Δ_4, etc., reckoned in units of the last decimal place. These maxima are independent of the value of h. From these we may write

$$a_1=\frac{\Delta_1\pm 1}{2h}-a_3h^2+\cdots=\frac{\Delta_1'\pm 9}{12h}+4a_5h^4+\cdots,$$
$$a_2=\frac{\Delta_2\pm 2}{2h^2}-a_4h^2+\cdots,\quad a_3=\frac{\Delta_3\pm 3}{12h^3}-5a_5h^2+\cdots,$$

expressions in which the error may be made very small by taking a large value of h. In all ordinary cases the values of the successive coefficients decrease rapidly, so that it is not necessary to find many of them. The accuracy of any one may be increased by using two values of h and thus confining the value of the required coefficient to the smaller interval common to two intervals that overlap. By using alternative forms, such as the second given for a_1, it is possible to eliminate from the formula for one coefficient all later coefficients that would have any appreciable effect, for the value of h used.

We put these relatively accurate values of a_1, a_2, $\cdots$ in the power series, and write $a_0=f(x_0+h)-s(h)$, where $s(h)=a_1h+a_2h^2+\cdots$. The error in the tabular value of $f(x_0+h)$ is not more than $\frac{1}{2}$. The error in $s(h)$ may be kept very small if the values of h are taken much smaller than those used in determining the coefficients. By using a number of values for h we get a number of overlapping intervals within which a_0, the true value of the function, must lie. The part common to these will in general be much smaller than any one, and we may thereby increase the accuracy to a considerable extent, frequently enough to get another decimal place, sometimes more than this.

As an illustration, let us seek from a five-place table a more accurate value for log 1.7 than .23045, the one given. Using Δ_1' with $h=103$ and 102 we get $25.545<a_1<25.548$ (using for units the least count of the table). Likewise we may get $.00746<-a_2<.00758$ and $.00000290<a_3<.00000305$. The value of a_4 is too small to determine even its sign from values of h in the neighborhood of 100. With mean values of these coefficients we form $f(x_0+h)-s(h)$ for various values of h. This gives 23044.43 for $h=3$, and 23045.38 for $h=-13$. These would give $23044.88<a_0<23044.93$ if the coefficients were strictly correct. To be sure of the limits, however, we should use the most unfavorable values of the coefficients in each case. These give $.2304485<\log 1.7<.2304494$, or $\log 1.7=230449$ correct to six decimals.

The probable number of values of h necessary to consider in order to increase the accuracy tenfold is discussed in Problem No. **3160** in this issue of the MONTHLY.

Limitations of space forbid the presentation of methods for systematizing, refining, and abbreviating the necessary computations.

3. A somewhat more accurate result may often be obtained by applying to the earlier interpolation method corrective terms based on the coefficients of the power series. Thus

$$a_0=\frac{hY+Hy}{h+H}-a_2hH-a_3hH(H-h)\cdots,$$

where $Y=f(x_0+H)$ and $y=f(x_0-h)$. In the last example if we use $h=13$, $H=23$ for a lower limit, and $h=25, H=3$ for an upper limit, we get $.2304486<\log 1.7<.2304493$, a result slightly better than the preceding.

Of these two methods the first is always, and the second frequently, within the realm of practical computation, that is will involve less work than computing the value of the function *de novo.*

4. Another method, having a wider application, is that of trial and error. This is more laborious, but can be applied to a short table, where the large values of h used in the preceding method for finding the coefficients, are not available. This method consists of subtracting from the tabulated values the values of a polynomial so chosen (by inspection) as to give small residues near the point investigated. This process is repeated until a certain range of residues is obtained lying between $-\frac{1}{2}$ and $\frac{1}{2}$. The subtraction is then continued of polynomials so chosen as to extend the range of such residues in each direction. The residues for the special value investigated will then approximate to the tabular error.

III. Note on the Convergence of Fourier Series[1]

By Dunham Jackson, University of Minnesota

In a recent number of the Monthly, Franklin has given a particularly clear and simple proof of the convergence of Fourier series under hypotheses general enough to cover the most important applications.[2] The conditions on the function to be represented may be essentially paraphrased by saying that it has the period 2π, and that a period interval can be divided into a finite number of subintervals, in each of which, considered by itself, the function has a continuous derivative, if suitably defined at the ends of the subinterval, though there may be a discontinuity in passing from one subinterval to the next.

The purpose of this note is to arrive at the same result by a proof which is believed to be still simpler in the character of its reasoning, though perhaps not very much shorter when spread on paper.

It is convenient to keep the notation and the identical language of Franklin's paper to a point a little above the middle of page 477, through the lines

"Thus the equation in question will follow from . . . .

$$\lim_{n=\infty} \int_0^\pi [f(u+x)-f(x+)]s_n du = 0 \text{ ."}$$

Then the proof proceeds as follows:

Let $\varphi(x)$ be an arbitrary function which is bounded and integrable from $-\pi$ to π (but not necessarily periodic nor defined in any way outside this interval), let α_n, β_n be its Fourier coefficients:

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t)\cos nt\,dt \text{ ,} \qquad \beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(t)\sin nt\,dt \text{ ,}$$

and let

$$\sigma_n = \tfrac{1}{2}\alpha_0 + \sum_{m=1}^{n} (\alpha_m \cos mx + \beta_m \sin mx) \text{ .}$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [\varphi(x) - \sigma_n(x)]^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} [\varphi(x)]^2 dx - \left[\tfrac{1}{2}\alpha_0^2 + \sum_{m=1}^{n} (\alpha_m^2 + \beta_m^2) \right] \text{ .}$$

As the left-hand member can not be negative, the sum on the right must remain bounded as n increases indefinitely, the series

[1] Presented to the American Mathematical Society, October 31, 1925.

[2] P. Franklin, A simple discussion of the representation of functions by Fourier series, this Monthly, (1924) pp. 475-478.

$$\sum_{m=1}^{\infty} (\alpha_m^2+\beta_m^2)$$

must converge, and it must be that[1] $\lim_{n=\infty} \alpha_n = 0$, $\lim_{n=\infty} \beta_n = 0$. Since $\varphi(x)\cos\frac{1}{2}x$ and $\varphi(x)\sin\frac{1}{2}x$ satisfy the conditions imposed on $\varphi(x)$, the integrals

$$\int_{-\pi}^{\pi} \varphi(t)\sin\tfrac{1}{2}t\cos nt\,dt\ , \qquad \int_{-\pi}^{\pi} \varphi(t)\cos\tfrac{1}{2}t\sin nt\,dt$$

approach zero as n becomes infinite, and their sum

$$\int_{-\pi}^{\pi} \varphi(t)\sin(n+\tfrac{1}{2})t\,dt$$

does the same.

If $\psi(x)$ is bounded and integrable from 0 to π, the function $\varphi(x)$ which is equal to $\psi(x)$ for $0<x\leqq\pi$, equal to $-\psi(-x)$ for $-\pi\leqq x<0$, and equal to 0, say, for $x=0$, is bounded and integrable from $-\pi$ to π, and the integral

$$\int_{-\pi}^{\pi} \varphi(t)\sin(n+\tfrac{1}{2})t\,dt = 2\int_{0}^{\pi} \psi(t)\sin(n+\tfrac{1}{2})t\,dt$$

still has the limit zero.

But, under the hypotheses of the theorem, the function

$$\psi(u) = \frac{f(x+u)-f(x+)}{2\sin u/2} = \frac{f(x+u)-f(x+)}{u}\cdot\frac{u/2}{\sin u/2}$$

is bounded and integrable for $0<u\leqq\pi$, since it has essentially the same discontinuities as $f(x+u)$ in the open interval, and approaches a limit as u approaches zero from the right. Consequently[2]

$$\lim_{n=\infty}\int_{0}^{\pi} [f(u+x)-f(x+)]s_n du = \lim_{n=\infty}\int_{0}^{\pi} \psi(u)\sin(u+\tfrac{1}{2})u\,du = 0\ .$$

It may be pointed out that this proof does not require that the derivative of $f(x)$ be continuous.

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[1] *Cf., e. g.*, M. Bôcher, Introduction to the theory of Fourier's series, *Annals of Mathematics*, vol. 7 (1906), pp. 81-152; p. 86.

[2] Cf. de la Vallée Poussin, *Leçons sur l'approximation des fonctions d'une variable réelle*, Paris, 1919; pp. 19-20.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

(a) *Ve-Po-Ad.* Chicago, Reliable Typewriter and Adding Machine Corp. Price $2.95.

(b) *Baby Calculator.* Chicago, Baby Calculator Sales Co. Price $2.50.

These two pocket adding machines are constructed on exactly the same principle. For each column of figures there is a slider, which is free to move vertically. This bears the digits 0 to 9, some one of which is visible at any time through a window in the fixed face of the machine, and indicates the result arrived at in that column. The slider has teeth projecting on both sides, placed at the same interval from each other as the digits . Those projecting on the left are accessible through a vertical slot in the face, situated below the window for that column and slightly to the left of it. Adjacent to this slot the face bears the digits 1 to 9 at the same intervals as those on the slides.

The machine is operated by placing the point of a stylus between two teeth and pushing the slides up or down. To add a given digit the point is placed in the interval opposite where this digit appears on the face, and is then pushed to the bottom or to the top of the slot, the choice being indicated to the eye by a difference in coloration of the teeth. The latter case is that where "carrying" is necessary. This is accomplished by what is the most characteristic device of the machines. Each slot is widened toward the left at the top for a distance corresponding to one unit. The operation requires that whenever the stylus is brought to the top of the slot it is carried on around to the left and down, thereby engaging the right hand teeth of the next slides to the left, moving it one unit, and thus "carrying one." A little more manipulation is necessary in case the column to the left happens to register 9.

The Baby Calculator allows for the addition of seven figure numbers to a seven figure total; the Ve-Po-Ad, of eight figure numbers to a nine figure total. The extreme column on the left in the Ve-Po-Ad is used only for carrying. Both instruments are unnecessarily hard on the eyes. In the Ve-Po-Ad, this trouble is due to the figures on the sliders. These are red on a shiny metallic background, and are almost invisible in most lights that do not give a glare from the metallic surface. There is the same sort of objection to the figures on the face of the Baby Calculator, which are shiny brass on shiny black. The sliders of the Baby Calculator and the face of the Ve-Po-Ad are optically satisfactory. The Ve-Po-Ad presents a more pleasing appearance, and is easier to operate, the Baby Calculator having a decidedly stiff action. This becomes slightly easier with use but not much. The Ve-Po-Ad has a permanent cover

of some leather-like material, which opens like a note-book. This adapts it better to carrying in the pocket. It is not as suitable to the vest pocket, however, as the Baby Calculator, being too wide. In fact the names would be more suitable if interchanged.

The operation of subtraction is provided for by a second set of figures for each column, which appears on the face of the machine. They are distinguished from the addition figures by being made much smaller. In subtracting, these figures are used in the same way as the larger ones for addition, except for the procedure when the stylus reaches the top or bottom of the slot. At the top it is not to go around to the left and down as in addition. When it reaches the bottom, "borrowing" is necessary, and this must be performed separately, the stylus being taken out and removed to the widened part at the top of the slot. It seems unfortunate that a similar widening at the bottom is not provided for this purpose, and its absence probably increases the time taken in subtraction nearly fifty per cent.

The instruction pamphlet for the Ve-Po-Ad deals with addition only. That for the Baby Calculator gives full directions for subtraction. Its treatment of multiplication and division, however, is very brief and inadequate. The user must develop the technique of these operations for himself. The directions for division are particularly poor, the method given being solely repeated subtraction. This, while satisfactory for the larger and more mechanically operated machines, would be extremely tedious with these small ones. The ordinary process, such as is used on paper, is much shorter. In both multiplication and division of large numbers, the work may be arranged so that the figures of the dividend or the multiplicand are gradually removed to make room for those of the product or quotient. In multiplication this may be done either from left to right or from right to left. For repeated multiplications it is well to use both methods, as corresponding significant figures are moved to the left in the one case, to the right in the other.

The extent to which one of these machines is a time saver depends on a variety of circumstances. It seems certain, however, that it is a great labor saver. Even if no saving of time at all is effected, the concentration necessary to perform the operations mechanically is much less than that required to perform them mentally. For many purposes the use of two of these machines in conjunction is very convenient. Sometimes the second may be used for subsidiary computations. More often the advantage consists in carrying intermediate results on one machine when positive, on the other when negative, in the case of substituting in a polynomial or similar formula.

L. S. DEDERICK.

History and Synopsis of the Theory of Summable Infinite Processes. By L. Smail. Eugene, Ore., University of Oregon Press, 1925. vi+175 pages. Price $2.00

This book contains first a three-page section entitled "Early History." In this the author goes back to views of Leibniz and Euler on $1-1+1-1+\cdots$ and to those of Cauchy and Abel on convergence, from them bringing the history of summable series down to Borel. Little of interest is presented. The facts stated are n the main well known to every mathematician interested in infinite series and no one else would be in the least interested in the book. However, the pages consumed are few. In the opinion of the reviewer it is a pity that an orderly historical account of later developments is not given.

The second and main part of the book entitled "Synopsis" consists of one hundred and seventy pages of abstracts of papers and books by one hundred and fifteen authors, arranged according to year of publication. Something like two hundred and seventy-five titles are cited running from 1880 to 1923. As a bibliography it is more complete than anything already in the field. Ford in his "Studies on Divergent Series and Summability" published in 1916 cites ninety-seven authors and about two hundred papers. He includes papers on asymptotic series.

Many of the abstracts were compared by the reviewer with the corresponding abstracts in the *Jahrbuch über die Fortschritte der Mathematik.* In the case of a few American dissertations no corresponding abstract exists in the *Jahrbuch*; and in many instances, particularly where articles in English are concerned, Smail gives the more lengthy and better abstracts. In a few instances the reverse is true. However, as it will probably be several years before the *Jahrbuch* is carried through 1923, Smail presents a more complete piece of work than would be afforded at the present time by a simple bibliography in conjunction with the *Jahrbuch.*

On the whole we have a very carefully prepared volume and one that will prove useful to every one interested in summable processes.

TOMLINSON FORT.

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

American Journal of Mathematics, volume 47, no. 3, July, 1925: "A configuration of Thirteen Pencils of Cubics and Cubics with Three Real Inflexions" by M. B. Turner, 149-162; "A Mathematical Theory of Competition" by C. F. Roos, 163-175; "Imprimitive Substitution Groups" by G. A. Miller, 176-180; "Classifications of Monoidal Involutions having a Fixed Tangent Cone" by Marian M. Torrey, 181-206; "Self-Projective Rational Septimics" by R. M. Winger, 207-223.

Annals of Mathematics, second series, volume 26, no. 4, June, 1925: "On continued fractions in the theory of binary forms" by A. Arwin, 247-272; "A property of sequences of Laplace" by H. L. Olson, 273-277; "Green's Lemma" by H. E. Bray, 278-286; "A class of minimum problems and the linear independence of functions of one variable" by O. Dunkel, 287-308; "Tables of quadratic forms" by A. E. Cooper, 309-316.

Bulletin of the American Mathematical Society, volume 31, no. 8, October, 1925: "Concerning the Complementary Intervals of Countable Closed Sets" by J. R. Kline, 409-410; "On Sets of Three Consecutive Integers which are Quadratic Residues of Primes" by A. A. Bennett, 411-412; "Groups in which the Normalizer of every Element except Identity is Abelian" by Louis Weisner, 413-416; "Note on Gibbs' Phenomenon" by C. N. Moore, 417-419; "A Historical Note on Gibbs' Phenomenon in Fourier's Series and Integrals" by H. S. Carslaw, 420-424; "A General Form of the Suspension Bridge Catenary" by Ira Freeman, 425-429; "On the Solution of Diophantine Equations by means of Ideals" by G. E. Wahlin, 430-444.

Science, new series, volume 62, no. 1606, October, 1925; "Arithmetization in the History of Mathematics" by G. A. Miller, 328.

The Tohoku Mathematical Journal, volume 25, nos. 1, 2, May, 1925: "On a series of functions formally analogous to Fourier's Series" by W. P. Udinsky, 1-23; "Determination of plane algebraic curves which are invariant under involutory Cremona transformations" by A. Emch, 63-76.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 2910 Harris Park Ave., Austin, Texas

CLUB ACTIVITIES

DENSION MATHEMATICS CLUB, Dension University, Granville, O.
[*1924*, 453]

The following are the subjects and speakers of the Denison Mathematics Club for the year 1924-1925:

January 15, 1924. "The perpetual calendar" by Lester Hunt. "What is wrong with the present-day calendar?" by Leland Powell. "The advantages of the proposed calendar" by Samuel Treharne.
A prize problem is given out at certain meetings and students are to bring in solutions at the next meeting. A decision as to who has the best solution is made. At this meeting Dr. Wiley explained the prize problem.

February 5. Short play on "The third dimension" was given. Social meeting. Interesting problems and puzzles were worked.

February 19. "Do negative numbers have logarithms?" by Leslie Bone, Hazel Dunlap and Kendrick Holt.

March 18. "Rubbing elbows with infinity" by Dr. Wiley.

May 2. Sixth annual banquet at Hotel Warden, Newark, Ohio.

May 13. "Probability" by Dr. Wiley.

October 7. "Mathematical fallacies" by Mr. Gay and Mr. Bash. Dr. Barnum from Constantinople, who had exchanged places for the year with Dr. Wiley was introduced.

October 21. "Duality" by Dr. Barnum.

November 11. "Kinds, forms and constituents of nebulae" illustrated with lantern slides by Turpin Bomeister.

December 16. Christmas social. Stunts, puzzles and a mathematical spelldown.

February 3, 1925. "New ways with series" by Professor Leman.

March 3. Prize problem stated by Professor Leman. Explanations and solutions are to be brought in later.

March 17. Miss Peckham addressed the club.

April 21. "How straight is a straight line?" by Mr. Bone.

May 5. "Applications of mathematics" by Miss Tippets.

May 19. Explanations of prize problem. Announcement of winners. Plans for club banquet on June 5.

(Report by Miss Josephine Deeds, Secretary)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OKLAHOMA, Norman, Okla.

[*1923*, 335]

The program of the Mathematics Club of the University of Oklahoma for the year 1924-1925 was the following:

October 23, 1924. "The theory of the number system" by Mr. Allen E. Andersen.

December 11, 1925. "The slide rule" by Professor E. R. Page.

January 15. "Ancient Egyptian and Babylonian mathematics" by Eugene Dunlap. "The regular heptagon" by Orlan Harden.

February 12. "Relation of physics to mathematics" by Professor W. Schriever.

February 26. "Transcendental numbers" by Hannah E. Brauer.

March 12. "Perfect numbers" by Eugene Springer. "Irrational numbers" by Mrs. Meagher.

March 26. "The electron theory" by Mr. D. E. Roller.

(Report by Frances M. Wright)

THE MATHEMATICS CLUB OF WELLESLEY COLLEGE, Wellesley, Mass.

[*1924*, 497]

The officers elected for the year 1924-1925 were: Doris Alexander '25, president; Loretta Davis '25, vice-president; Mildred Larimer '25, senior executive; faculty executive, Professor Helen Merrill; Eleanor Loomis '26, secretary and treasurer; Dorothy Dodd '26, junior executive.

At the club meeting, the following programs were presented:

October 21, 1924. "My experiences at the International Congress of Mathematicians" by Professor Clara Smith.

December 12. Mathematical puzzles and charades by juniors, seniors and faculty.

February 13, 1925. "Growth and development of number systems" by Isobel Hutchison '25. "Survey of the history of Greek mathematics" by Margaret Harris '25. "Problems of ancient mathematics" by Loretta Davis '25. "A famous woman mathematician" by Mildred Larimer '25.

March 13. "History of mathematics from the Renaissance to Vieta" by Nina Hammond '26. "History of mathematics from Vieta to Newton" by Ruth Mason '26. "History of mathematics from Newton to Euler" by Margaret Lane '26.

April 14. "Old and rare mathematics books" by Professor Lennie P. Copeland.

May 15. "History of the development of analytic methods in modern mathematics" by Professor Clara Smith. "Development of geometrical methods in modern mathematics" by Professor Mabel Young.

The following officers were elected for the year 1925-1926: Elizabeth Maxon '26, president; Nina Hammond '26, vice-president; Charlotte Banta '26, senior executive; Professor Lennie Copeland, faculty executive.

(Report by Miss Maxon)

NEWTONIAN SOCIETY, State College of Washington, Pullman, Wash.

[*1924*, 453]

The officers of the Newtonian Society for the year 1924-1925 were: Kathryn Maloney, president; Marion Upton, secretary-treasurer; Frances Helmer, reporter.

The following papers were presented at the meetings in 1924-1925.

January 20, 1925. "Formal logic" by Professor C. A. Isaacs.

February 17 "Determination of π" by Fern Bolich. "Life of Newton" by Marion Upton.

March 3. "Mathematical notation and terminology" by Professor Colpitts.
March 17. "Common fractions" by Ruth Himmelsbach. "Simple continued fractions" by Walter Bond.

Special meetings were held on December 8, 1924 and March 31, 1925.

(Report by Professor Isaacs)

Pi Mu Epsilon, Syracuse University, Syracuse, N. Y.

[*1924*, 494]

The officers for the year 1924-1925 were: Professor F. N. Bryant, director; Miss May J. Sperry, instructor, vice-director; Julia Bower '25, secretary; treasurer, Reginald Steele; Dorothy Park, librarian; Helen O'Donnell '26, Kenneth Robertson '25, Warren Lyon, instructor, Louis Rees, instructor, additional members of the executive committee; Dr. W. G. Bullard, Dr. W. F. Taylor, Vera Keeney '25, Ruth Woodworth '25, Julia Bower '25, scholarship committee.

The programs for the year were:

January 6, 1925. "Applications of complex quantities in engineering" by Mr. Clarke. "The planimeter" by Mr. Bothwell. "The slide rule" by Mr. Johnson.
February 10. "Systems of numeration" by Miss Garrett. "Life of Laplace" by Miss Keeney. "The three great problems of the Greeks" by Miss Woodworth.
March 3. "How to find the contents of a horizontal storage tank" by Mr. Hosier. "Derivation of the kinetic theory of gases" by Mr. McCarthy. "Stadia surveying" by Mr. Kaufmann.
April 3. "Magic figures" by Miss Keeney. "The papyrus of Ahmes" by Miss Fayle. "Early days and former members of the club" by Dr. Decker.
April 28. "Mortality tables" by Mr. Albright. "Statistical study of the relation between grades and cuts" by Miss Gilbert and Miss Shipston.

Some interesting social meetings have been held:

December 12, 1924. Initiation. A successful "get-acquainted" program was given by Liberal Arts.
February 10, 1925. "Mathematical magic" terminated by a flashlight picture of the Fraternity for the 1926 "Onondagan." The meeting adjourned to the drafting room of the College of Applied Science where Pi Mu Epsilon cross word puzzles were worked out.
March 3. A clever mock-meeting was presented by the Engineers.

(Report by Professor E. D. Roe, Jr.)

PROBLEMS AND SOLUTIONS

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N. B. Problems containing results believed to be new, or extensions of old results are especially suoght. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3156. Proposed by R. E. Gaines, University of Richmond, Va.

A *given* ellipse always touches a fixed straight line at a given point; find the locus of its center. Find the locus of the focus of a parabola under similar conditions.

3157. Proposed by Dr. Bernardo Ig. Baidaff, Buenos Aires, Argentina.

Calculate

$$\underset{p}{(}x\ \underset{p-1}{(}\ x \cdots \underset{2}{(}x\ \underset{1}{(}\ xe^x \underset{1}{)}'\ \underset{2}{)}' \cdots \underset{p-1}{)}'\ \underset{p}{)}', \tag{1}$$

$$\underset{p}{(}x\ \underset{p-1}{(}\ x^2 \cdots \underset{3}{(}x^{p-2}\underset{2}{(}x^{p-1}\underset{1}{(}x^p\underset{1}{)}'\ \underset{2}{)}''\ \underset{3}{)}''' \cdots \underset{p-1}{)}^{(p-1)}\underset{p}{)}^{(p)}, \tag{2}$$

where the accents ′, ″, ⋯ mean the first, second, ⋯ derivatives, and where the parentheses are paired as indicated by the numbers under them.

3158. Proposed by Otto Dunkel, Washington University.

Show that the absolute values of the imaginary roots (also of the negative roots when n is even) of the equation

$$x^{n+1}-(k+1)x^n+k=0, \quad k>0, \quad n\geqq 2,$$

lie between $\sqrt[n]{k/(k+2)}$ and unity; also that the absolute values of the imaginary roots are greater than that of the negative root of (a) the given equation when n is even, of (b) the equation obtained by replacing n by $n-1$ when n is odd.

If $k\geqq 1$ the absolute values of the imaginary roots (and negative roots) lie between .61 and 1.

3159. Proposed by R. H. Sciobereti, Berkeley, California.

Determine and discuss the nature of the curve described by the extremity of the shadow of a vertical rod on the horizontal plane.

3160. Proposed by L. S. Dederick, University of British Columbia.

All values of a continuous variable between 0 and 10 are equally likely. What is the most probable number of values that must be selected at random before obtaining two that differ by more than 9?

Is the result the same if the phrase "most probable" is taken literally as if it has the common meaning as applied to errors, as likely as not to be exceeded?

3161. Proposed by C. K. Robbins, Purdue University.

A non-degenerate conic and its tangents are inverted in $x^2+y^2=1$.

Show that the locus of the centers of the circles, which are the inverses of the tangents to the conic, is a conic.

3162. Proposed by J. L. Riley, Ouachita College, Arkadelphia, Ark.

Give an integral solution of $z^2=ay^2+x^2$.

3163. Proposed by Norman Anning, University of Michigan.

" . . . . the fair pearl necklace of the Qúeen,
That burst in dancing, and the pearls were spilt;
Some lost, some stolen, some as relics kept.
But nevermore the same two sister pearls
Ran down the silken thread to kiss each other
On her white neck—" —From *Vivien* by Tennyson.

Suppose the clasps alone remained upon the string while 50 pearls were spilt and 40 found; what is the chance of stringing these afresh to fit the poet's fancy?

SOLUTIONS

3120 [1925, 137] Proposed by V. M. Spunar, Chicago, Illinois.

If ρ_1 and ρ_2 be the greatest and least radii of curvature of a curved surface at a given point; φ and ψ, the angles which the normal to the surface at the given point makes with the axes of x and y, show that

$$1/\rho_1+1/\rho_2 = \frac{d}{dx}(\cos\psi)+\frac{d}{dx}(\cos\psi)$$

SOLUTION BY R. H. SCIOBERETI, Berkeley, California.

Let l, m, n, be the direction cosines of the normal to the surface at point $M(x, y, z)$, the system of reference being three rectangular axes Ox, Oy, Oz. From the classical theory of curved surfaces it is well known that:

$$l=\cos\phi=\frac{p}{\sqrt{1+p^2+q^2}},\quad m=\cos\psi=\frac{q}{\sqrt{1+p^2+q^2}},\qquad n=\frac{1}{\sqrt{1+p^2+q^2}}$$

where

$$p=\frac{\partial f}{\partial x}\qquad\text{and}\qquad q=\frac{\partial f}{\partial y}.$$

Furthermore, the principal radii of curvature at the point M are the roots of the following quadratic equation:

$$(rt-s^2)\rho^2-\sqrt{1+p^2+q^2}[(1+p^2)t-2pqs+(1+q^2)r]\rho+(1+p^2+q^2)^2=0\ ,$$

where r, s, and t have their usual meaning, *i.e.*

$$r=\frac{\partial p}{\partial x}\,,\qquad s=\frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}\,,\qquad t=\frac{\partial q}{\partial y}\,.$$

Now from the theory of symmetric functions, the sum of the reciprocals of the roots will be:

$$1/\rho_1+1/\rho_2=\frac{\sqrt{1+p^2+q^2}[(1+p^2)t-2pqs+(1+q^2)r]}{(1+p^2+q^2)^2}$$

which may be written

$$\frac{\left(1+p^2+q^2\right)\frac{\partial q}{\partial y}-q\left(p\frac{\partial p}{\partial y}+q\frac{\partial q}{\partial y}\right)}{(1+p^2+q^2)^{3/2}}+\frac{\left(1+p^2+q^2\right)\frac{\partial p}{\partial x}-p\left(p\frac{\partial p}{\partial x}+q\frac{\partial q}{\partial x}\right)}{(1+p^2+q^2)^{3/2}}\,,$$

or

$$(1+p^2+q^2)^{-1/2}\frac{\partial q}{\partial y}-q\frac{\left(p\frac{\partial p}{\partial y}+q\frac{\partial q}{\partial y}\right)}{(1+p^2+q^2)^{3/2}}+(1+p^2+q^2)^{-1/2}\frac{\partial p}{\partial x}-p\frac{\left(p\frac{\partial p}{\partial x}+q\frac{\partial q}{\partial x}\right)}{(1+p^2+q^2)^{3/2}}$$

or

$$\frac{\partial}{\partial y}\left[\frac{q}{\sqrt{1+p^2+q^2}}\right]+\frac{\partial}{\partial x}\left(\frac{p}{\sqrt{1+p^2+q^2}}\right);\ \text{hence}$$

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{\partial}{\partial x}(\cos\varphi)+\frac{\partial}{\partial y}(\cos\psi)\,.$$

NOTE. The last equation in the statement of the problem seems to be incorrect, and ought to be,

$$1/\rho_1+1/\rho_2=\frac{\partial}{\partial x}(\cos\varphi)+\frac{\partial}{\partial y}(\cos\psi)$$

Also solved by CONSTANCE R. BALLANTINE.

3121 [1925, 137]. **Proposed by A. A. Bennett, University of Texas.**

Find the smallest positive integer x, not a multiple of 31, such that x^2+1 shall have two factors whose difference is divisible by 31.

SOLUTION BY CONSTANCE R. BALLANTINE, New York University.

The quadratic residues of 31 are:

$$0,\ 1,\ 2,\ 4,\ 5,\ 7,\ 8,\ 9,\ 10,\ 14,\ 16,\ 18,\ 19,\ 20,\ 25,\ 28.$$

From these we select each r such that $r+1$ is also a residue, obtaining

$$x^2 \equiv 1,\ 4,\ 7,\ 8,\ 9,\ 18,\ 19 \pmod{31}.$$

($x^2\equiv 0$ is ruled out by the conditions of the problem.)

Let $x^2+1=yz$, $y=t+31u$, and $z=t+31v$, where $0<t<31$. Then

$$x^2+1=t^2+31t(u+v)+961uv\equiv 2,\ 5,\ 8,\ 9,\ 10,\ 19,\ 20 \pmod{31};$$

whence $t=3, 6, 8, 9, 12, 14, 15, 16, 17, 19, 22, 23, 25, 28$.

If the prime p divides t, we have

$$961uv-1\equiv r \pmod{p},$$

where r is a quadratic residue of p. Since 3 divides more of the possibile values of t than any other odd prime, let us examine the congruence

$$961uv-1\equiv 0,\ 1 \pmod{3},$$

or

$$uv\equiv 1,\ 2 \pmod{3}.$$

Since u and v are integers and $u\neq v$, $u=1$, $v=2$ is the smallest set of values possible. This gives us for $t=3$,

$$yz=34\cdot 65=2210=1+47^2.$$

Thus $x=47$ is a solution.

Since 3 is the smallest value of t, a smaller value for x could be obtained only by taking $u=0$ or $v=0$.

This requires that -1 be a quadratic residue of each prime factor of t, which limits the values of t to 8, 16, 17, and 25.

Taking $u=0$, $t=8$, we have

$$64+248v=x^2+1<2210,$$

if we are to have $x<47$. Hence $v<9$. Since $x^2=248v+63$, we have

$$2v\equiv 0,\ 1 \pmod{3}$$
$$v\equiv 0,\ 2 \pmod{3}.$$

Also,

$$3v\equiv 0,\ 1,\ 2,\ 4 \pmod{7},$$
$$v\equiv 0,\ 5,\ 3,\ 6 \pmod{7},$$

and

$$8v+3\equiv 0,\ 1,\ 4,\ 9,\ 6,\ 5 \pmod{10} \text{ or}$$
$$v\equiv 1,\ 2,\ 4 \pmod{10}.$$

Therefore, $t=8$ gives no solution.

Taking $u=0$, $t=16$, we have

$$256+496v<2210,$$
$$v<4.$$

To make $496v+255$ a square, we must have

$$v\equiv 0,\ 1 \pmod{3},$$
$$6v+3\equiv 0,\ 1,\ 2,\ 4 \pmod{7},$$
$$v\equiv 1,\ 2,\ 3,\ 6 \pmod{7}.$$

Hence $v=1$ or 3.

These values of v give $yz=752$ and $yz=1744$ respectively. Since neither 751 nor 1743 is a square, $t=16$ gives no solution.

Taking $u=0$, $t=17$, we have

$$289+527v<2210,$$
$$v<4.$$

To make $527v+288$ a square, we must have

$$2v\equiv 0,\ 1 \pmod{3},$$
$$v\equiv 0,\ 2 \pmod{3},$$
$$2v+1\equiv 0,\ 1,\ 2,\ 4 \pmod{7},$$
$$v\equiv 0,\ 3,\ 4,\ 5 \pmod{7}.$$

The one possible value of v is 3, but this gives $yz=1870$, and 1869 is not a square.

For $u=0$, $t=25$,

$$625+775v<2210,$$
$$v<3\,.$$
$$775v+624\equiv 0,\ 1\ (\text{mod } 3),$$
$$v\equiv 0,\ 1\ (\text{mod } 3)\,.$$

But $v=1$ gives $yz=1400$, and 1399 is not a square.

Thus 47 is proved to be the least value of x.

Also solved by F. L. Wilmer and E. E. Whitford.

3123 [1925, 138]. **Proposed by B. F. Finkel, Drury College.**

A circular hole, radius r, in the bottom of a flat-bottomed water-tank is covered with a weightless spherical rubber shell, radius R. Water is then poured into the tank to the depth h. What is the ratio of R to r when the shell is just on the point of rising?

Solution by E. M. Berry, Purdue University

The force F_1 tending to keep the shell from rising equals the weight of a vertical column of water having the area of the hole as base and whose altitude is h. This gives

$$F_1=\omega\pi r^2 h \tag{1}$$

where ω is the weight of a unit volume of water.

The force F_2 tending to cause the shell to rise is equal to the weight of water displaced.

Taking the origin at the center of the sphere and the element of volume as a thin disk parallel to the bottom of the tank, we have

$$F_2=\omega V=\pi\omega\int_{z_1}^{z_2}(R^2-z^2)dz \tag{2}$$

where V is the volume of water displaced.

Case 1—The water does not cover the shell, then

$$z_2=h-\sqrt{R^2-r^2}, \qquad z_1=-\sqrt{R^2-r^2}$$

and we get from (2)

$$F_2=\omega\pi(r^2h-\tfrac{1}{3}h^3+h^2\sqrt{R^2-r^2}).$$

The shell is just on the point of rising when $F_1=F_2$; hence we have

$$h=3\sqrt{R^2-r^2} \tag{3}$$

Case 2—The water covers the shell entirely, then $z_2=R$ and $z_1=-\sqrt{R^2-r^2}$ and (2) becomes

$$F_2=\tfrac{1}{3}\omega\pi\left[2R^3+(2R^2+r^2)\sqrt{R^2-r^2}\right].$$

Putting $F_1=F_2$ we get

$$3r^2h=2R^3+(2R^2+r^2)\sqrt{R^2-r^2}. \tag{4}$$

We can plot equation (4) by putting $r/R=k$ and solving for R,

$$R=\frac{3k^2h}{2+(2+k^2)\sqrt{1-k^2}}, \quad r=kR.$$

as parametric equations equivalent to (4).

Case 3—The water just comes to the top of the shell, $h=R+\sqrt{R^2-r^2}$ and equations (3) and (4) reduce to the same thing $r=\sqrt{3}R/2$ giving us

$$R=2h/3, \qquad r=h/\sqrt{3}. \tag{5}$$

If we take r and R as coördinates of the two curves, the curves (3) and (4) are tangent at this point as can be seen by finding dr/dR for each curve at this point. Hence equation (3) should be used for

$R \geqq 2h/3$ or $\sqrt{3}/2 \leqq k < 1$ and equation (4) for $0 < R \leqq 2h/3$ or $0 < k \leqq \sqrt{3}/2$. Thus if h is a constant, R can have all positive values and the ratio k ranges from 0 to 1.

It should be noticed that if R and r are fixed and h varied the ball would tend to rise for h less than the value given by (3) or (4) and would stay down for values of h greater than this value.

Also solved by PHILIP FITCH and H. B. WILCOX.

3125 [1925, 137]. **Proposed by W. J. Sidis, New York City**

If n is a prime number and r is a prime number not of the form $kn+1$, then in the scale of radix r, a perfect nth power can be found ending in any given digit. Also, provided the given last digit is not 0, a perfect nth power can be found ending in any given set of digits.

SOLUTION BY CONSTANCE R. BALLANTINE, New York University.

Let $x^n = a_0 + a_1\, r + a_2\, r^2 + \ldots\ldots$ Then if a_0 is given, the possibility of finding x depends on the possibility of solving the congruence

$$x^n \equiv a_0 \pmod{r}\ .$$

That the congruence can be solved, under the given conditions, follows at once from a theorem of Legendre (Dickson, *History of the Theory of Numbers* I, 205). The only solution is

$$x \equiv a_0^{\,l} \pmod{r}$$

where $ln - q(r-1) = 1$, the g. c. d. of n and $r-1$.

Suppose we have found $y_1, y_2, \ldots y_{k-1}$ such that

$$x \equiv a_0^{\,l} + y_1\, r + y_2\, r^2 + \ldots + y_{k-1} r^{k-1} \pmod{r^k}$$

implies

$$x^n \equiv a_0 + a_1 r + a_2 r^2 + \ldots\ldots + a_{k-1} r^{k-1} \pmod{r^k}. \tag{1}$$

Then we can find y_k such that

$$x \equiv a_0^{\,l} + y_1 r + y_2 r^2 + \ldots\ldots\ldots + y_{k-1} r^{k-1} + y_k r^k \pmod{r^{k+1}} \tag{2}$$

implies

$$x^n \equiv a_0 + a_1 r + a_2 r^2 + \ldots\ldots + a_{k-1} r^{k-1} + a_k r^k \pmod{r^{k+1}}. \tag{3}$$

For, raising (2) to the nth power, we shall have in view of (1)

$$x^n \equiv a_0 + a_1 r + \ldots\ldots\ldots + a_{k-1} r^{k-1} + (n a_0^{\,l(n-1)} y_k + b) r^k \pmod{r^{k+1}} \tag{4}$$

where

$$b = \Sigma \left(\frac{n!}{i_0! i_1! i_2! \ldots .\ i_{k-1}!} \right) a_0^{\,l(n-i_0)} y_1^{\,n-i_1} y_2^{\,n-i_2} \ldots .\ y_{k-1}^{\,n-i_{k-1}},$$

the summation extending over all $i_0, i_1, \ldots, i_{k-1}$ for which $i_1 + 2i_2 + \ldots + (k-1) i_{k-1} = k$, $i_0 + i_1 + \ldots . + i_{k-1} = n$.

From (3) and (4) we obtain

$$n a_0^{\,l(n-1)} y_k r^k \equiv (a_k - b) r^k \quad \pmod{r^{k+1}};$$

whence

$$n a_0^{\,l(n-1)} y_k \equiv a_k - b \qquad \pmod{r},$$

which is solvable if

$$a_0 \not\equiv 0 \qquad \pmod{r}\ .$$

3129 [1925, 204]. Proposed by C. N. Schmall, New York City.

Three hyperbolas are described each touching one side of a given triangle and having the remaining sides as asymptotes. Show that the product of their three latera recta is equal to the cube of the diameter of the inscribed circle of the triangle.

SOLUTION BY C. A. SHOOK, Yale University

Let ABC be any triangle with angles A, B, and C and sides a, b, and c. Choose rectangular axes with origin at A, x-axis bisecting the angle A, and such that C is in the first and B in the fourth quadrant. Then,

$$C\equiv\left(b\cos\frac{A}{2},\ b\sin\frac{A}{2}\right)\text{ and } B\equiv\left(c\cos\frac{A}{2},\ -c\sin\frac{A}{2}\right)$$

If $K=\dfrac{b-c}{b+c}$, the slope of BC is $\dfrac{1}{K}\tan\dfrac{A}{2}$,

and its equation is,

$$x\tan\frac{A}{2}-Ky=b\sin\frac{A}{2}(1-K)\ . \tag{1}$$

The hyperbola,

$$\frac{x^2}{\beta^2\cot^2\dfrac{A}{2}}-\frac{y^2}{\beta^2}=1\ , \tag{2}$$

will have AB and AC as asymptotes. We must determine β so that (2) is tangent to (1).

The slope of the hyperbola at any point is $(x/y)\tan^2 A/2$ so that at the point of tangency,

$$\frac{x}{y}\tan^2\frac{A}{2}=\frac{1}{k}\tan\frac{A}{2}$$

From (1) and (3), the point of tangency is found to be,

$$\left[\frac{1}{2}(b+c)\cos\frac{A}{2},\ \frac{1}{2}(b-c)\sin\frac{A}{2}\right],$$

β is easily determined by substituting these co-ordinates in (2). There results $\beta=\sqrt{bc}\sin\dfrac{A}{2}$.

Now the latus rectum $=L_a=2\beta\tan\dfrac{A}{2}=2\sqrt{bc}\sin\dfrac{A}{2}\tan\dfrac{A}{2}$.

Calling r the radius of the inscribed circle and using the well-known values for $\sin A/2$ and $\tan A/2$,

$$L_a=\frac{2r\sqrt{(s-b)(s-c)}}{s-a}\ .$$

L_b and L_c are found by permuting a, b, and c.
Then $L_a\cdot L_b\cdot L_c=8r^3=(\text{diameter})^3$.

Also solved by LEONARD CARLITZ, MICHAEL GOLDBERGER, M. S. KNEBELMAN, and AUGUST SÖRENSON.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. KUHN, Ohio State University, Columbus, Ohio.

At Miami University, Mr. E. E. Erickson of Iowa University has been appointed assistant professor. Assistant Professors R. A. Sheets and G. W. Spenceley have been promoted to associate professorships. Mrs. G. W. Spenceley is supplying for Professor Sheets who has been compelled by ill health to give up teaching for the year.

At Central Wesleyan College, Warrenton, o., Professor J. W. Blincoe has been appointed head of the department of mathematics to succeed Professor H. V. Knorr who has resigned.

Professor W. W. Rankin, head of the mathematics department at Agnes Scott Còllege, Decatur, Ga., has been appointed to a professorship in Duke University and will begin work there next September.

Professor I. M. DeLong of the University of Colorado represented the Mathematical Association at the inauguration of President C. C. Mierow at Colorado College on December 5, 1925.

Princeton University has received a gift of $150,000 from Mr. T. D. Jones, of Chicago, to be used to endow a chair in the department of mathematics to be called the Henry Burchard Fine professorship of mathematics, in honor of Professor Fine, dean of the department of science.

Dr. M. C. Foster, of Yale University, has been appointed assistant professor of mathematics at Williams College.

Dr. J. H. Taylor, of Princeton University has been appointed assistant professor of mathematics at Lehigh University.

Assistant Professor P. H. Linehan of the College of the City of New York has been promoted to an associate professorship of mathematics.

At Clark University, Professor F. B. Williams, professor of mathematics since 1908, will be absent on sabbatical leave for the second half of the current year. He will spend his time in Eurpoe. Mr. R. R. Smith, A.B., at present instructor in mathematics at the Newton, Mass., High School, has been appointed assistant professor in mathematics and education.

Mr. R. L. Flanders, of Norwich University, has been appointed assistant professor of civil engineering at the Oklahoma Agricultural and Mechanical College.

Mr. L. C. Bagby has been appointed assistant professor of mathematics at the University of South Dakota.

Professor W. P. YANCEY, profesor of mathematics and physics at St. Ambrose College, Davenport, Iowa, died on March 13, 1925. He held the degree of A.M. from Woodstock College, and had been in his position at St. Ambrose since 1918. He served also as vice-president of the Tri-City Chemist Club and as director of St. Ambrose Science Club.

Professor L. A. STOUT, for many years head of the department of mathematics in Dakota Wesleyan University, Mitchell, S. Dak., died Jan. 31, 1926.

Professor G. N. ARMSTRONG of Ohio Wesleyan University died January 7, 1926.

Professor M. T. Peed of Emory University, Oxford, Georgia, died August 28, 1925. His work at Emory extended over a period of thirty-six years. He was a member of the American Mathematical Society, the Mathematical Association and of the Phi Beta Kappa Society.

Mr. W. E. HEAL, of the U. S. Coast and Geodetic Survey, died October 9, 1925, at the age of 69.

The death is announced of Mr. G. P. ALDRICH, of the State University of Iowa.

Wells College is giving an honor course in mathematics for students of marked ability. During the two upper years the honor candidate has calculus, projective geometry, differential equations, higher plane curves and a seminar in which subjects are assigned for individual study and written reports. The examination at the end of the senior year covers all honor work in mathematics and is given by a committee of three consisting of two outside mathematicians and the head of the department of mathematics at Wells.

The Louisiana-Mississippi Section of the Association will hold its third annual meeting March 12 and March 13 at New Orleans under the auspices of Tulane University, Newcombe College, and the Academy of Science at New Orleans. Professor FLORIAN CAJORI of the University of California will be the guest of the Section on this occasion.

At the Louisiana State University, Professor S. T. SANDERS has just returned after an extended absence during which he has been studying at the University of Chicago. During his absence Professor I. C. NICHOLS has been in charge of the Department. The site of the University has recently been changed to a place two miles south of the city of Baton Rouge where ample grounds and new buildings provide a fine opportunity for the growth of the institution. Mr. RALPH O'QUIN, instructor in the Department, has spent several summers in graduate study at the University of Texas.

The Royal Academy of Belgium has awarded the prize for its 1925 mathematical competition to Professor W. C. GRAUSTEIN of the department of mathematics of Harvard for his memoir on the geometry of surfaces. The prize has been won by an American only once before, when, in 1909, it was awarded to Professor E. J. WILCZYNSKI of the University of Chicago. Professor Wilczynski has been seriously ill for more than two years. He is now at a private sanitarium in Denver, Colorado.

Anticipating the official announcement of the new Chauvenet Prize given by the Association for excellence in expository contributions in mathematics, which will appear in the Secretary's report of the Kansas City meeting, readers of the MONTHLY will be pleased to know that the first award for the period 1919-1924 was made to G. A. BLISS for his paper in the *Annals of Mathematics* last year on "Algebraic Functions and their Divisors." The hundred dollars in cash was provided for this first award by a friend of the Association, but will be appropriated from the Association treasury hereafter for each five-year period. It was felt by the donor and the members of the committee who determined the terms of award that such a prize would tend to stimulate expository production and that more such contributions are greatly needed.

The second number of the Carus Mathematical Monographs is now in press and will be ready for delivery early in February. This number is entitled *Analytic Functions of a Complex Variable* by Professor D. R. CURTISS of Northwestern University. It will be published in the same style as the first monograph and will have about the same size. It will be sold to members of the Association at the cost price of $1.25, one copy to each member who sends in his application for the same to the Secretary of the Association. The distribution to the public will take place as before through the OPEN COURT PUBLISHING Company and the regular sale price will be $2.00 per volume.

The distribution of the first monograph to the members of the Association has reached about 900. Other members who have not yet subscribed may still do so by making application to the Secretary. It would be a great tribute to the donor and a satisfaction to the officers of the Association, as well as a strong stimulus for the sales to the outside public, if practically every member should become a subscriber to the entire series of Carus monographs.

The National Council of Teachers of Mathematics has just published a *Yearbook* on the general theme: "Progress in the teaching of mathematics in the United States in the last twenty-five years." It contains contributions from D. E. SMITH, Columbia University, on "A general survey of mathematics in our high schools"; E. H. MOORE, on "The foundations of mathematics" (a

reprint of his presidential address); W. D. REEVE, Columbia University, on "Improving tests in mathematics"; RALEIGH SCHLORING, University of Michigan, on "Suggestions for the solution of an important problem"; FRANK CLAPP, University of Wisconsin, on "Some recent investigations in arithmetic"; WILLIAM BETZ, Rochester, N. Y., on "Mathematics of the junior high school"; H. E. SLAUGHT, University of Chicago, on "Mathematics and the public"; MARIE GUGLE and others, Columbus Schools, on "Some recreational values secured in our secondary schools through mathematical clubs"; E. W. SCHREIBER, Maywood, Ill., on "Mathematics books published in recent years for secondary schools and for teachers of mathematics."

The National Council has undertaken to perpetuate the work of the National Committee on Mathematical Requirements which latter was initiated and sponsored by the Mathematical Association of America. The publication of this *Yearbook* is a step in the fulfillment of this responsibility. It deserves the support of all who are interested in the field of secondary mathematics.

The *Yearbook* is a volume of 150 pages and will be sold at the cost price of $1.10, postpaid. Orders may be sent to C. M. AUSTIN, Oak Park High School, Oak Park, Ill. The edition is limited to two thousand copies.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of..Dollars, to be known as the..Fund, and to be used

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The Association needs funds for scientific publications and for the promotion of scientific activities.

CONTENTS.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.
BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.
BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.
Eleventh Annual Meeting, Philadelphia, Pa., December, 1926.
The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.
INDIANA, Purdue University, May.
IOWA, Cedar Rapids, April.
KANSAS, Combined with National Meeting, Kansas City, Mo.
KENTUCKY, Berea College, May 1.
LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.
MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Baltimore, Md., May 8
MICHIGAN, Ann Arbor, Mich., April 1.
MINNESOTA, Northfield, Minn., May 22.
MISSOURI, Kansas City, Mo., November.
NEBRASKA, Bethany, Neb., May.
OHIO, Columbus, Ohio, April 2.
ROCKY MOUNTAIN, Fort Collins, Colo., April 16-17.
SOUTHEASTERN, Atlanta, Ga., March.
SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.
TEXAS, Not yet determined.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

REGISTER OF OFFICERS AND MEMBERS

VOLUME XXXIII, 1926

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PRICE 50 CENTS

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MENASHA, WIS., AND ANN ARBOR, MICH.

Application has been made for transfer to Menasha, Wis.

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;

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The Chauvenet Prize

The ASSOCIATION has just established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The first award, covering the five years preceding 1925, was made to GILBERT AMES BLISS for his paper on "Algebraic Functions and their Divisors" published last year in the *Annals of Mathematics*. The committee of award consisted of Professor W. C. Graustein, Harvard University, Anna J. Pell-Wheeler, Bryn Mawr College, and E. B. VanVleck, Chairman, University of Wisconsin.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION,—one more of the many good reasons for membership.

THE

MATHEMATICAL ASSOCIATION

OF AMERICA

(INCORPORATED)

REGISTER OF OFFICERS AND MEMBERS

FOR THE ACADEMIC YEAR

1925-1926

MENASHA, WIS. AND ANN ARBOR, MICH.
PUBLISHED BY THE ASSOCIATION
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For a list of all officers and Council Members or Trustees for preceding years see page 56.

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ADDENDA JANUARY 1, 1926.

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* Removed from state. † Died Aug. 28, 1925.

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Closed for printing November 15, 1925.

Note. This list gives data for the members of the Association as taken from the register issued during November, 1924, unless changes in position, mailing address, etc., have been reported to the Secretary. Records may be inexact or incomplete through the failure of members to return fresh information to the Secretary, but great efforts have been put forth to be full and correct. When members are known to be occupied in other lines than mathematics, this fact is indicated.

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WELLMAN. Kreth.

KANSAS. (48)

BALDWIN CITY. P. L. Evans, Garrett.
COFFEYVILLE. Steininger.
EMPORIA. Philips, Rumney.
HAYS. Colyer.
HESSTON. Driver.
HIGHLAND. T. B. Henry.
IOLA. R. H. Carpenter, F. L. Marshall.
KANSAS CITY. Dougherty, Helwig.
LAWRENCE. Ashton, Babcock, Black, H. K. Hughes, H. E. Jordan, Marm, U. G. Mitchell, Reagan, G. W. Smith, Stouffer, H. M. Walker, J. J. Wheeler.
LINDSBORG. Lindberg.
MANHATTAN. W. H. Andrews, Holroyd, Hyde, Janes, C. F. Lewis, Lyons, Mossman, Remick, Stratton, A. E. White.
NEWTON. Richert.
OTTAWA. J. L. Bennett, Loewen.
PARSONS. Farner.
PITTSBURG. W. H. Hill, Shirk.
ST. MARYS. Case.
STERLING. T. Bell.
TOPEKA. Harshbarger, McLatchey.
WICHITA. Hoare, W. O. Mendenhall.
WINFIELD. H. S. Myers.

KENTUCKY. (21)

BEREA. Guilliams, Hutcherson, Peck.
DANVILLE. Crooks, Fehn.
GEORGETOWN. Clayton, C. H. Richardson.
LEXINGTON. Andrew, Boyd, M. E. Clarke, J. M. Davis, Downing, Le Stourgeon, Maney, E. L. Rees.
PARIS. R. I. Pepper, F. A. Scott.
RUSSELLVILLE. Saunders.
STANFORD. E. V. Watts.
VERSAILLES. Kienzle.
WILLIAMSBURG. Vallandingham.

LOUISIANA. (30)

ALEXANDRIA. Touchstone.
BATON ROUGE. Guilbeau, Nichols, O'Quinn, S. T. Sanders, Shoptaugh, Webber, Welch.
CLINTON. Petty.
ELTON. B. A. Tucker.
HOMER. Z. Allen.
LAFAYETTE. Stokes.
NATCHITOCHES. Maddox, Sparks.
NEW ORLEANS. H. E. Buchanan, Cope, Daly, Dinwiddie, A. M. Howe, Lastrapes, Latimer, Many, J. H. Meyer, Spencer.
PINEVILLE. C. D. Smith.
RUSTON. Pugh.
SHREVEPORT. Hardin, Maizlish.
VIVIAN. J. H. Manning.
ZACHARY. Peters.

MAINE. (9)

BRUNSWICK. Hammond, Moody.
LEWISTON. Ramsdell.
ORONO. Bryan, Hadlock, J. N. Hart.
WATERVILLE. Ashcraft, B. E. Carter, Trefethen.

MARYLAND. (49)

ANNAPOLIS. Bingley, Bramble, J. A. Bullard, Capron, Clements, Dillingham, Eppes, Garrison, Gaver, Mayer, Rawlins, Robert, Root, J. B. Scarborough, J. Tyler.
BALTIMORE. Bacon, H. Barton, Cary, Cohen, C. W. R. Crum, Harry, Hulburt, F. P. Lewis, F. Morley, Murnaghan, Musselman, C. A. Nelson, Rainich, L. J. Reed, Richeson, A. C. Robinson, Schad, Stanwick, Thomsen, Torrey, Wagner.
CHELTENHAM. Hartnell.
CHESTERTOWN. J. S. W. Jones.

CHEWSVILLE. Waltz.
COLLEGE PARK. Gwinner, Spann, Taliaferro.
EMMITSBURG. Burke.
FREDERICK. K. S. Arnold, L. O. Brown, Packer.
RIVERDALE. W. C. Myers.
ROLAND PARK. Morrow.
WESTMINSTER. B. I. Hart.

MASSACHUSETTS. (92)

AMHERST. Esty, F. C. Moore, Olds, C. S. Porter, W. R. Porter.
BELMONT. Douglass, Downey, Rutledge.
BOSTON. C. A. Benander, V. I. Benander, Brigham, Bruce, Gould, Laurentine, Mode, Osborne, M. M. Smith, Spear, Edwin B. Wilson.
BROOKLINE. A. L. Miller.
CAMBRIDGE. F. H. Bailey, Beatley, Birkhoff, Bradley, Coolidge, W. L. Crum, Franklin, Graustein, E. V. Huntington, Jenkins, M. F. Jordan, Kellogg, Kennelly, C. L. E. Moore, Osgood, Passano, F. W. Perkins, H. B. Phillips, L. H. Rice, H. W. Tyler, Walsh, D. E. Whitford, Winters, F. S. Woods, Zeldin.
CHESTNUT HILL. Archer.
DORCHESTER. Quigley.
EVERETT. Bryant.
HOLYOKE. Moriarty.
LAWRENCE. Lord.
LYNN. G. W. Evans.
MELROSE. Hussey.
NATICK. Willis.
NORTHAMPTON. S. R. Benedict, Eversull, Munroe, Rambo, R. G. Wood.
NORTON. G. A. Shook.
PITTSFIELD. Washburne.
SOUTH HADLEY. Doak, E. N. Martin, S. E. Smith.
SPRINGFIELD. Hickox, Marsh.
TUFTS COLLEGE. Bush, Mergendahl, Ransom.
VINEYARD HAVEN. H. M. Manning.
WELLESLEY. Anderton, Copeland, Merrill, C. E. Smith, Stark, Vivian, M. M. Young.
WEYMOUTH. Loud.
WILLIAMSTOWN. Agard, Hardy, V. H. Wells.
WOLLASTON. Dennison, Gardner.
WORCESTER. E. C. Brown, Gay, Hollis, Lazott, Melville, R. K. Morley, Phinney, H. Rice, A. H. Wheeler, F. B. Williams.

MICHIGAN. (65)

ALBION. F. E. Field, Sleight.
ALMA. Clack.
ANN ARBOR. Anning, Blessing, Bradshaw, Coe, Craig, Denton, Dostal, P. Field, S. E. Field, W. B. Ford, J. W. Glover, T. H. Hildebrandt, Hopkins, M. F. Johnson, Karpinski, Kazarinoff, Lemon, Love, Markley, O. J. Peterson, Poor, Raiford, C. Reid, Romig, Rood, Rouse, Running, Schorling, Shohat, Ziwet.
BAY CITY. Shellenberger.
DETROIT. Baldwin, Barlow, Darnell, Mullen, A. L. Nelson, Worden.
EAST LANSING. Crowe, L. C. Emmons, Evers, V. G. Grove, R. N. Haskell, H. L. Olson, Plant, Speeker.
HIGHLAND PARK. Thome.
HILLSDALE. Herron, Penrod.
HOLLAND. Lampen.
KALAMAZOO. H. Blair, J. P. Everett, T. O. Walton.
MARQUETTE. Spooner.
MONROE. Paula.
MOUNT PLEASANT. Pearce.
ROYAL OAK. Schoonover.
STURGIS. Steirnagle.
YPSILANTI. Barnhill, Gee, Lindquist, Lyman, Matteson.

MINNESOTA. (37)

COLLEGEVILLE. Winklemann.
DULUTH. Brigetta.
MANKATO. Chapman, A. V. Robbins.
MINNEAPOLIS. Beal, Brink, Brooke, Bussey, Dalaker, Earl, Gibbens, W. L. Hart, Hartig, D. Jackson, Kirchner, Priester, Shuman, Shumway, H. L. Smith, Sublette, Thorp, Underhill, Warne, Wilcox.
MOOREHEAD. K. Leonard.
NORTHFIELD. Clement, Gingrich, Nordgaard, M. B. White.
ST. CLOUD. Bemis.
ST. PAUL. R. A. Johnson, Kingery, Moench, Reuterdahl, F. J. Taylor.
VIRGINIA. C. L. Hancock.
WINONA. Bogard.

MISSISSIPPI. (13)

A. AND M. COLLEGE. H. Fox, B. M. Walker.
BLUE MOUNTAIN. Hutchins.
CLINTON. Hitt.
COLUMBUS. S. P. Walker.
GULFPORT. Maiden, Mauldin.
HATTIESBURG. Sharp.
JACKSON. Babbitt, Horrell, Smylie.
UNIVERSITY. Hume, R. Torrey.

MISSOURI. (39)

CAPE GIRARDEAU. B. F. Johnson, Knepper.
COLUMBIA. E. F. Allen, Callaway, Ingold, Jaeger, Wahlin, W. D. A. Westfall.
FULTON. Sweazey, M. A. Wood.
KANSAS CITY. Cutting, Epperson, Luby, Pierson.
KIRKSVILLE. Cosby, Jamison.
LIBERTY. Fleet.
PARKVILLE. R. A. Wells.
ROLLA. Hinsch.
SPRINGFIELD. Finkel, W. N. Thompson.
ST. CHARLES. Karr.
ST. LOUIS. Ammerman, Brennan, Dunkel, Gerst, A. H. Huntington, E. W. King, Nauer, Osborn, Rider, Roever, Shannon, E. Stephens, J. M. Young.
TARKIO. Jenison.
WARRENSBURG. J. H. Scarborough.
WARRENTON. Knorr.
WEBSTER GROVES. M. B. Clarke.

MONTANA. (5)

GLASGOW. Calderwood.
HELENA. O'Neill.
MISSOULA. Carey, Lennes, A. S. Merrill.

NEBRASKA. (22)

BETHANY. Osterhout.
CERESCO. L. C. Walker.
CRETE. J. N. Bennett.
GRAND ISLAND. H. Anderson.
HASTINGS. McDill.
KEARNEY. Hanthorn.
LINCOLN. Brenke, Candy, Congdon, Gaba, M. F. Jackson, Kirker, Opp, T. A. Pierce, Runge.
OMAHA. H. A. Campbell, Frankish, T. I. Porter, Wilmer.
PERU. A. L. Hill.
UNIVERSITY PLACE. Howie.
YORK. Feemster.

NEVADA. (1)

RENO. C. Haseman.

NEW HAMPSHIRE. (14)

DOVER. Huntley.
DURHAM. G. N. Bauer, Slobin, Wilbur.
EXETER. Sweet.
HANOVER. Beetle, Bill, B. H. Brown, C. H. Forsyth, Mathewson, Morgan, Silverman, C. E. Wilder, J. W. Young.

NEW JERSEY. (35)

BELLEPLAIN. Durell.
CARTERET. G. H. Hill.
CONVENT STATION. Maskell.
CRAWFORD. Knedler.
EAST ORANGE. Koch, Mallory, F. H. Robinson.
JERSEY CITY. J. P. Smith.
LAMBERTVILLE. Okean.
LEONIA. Gafafer, M. S. Taylor.
MOONESTOWN. R. R. Wood.
MORRIS PLAINS. A. E. Johnson.
NEWARK. Conkling.
NEW BRUNSWICK. Carlson, Dresser, Garretson, R. Morris, R. Thompson, A. A. Titsworth.
PATERSON. Caster.
PRINCETON. E. P. Adams, Alexander, Eisenhart, Fine, W. Gillespie, Hille, Lefschetz, H. D. Thompson, Veblen, Wedderburn, Willson.
RUTHERFORD. McMackin.
SUMMIT. H. E. Webb.
TRENTON. Colliton.

NEW MEXICO. (4)

ALBUQUERQUE. Barnhart.
EAST LAS VEGAS. Rodgers.
SILVER CITY. Hunter.
SOCORRO. Reece.

NEW YORK. (182)

ALBANY. Birchenough, G. M. Conwell.
ALFRED. Seidlin, W. A. Titsworth.
ANNANDALE-ON-HUDSON. Packard.
AURORA. Hollcroft.
BALDWIN. C. C. Grove.
BEECHHURST. E. Berger.
BROOKLYN. Angelica, Bergstresser, W. J. Berry, Bowden, Emery, Kreines, Langman, Lehmann, Littauer, Locke, Rosanoff, Schuyler, Tanzola, Thecla, G. F. Wilder.
BUFFALO. Cusick, Harrington, T. H. Milne, Pound, Sherk.
CLINTON. H. S. Brown, Carruth, Ferry, A. L. Fitch.
DE KALB JUNCTION. H. M. Phillips.
EAST ELMHURST. Hanson.
ELMIRA. Suffa, F. W. Wright.
FLUSHING. P. H. Graham, Oglesby.
GENEVA. W. H. Durfee, W. P. Durfee, Hubbs.
HAMILTON. Aude, DoBell, A. W. Smith.
ITHACA. Agnew, Beisel, Boothroyd, Carver, Farnum, D. C. Gillespie, Hurwitz, B. F. Kimball, MacCreadie, H. B. Meek, F. W. Owens, H. B. Owens, Poritsky, Ranum, Schoonmaker, Shaub, V. Snyder, Tanner, Wiener.
JACKSON HEIGHTS. Harper.
LARCHMONT. Hekimian.
MOUNT VERNON. Breckenridge.
NEWBURGH. C. W. Miller.
NEW YORK. J. Allen, Auerbach, Autenrieth, Ballantine, Berkeley, V. Blair, W. M. Bond, Brahdy, Brewster, T. H. Brown, Burdick, R. W. Burgess, G. A. Campbell, Chellborg, J. R. Clark, R. F. Clark, J. G. Coffin, Cooley, C. H. Douglas, Eckersley, Edmonson, Eis le, Fiske, Fite Fort, Foster, Frankel, Fry. Hawkes, R. Henderson, Himwich, Hirsch, Hodgdon, Hoyt, Jablonower, Joffe, Kasner, Kunte, Langellotti, Larkin, Linehan, J. J. McCarthy, McGuire, Meder, Mirick, Molina, A. I. Moore, Mullins, Paaswell, Pedersen, Penn, Phelps, Pooler, Post, Pride, Quilty, Reddick, Reeve, Ritt, Saurel, Schmall, Schub, Seto, Shewhart, Siceloff, Simons, D. E. Smith, R. F. Smith, R. R. Smith, Spies, W. A. Stevens, H. Thompson, Thorne, Tilly, A. B. Turner, Upton, Waldo, E. Walker, Webster, Wechsler, E. E. Whitford, W. O. Wiley, Woodyard.
OLEAN. S. L. Fitch, Lowry.
PARISH. Church.
PLEASANTVILLE. B. G. Westfall.
POUGHKEEPSIE. Cowley, Cummings, M. E. Wells.
ROCHESTER. Betz, Gale, H. Harding, T. R. Long, Silberstein, Watkeys, Weisner.
SCARSDALE. MacNeish.
SCHENECTADY. D. S. Morse, Newkirk, A. D. Snyder, Vedder.
SYRACUSE. W. G. Bullard, Carroll, Decker, Lindsey, Secy. Pi Mu Epsilon Frat., Roe, M. Sperry, W. E. Taylor.
TARRYTOWN. R. G. Putnam.
TROY. Crockett.
WEST POINT. C. P. Echols.
YONKERS. Hubert, F. W. John, Yanosik.

NORTH CAROLINA. (19)

CHAPEL HILL. Browne, Cain, A. Henderson, M. A. Hill, Hobbs, Lasley.
CHARLOTTE. O. M. Jones.
DAVIDSON. J. L. Douglas.
DURHAM. M. R. Richardson, Robison.
ELON COLLEGE. Amick.
GREENSBORO. G. W. Mendenhall, F. S. Mitchell, Pegram, Strong.
GREENVILLE. M. D. Graham.
GUILFORD COLLEGE. Pancoast.
JAMESTOWN. Ragsdale.
MURFREESBORO. Caldwell.

NORTH DAKOTA. (5)

FARGO. Householder, I. W. Smith.
MINOT. De La.
UNIVERSITY. Hitchcock.
VALLEY CITY. J. B. Meyer.

OHIO. (99)]

ADA. Fairchild, Whitted.
AKRON. P. C. Jones.
ALLIANCE. Trott.
ATHENS. Borger, F. W. Reed.
BEREA. Dustheimer.
BOWLING GREEN. Overman.
BLUFFTON. Hirschler.
CHILLICOTHE. Cornetet.
CINCINNATI. I. A. Barnett, Brand, Garabedian, H. Hancock. Kindle, Lubin, C. N. Moore, Salkover, E. S. Smith, Wilczewski, Yowell.
CLEVELAND. Focke, W. W. Johnson, B. W. Jones, McBane, J. E. Merrill, M. Morris, Nassau, Palmié, Simon, C. F. Thomas, Wilkins.
COLUMBUS. C. L. Arnold, Atwater, Bareis, Blumberg, Bohannan, Cottingham, Harmount, R. S. Kimball, Kuhn, MacDuffee, McCoy, Manson, C. C. Morris, Preston, Rasor, Rickard, B. Sanders, Singer, R. L. Wilder, Wildermuth
DAYTON. Hartwick.
DEFIANCE. A. G. Caris.
DELAWARE. G. N. Armstrong, Crane, R. L. Newlin.
GAMBIER. R. B. Allen, Denston.
GRANVILLE. F. B. Wiley.
HARRIETTSVILLE. G. S. Jones.
HILLIARD. J. H. Weaver.
HIRAM. E. H. Clarke, Jerome.
KENT. Manchester.
MARIETTA. Coar, Rea.
NEW CONCORD. C. E. White.
OBERLIN. Cairns, Carr, A. H. Fox, Sinclair, Yeaton.
OXFORD. J. P. Albert, W. E. Anderson, Baudin, M. B. Carter, Lange, Sheets, Spenceley, Tappan.
PAINESVILLE. A. D. Lewis.
ROSS. Haldeman.
SPRINGFIELD. Tripp.
STEUBENVILLE. Horn.
TIFFIN. J. Pierce, Rhodes.
TOLEDO. Brandeberry, Dancer, Mercedes.
URBANA. Stout.
WASHINGTON C. H. Haigler.
WESTERVILLE. B. C. Glover.
WILBERFORCE. Tinner, Waits.
WILMINGTON. Spinks.
WOOSTER. Williamson, Yanney.
ZANESVILLE. Riesbeck.

OKLAHOMA. (22)

ALVA. H. L. Hall.
CHICKASHA. Hawkins.
DURANT. Work.
MIAMI. J. B. Steed, C. S. Whitney.
NORMAN. Court, Barbour, Hassler, D. McFarland, McGilvray, Meacham, S. W. Reaves, F. M. Wright.
OKLAHOMA CITY. Meador, L. V. Robinson.
SHAWNEE. W. T. Short.
STILLWATER. R. L. Flanders, Gundersen.
TULSA. Roman, G. R. West.
WEATHERFORD. McCormick.
WILSON. R. O. Webb.

OREGON. (11)

CORVALLIS. Beaty, C. L. Johnson, Van Fleet, G. A. Williams.
EUGENE. De Cou, McAlister, W. E. Milne, Wills.
PORTLAND. Griffin, Merriss, J. M. Short.

PENNSYLVANIA. (104)

ALLENTOWN. C. S. Allen, Bauman.
ANNVILLE. Redditt.
BEAVER FALLS. Cleland.
BETHLEHEM. A. A. Bennett, Knebelman, Lyle, McDonough, Rau, J. B. Reynolds, Weida.
BRYN MAWR. A. P. Wheeler, S. M. Wolfe.
CAMP HILL. Foberg.
CARLISLE. Landis.
COLLEGEVILLE. Clawson.
CRAFTON. B. C. Patterson.
CYNWYD. Sensenig.
DERRY. Minister.
DEVON. J. A. Clarke.
EASTON. Benner, Doushkess, W. S. Hall, Hatch, Marquard, W. M. Smith.
EDINBORO. Heinaman.
GERMANTOWN. Mullikin.
GETTYSBURG. Arms, Uhler.
GROVE CITY. Ramsey.
HARRISBURG. Whited.
HAVERFORD. L. W. Reid, A. H. Wilson.
HUNTINGTON. C. S. Shively.
IRWIN. A. A. Jones.
LANCASTER. R. L. Charles, W. F. Long.
LANSDOWNE. Chambers, Glenn, Gummere.
LATROBE. Seubert.
LEWISBURG. Bartol, H. S. Everett, Gold, Lindemann.
LINCOLN UNIVERSITY. W. L. Wright.
LOCK HAVEN. High.
MEADVILLE. Akers, Wagner.
MECHANICSBURG. N. B. Freeman.

MILLERSVILLE. Seiverling.
NEW WILMINGTON. McCain.
PARKESBURG. Lufkin.
PHILADELPHIA. W. L. Ayres, P. A. Caris, Crawley, Eshleman, H. B. Evans, F. H. Jackson, F. John, Kline, Levita, Linton, Partridge, Rittenhouse, Rosengarten, Safford, Shippy.
PITTSBURGH. Baird, Barrett, Bishop, Burley, Geckeler, R. P. Johnson, J. H. Mathews, Riggs, Rosenbach, Simester, Swartzel, Taber, J. S. Taylor, Whitman.
SLIPPERY ROCK. Lady.
SOUTH BETHLEHEM. MacNutt.
SHIPPENSBURG. Kieffer.
STATE COLLEGE. Bushyager, T. Cohen, J. E. Davis, Gravatt, L. S. Johnston, E. D. McCarthy, Shibli, J. M. West, F. G. Williams.
SWARTHMORE. Marriott, J. A. Miller.
SWISSVALE. Foraker.
WASHINGTON. Atchison, Bert, Cardin, R. W. Thomas.
WEST PHILADELPHIA. Latshaw.

PHILIPPINE ISLANDS. (4)

LAGUNA. Salvosa.
MANILA. Gokhale, Tan, Tienzo.

PORTO RICO. (1)

MAYAGUEZ. Sanchez.

RHODE ISLAND. (14)

NEWPORT. Chase.
PROVIDENCE. C. R. Adams, Archibald, Burwell, Chace, Currier, Gilman, Hickson, H. P. Manning, M. Morse, R. G. D. Richardson, Sauté, Suesman, Watt.

SOUTH CAROLINA. (13)

CHARLESTON. O. J. Bond, R. H. Coleman.
COLUMBIA. Coker, J. B. Coleman, J. B. Jackson, W. L. Williams.
GREENVILLE. Earle, R. B. Wood.
GREENWOOD. Weber.
HARTSVILLE. C. M. Reaves.
ROCK HILL. Pope, G. T. Pugh.
SALUDA. Ramage.

SOUTH DAKOTA. (9)

ABERDEEN. Hobart.
BROOKINGS. I. L. Miller.
HURON. Titt.
RAPID CITY. Bowles, McLaury.
STURGIS. Horlocker.
VERMILION. Bagby, McKinney.
YANKTON. Faught.

TENNESSEE. (13)

JACKSON. Hess, Walden.
KNOXVILLE. J. D. Bond, Brezler, Ghormley.
MARYVILLE. Knapp.
NASHVILLE. R. V. Blair, S. I. Jones, Miser.
NORMAL. P. L. Armstrong.
PULASKI. Mize.
RIPLEY. Edward B. Wilson.
SEWANEE. S. M. Barton.

TEXAS. (83)

ABILENE. Burnam.
AUSTIN. Batchelder, H. Y. Benedict, Cooper, Decherd, Dodd, Duncan, Ettlinger, Gehman, H. L. Holmes, Horton, Jacobs, Lubben, Mayne, D. E. Mitchell, R. L. Moore, Mullings, Phenix, M. B. Porter, P. K. Rees, W. A. Rees, Rupp, Rutherford, Smail, Vandiver.
BEAUMONT. M. A. Campbell.
BOERNE. Hathaway.
BROWNSVILLE. de la Garza.
BROWNWOOD. Gayden, McClelland.
CAMERON. Newton.
CANYON. L. G. Allen.
COLLEGE STATION. F. Ayres, A. A. Blumberg, H. A. Foster, Halperin, D. C. Jones, McKee, W. L. Porter, W. P. Stevens.
DALLAS. Dice, Hartsfield, E. H. Jones, Mahoney, Seale.
DENTON. M. C. Brown.
FORT WORTH. Hargett, Howard, E. R. Tucker.
GALVESTON. P. H. Underwood.
GEORGETOWN. Wunder.
HOUSTON. Bray, Dean, G. C. Evans, Ewing, L. R. Ford, Hickey, E. O. Lovett, Michal, Roos.
KINGSVILLE. H. Porter.
LUBBOCK. Ames, D. A. Flanders, E. T. Stafford, Whyburn.
MILFORD. Durham.
NACOGDOCHES. C. E. Ferguson.
PORT ARTHUR. G. S. Smith.
SAN ANTONIO. McNelly, J. E. Nelson, Roach, Rockwell.
SAN MARCOS. J. S. Brown, Kaderli, Sewell.
SHERMAN. May.
STEPHENVILLE. Hale, McSweeny, Marrs, Redden.
WACO. Harrell, W. A. Nelson.
WICHITA FALLS. B. T. Adams.

UTAH. (6)

LEWISTON. Van Orden.
SALT LAKE CITY. J. L. Gibson, Horsfall, Marthakis, Pehrson, Unseld.

VERMONT. (7)

BURLINGTON. Butterfield, Donahue, Millington, Swift, E. Thomas.
MIDDLEBURY. Hazeltine, L. R. Perkins.

VIRGINIA. (33)

ABINGDON. V. L. Wright.
ASHLAND. T. McN. Simpson.
BLACKSBURG. Brodie, Gudheim, Hatcher, O'Shaughnessy, J. E. Williams.
BRIDGEWATER. Shull.
CHARLOTTESVILLE. F. A. Wells.
CLIFTON STATION. O. Stone.
EMORY. J. S. Miller.
FARMVILLE. Taliaferro.
HOLLINS. Dickinson.
LANGLEY FIELD. Hemke.
LEXINGTON. Paxton, L. W. Smith, C. W. Watts, Witt.

LYNCHBURG. Larew, Pattillo.
MONTEREY. Colaw.
RADFORD. Bowers.
RICHMOND COLLEGE. Gaines.
SALEM. Carpenter.
SWEET BRIAR. Morenus, Searle.
UNIVERSITY. W. H. Echols, Luck, Sparrow, Thornton.
UNIVERSITY OF RICHMOND. I. Harris.
WILLIAMSBURG. Rowe, B. Russell.

WASHINGTON. (18)

DAVENPORT. P. A. Carlson.
EVERETT. Robb.
PULLMAN. G. H. Freeman, A. A. Grant, Isaacs.
SEATTLE. E. T. Bell, Biggerstaff, Caffrey, Cramlet, Jerbert, Moritz, Mullemeister, Neikirk, E. D. West, Winger.
TACOMA. Hanawalt.
WALLA WALLA. Bratton, Eells.

WEST VIRGINIA. (11)

BETHANY. Cramblet.
BRIDGEPORT. Slawter.
HUNTINGTON. Hackney.
INSTITUTE. Cox.
KEYSER. Harshbarger.
MORGANTOWN. M. Buchanan, Colwell, H. A. Davis, Eiesland, C. N. Reynolds, B. M. Turner.

WISCONSIN. (35)

BELOIT. H. H. Conwell, Huffer.
JANESVILLE. Battig.
MADISON. F. E. Allen, Bunyan, Dowling, Dresden, Fowlkes, W. W. Hart, Ingraham, Mickelson, E. B. Miller, Pollard, Skinner, Slichter, Van Vleck, W. Weaver.,
MILTON. A. E. Whitford.
MILWAUKEE. E. R. Beckwith, Ericson, P. H. Evans, Frumveller, Knight, Morrissy, Quarles, C. G. Simpson.
PLATTEVILLE. N. M. Johnston, Warner.
RIPON. Woodmansee.
RIVER FALLS. McMillan.
SOUTH MILWAUKEE. Hoar.
SUPERIOR. C. W. Smith.
WEST ALLIS. Roth.
WEST DE PERE. DeCleene.
WILLIAMS BAY. Justin.

WYOMING. (4)

LARAMIE. Bellamy, Fitterer, Gossard, Rechard.

FOREIGN MEMBERS. (Other than Canada.)

ARGENTINE. (1)

BUENOS AIRES. Baidaff.

BELGIUM. (1)

LIEGE. Van Hee.

BULGARIA. (1)

LOVETCH. E. M. Perry.

CHINA. (7)

CANTON. W. E. MacDonald.
CHANGSHA. Leavens.
NANKING. Loh.
PEKING. Heinz, Konantz.
SHANGHAI. Ely.
TANGSHAN. Patten.

FRANCE. (6)

BESANÇON. Lebeuf.
NANCY. Gérardin.
PARIS. Borel, Hadamard.
STRASBOURG. Fréchet, Linfield.

GERMANY. (1)

GÖTTINGEN. Coates.

GREAT BRITAIN. (5)

CAMBRIDGE. P. W. Wood.
EDINBURGH. Horsburgh.
HOVE. Chepmell.
OXFORD. Hardy, W. E. Robertson.

INDIA. (2)

ALLAHABAD CITY. Mitra.
CALCUTTA. Bose.

ITALY. (6)

BOLOGNA. Bortolotti, Enriques, Pincherle.
PALERMO. Cipolla.
PISA. Bianchi.
TURIN. Fubini.

JAPAN. (3)

PYENGYANG. Parker.
TOKYO. Mikami, Ono.

NEW ZEALAND. (1)

DUNEDIN. Martyn.

POLAND. (1)

WARSAW. Dickstein.

PORTUGAL. (1)

LISBON. da Cunha.

SOUTH AFRICA. (3)

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton.
RONDEBOSCH. Muir.

SPAIN. (1)

MADRID. de Toledo.

SWITZERLAND. (3)

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHATEL. DuPasquier.

SYRIA. (1)

BEIRUT. Jurdak.

TURKEY. (3)

CONSTANTINOPLE. A. F. Johnson, Mourad, Plapp.

UKRAINE. (1)

KIEFF. Kryloff.

RECAPITULATION OF MEMBERSHIP.

Individual members November 15, 1925	1,759	
Institutional members November 15, 1925	115	
Total membership November 15, 1925		1,874
Total membership November 1, 1922		1,476

CHARTER MEMBERSHIP.

Individual charter members	1,045	
Institutional charter members	52	
Total charter membership		1,097
Net gain in individual members	714	
Net gain in institutional members	63	
Total net gain over charter membership		777
Total net gain since November 1, 1922		398

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

Article I—Name, Purpose and Corporate Seal.

1. This organization shall be known as

The Mathematical Association of America (Incorporated).

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

Article III—Board of Trustees and Officers.

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President and Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief, the Manager and one other member, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall

announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-Presidents such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the Office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safe-keeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificate shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

Article IV—Meetings.

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections.

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

Article VI—Official Publications.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues.

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

5. New members entering the Association after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to McClintock's Male Annuitant Table based upon four (4) per cent interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the McClintock Male Four (4) Per Cent Table; and the reserve thus computed shall be held as a liability.

Article VIII—Amendments to the Articles of Association and By-Laws.

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION.

PRESIDENTS.

E. R. HEDRICK 1916
FLORIAN CAJORI 1917
E. V. HUNTINGTON 1918
H. E. SLAUGHT 1919
D. E. SMITH 1920
G. A. MILLER 1921
R. C. ARCHIBALD 1922
R. D. CARMICHAEL 1923
H. L. RIETZ 1924
J. L. COOLIDGE 1925

VICE-PRESIDENTS.

E. V. HUNTINGTON 1916
G. A. MILLER 1916
D. N. LEHMER 1917, 1918
OSWALD VEBLEN 1917
J. W. YOUNG 1918
R. G. D. RICHARDSON 1919
H. L. RIETZ 1919
HELEN A. MERRILL 1920
E. J. WILCZYNSKI 1920
R. C. ARCHIBALD 1921
R. D. CARMICHAEL 1921, 1922
B. F. FINKEL 1922
A. B. CHACE 1923
L. P. EISENHART 1923
J. L. COOLIDGE 1924
DUNHAM JACKSON 1924, 1925
A. A. BENNETT 1925

SECRETARY-TREASURER.

(Appointed by the Council or Board after 1918.)

W. D. CAIRNS 1916–

COMMITTEE ON PUBLICATIONS.

(Appointed by the Council or Board.)

H. E. SLAUGHT 1916–
R. D. CARMICHAEL 1916–1918
W. H. BUSSEY 1916–1918
R. C. ARCHIBALD 1919–1921
W. A. HURWITZ 1919–1921
A. A. BENNETT 1922
H. P. MANNING 1922
W. B. FORD 1923–
J. L. COOLIDGE 1923
A. J. KEMPNER 1924–

ELECTED MEMBERS OF THE COUNCIL OR BOARD.

D. N. LEHMER 1916–1918, 1922–
R. E. MORITZ 1916–1918
K. D. SWARTZEL 1916
OSWALD VEBLEN 1916, 1920–1922
R. C. ARCHIBALD 1916–1917, 1923–
FLORIAN CAJORI 1916, 1918–1923
M. B. PORTER 1916–1917
J. W. YOUNG 1916–1917, 1920–1922
B. F. FINKEL 1916–1921
E. H. MOORE 1916–1921, 1923–
J. N. VAN DER VRIES 1916–1918
ALEXANDER ZIWET 1916–1918
E. R. HEDRICK 1917–1922, 1924–
HELEN A. MERRILL 1917–1919
D. E. SMITH 1917–1919, 1921–
ELIZABETH B. COWLEY 1918–1920
G. A. MILLER 1918–1920, 1922–1924,
E. J. WILCZYNSKI 1918–1919, 1922–
L. P. EISENHART 1919–1922, 1925–
E. V. HUNTINGTON 1917, 1919–
E. L. DODD 1920
R. D. CARMICHAEL 1920, 1924–
A. A. BENNETT 1921
H. L. RIETZ 1921–1923, 1925–
C. F. GUMMER 1921–
A. B. CHACE 1924–
DUNHAM JACKSON 1923
CLARA E. SMITH 1923–

The Carus Mathematical Monographs

The CARUS MONOGRAPHS have already begun to fulfill their mission as intended by the generous donor, MRS. MARY HEGELER CARUS and her son DR. EDWARD H. CARUS.

The first edition of Monograph Number One, "Calculus of Variations," by Professor Bliss, will soon be exhausted.

Nearly 900 members of the Association have taken advantage of the distribution at cost. Those who neglected to do so at the start may still have the privilege by applying to the Secretary.

It would be a great tribute to the donor and an honor to the Association if every member would subscribe for the complete series.

It is believed that the Association is rendering a great service to mathematics by this enterprise, and a one hundred per cent support from the membership would constitute an appropriate vote of confidence in the undertaking.

Monograph Number Two, "Analytic Functions of a Complex Variable," by Professor Curtiss, is in press and will be distributed in February to those who have sent in their subscriptions to the Secretary. All distribution to the general public will be made through THE OPEN COURT PUBLISHING COMPANY, 112 South Michigan Avenue, Chicago, Illinois.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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MENASHA, WIS., AND ANN ARBOR, MICH.

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THE MATHEMATICAL ASSOCIATION OF AMERICA

The following sixty-one persons and three institutions, on application duly certified, have been elected to membership in the Association. Of these fifteen are a result of the joint campaign of the Society and the Association, and fourteen from the recent work of the Association's own Committee on Membership.

J. P. ALBERT, A.B. (Miami). Instr., Miami Univ., Oxford, Ohio.
G. P. ALDRICH, M. S. (Iowa). Instr., Univ. of Iowa, Iowa City, Iowa.
SISTER ALICE IRENE, A.M. (Columbia Coll.). Instr., Columbia College of St. Catherine, St. Paul, Minn.
H. T. R. AUDE, M.S. (Colgate). Asso. Prof., Colgate Univ., Hamilton, N. Y.
LEON BATTIG, A.B. (Wisconsin). Jr. and Sr. High Schools, Janesville, Wis.
B. R. BEISEL, B.S. (Allegheny). Instr., Cornell Univ., Ithaca, N. Y.
C. O. BEMIS, A.B. In Educ. (Minnesota). Teachers Coll., St. Cloud, Minn.
A. D. BUTTERFIELD, M.S. (Worcester Polyt.); A.M. (Columbia). Prof., Math. and Geodesy, Univ. of Vermont, Burlington, Vt.
A. G. CLARK, A.M. (Colorado). Asst. Prof., Colorado Agric. College, Ft. Collins, Colo.
J. R. CLARK, Ph.D. (Columbia). Lincoln School of Teachers Coll., New York, N. Y.
J. L. CLAYTON, B.S. (Georgetown). Instr., Georgetown Coll., Georgetown, Ky.
C. B. DANSBY, A.B. (Morehouse). Instr., Morehouse Coll., Atlanta, Ga.
C. H. DENNISON, (Mass. Inst. Tech.). Amer. Rubber Co., Cambridge, Mass.
H. G. DRESSER, B.S. (Tufts). Instr., Rutgers Univ., New Brunswick, N. J.
P. D. EDWARDS, A.M. (Indiana). Technical H. S., Indianapolis, Ind.
P. L. EVANS, A.B. (Baker). Instr., Baker Univ., Baldwin, Kans.
CORNELIUS EVERS, A.B. (Hope Coll.). Instr., Michigan Agric. Coll., East Lansing, Mich.
B. F. GAINES, M.S. (Cornell). Asst. Prof., Univ. of Florida, Gainesville, Fla.
H. L. HALL, A.B. (Indiana). Head of Dept., N. W. State Teachers Coll., Alva, Okla.
G. L. HORRELL, M.S. (Millsaps). Prof., Millsaps Coll., Jackson, Miss.
F. S. HEINAMAN, A.B. (Lebanon Valley). Dean, St. Normal Sch., Edinboro, Pa.
DEBORAH M. HICKEY, Student, Rice Inst., Houston, Tex.
W. A. JENKINS, A. M.(Michigan). Instr., Univ. of Mich., Ann Arbor, Mich.
JESSIE M. JEROME, A.M. (Hiram). Prof., Hiram College, Hiram, Ohio.
D. C. JONES, A.B. (Emory). A & M Coll., College Station, Tex.
WILLIAM JORDAN, A.B. (Oakland City Coll.; DePauw). Head of Dept., Oakland City Coll., Oakland City, Ind.
MARY N. KEITH, A.B. (Wellesley). Asst. Prof., Univ. of Redlands, Redlands, Calif.
NORA A. KIEFFER, A.M. (Columbia). Instr., Cumberland Valley State Normal School., Shippensburg, Pa.
M. S. KNEBELMAN, M.S. (Lehigh). Asst. Prof., Lehigh Univ., Bethlehem, Pa.
E. E. KNIGHT, A.M. (Illinois). Instr., State Normal Sch., Milwaukee, Wis.
C. H. LADY, A.B. (Columbia). State Normal Sch., Slippery Rock, Pa.
W. M. MAIDEN, A.M. (Virginia). Head of Dept., Gulf Coast Milit. Acad., Gulfport, Miss.
LAURA F. McDONOUGH, A.B. (Penna.). Head of Dept., Moravian Coll., Bethlehem, Pa.
W. B. MARQUARD, M.E. (Ohio State). Asso. Prof., Lafayette Coll., Easton, Pa.
E. D. MEACHAM, Ph.D. (Chicago). Prof., Univ. of Okla., Norman, Okla.
C. W. MILLER, B.S. (Bucknell). Instr., Newburgh Acad., Newburgh, N. Y.
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W. A. NEWLIN, M.S. (California). H. S. and Jr. Coll., Pasadena, Calif.
G. A. O'DONNELL, A.M. (Woodstock). Professor, Georgetown Univ., Washington, D. C.
E. B. PENROD, M.S. (Purdue); M.M.E. (Cornell). Prof., Hillsdale College, Hillsdale, Mich.
W. L. PORTER, A. B.(Howard Coll.). Asso. Prof., A. & M. Coll., College Station, Tex.
T. E. RAIFORD, A.M. (Michigan). Instr., Univ. of Mich., Ann Arbor, Mich.
B. P. REINSCH, Ph.D. (Illinois). Instr., Univ. of Ill., Urbana, Ill.
FRED REUSSER, M.S. (Iowa). Head of Dept., Buena Vista Coll., Storm Lake, Ia.
A. W. RICHESON, A.M. (Johns Hopkins). Teacher, Univ. of Maryland, Baltimore, Md.
C. F. ROOS, A.M. (Rice Inst.). Fellow, Rice Inst., Houston, Tex.
H. M. RUTHERFORD, Student, Rice Inst., Houston, Tex.
FAITH SAUNDERS, A.M. (Missouri). Logan Coll., Russellville, Ky.
V. Z. SHIPPY, B. S. (Penna.). Central H. S., Philadelphia, Pa.
NATHANIEL SMYLIE, A.M. (S.W.P. Univ.). Head of Dept., Belhaven College, Jackson, Miss.
C. M. SPARROW, Ph.D. (Johns Hopkins). Prof., Univ. of Virginia, University, Va.
ELIZABETH T. STAFFORD, M.S. (Brown). Instr., Univ. of Texas, Austin, Tex.
EDITH STEININGER, A.M. (Kansas). Head of Dept., Clay Co. Community H. S., Clay Center, Kans.
W. P. STEVENS, A.M. (Baylor Univ.). Instr., A. & M. College, College Station, Texas.
P. W. STONER, M.S. (California). High School, Pasadena, Calif.
S. P. WALKER, A.M. (Miss.; Chicago). Prof., Miss. State Coll. for Women, Columbus, Miss.
A. S. WIENER, Student, Cornell Univ., Ithaca, N. Y.
G. A. WILLIAMS, A.B. (Illinois). Instr., Oregon Agric. Coll., Corvallis, Ore.
N. R. WILSON, Ph.D. (Chicago). Prof., Univ. of Manitoba, Winnipeg, Man.

THE PRINCIPIA, St. Louis, Mo.
UNIVERSITY OF OREGON, Eugene, Ore.
MISSOURI SCHOOL OF MINES AND METALLURGY, Rolla, Mo.

W. D. CAIRNS, *Secretary-Treasurer.*

MEETING OF THE SOUTHERN CALIFORNIA SECTION

The second regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of Southern California, in Los Angeles, on Saturday, November 7, 1925, Professor Harry Bateman presiding.

There were forty-five present, including the following thirty-one members of the Association: O. W. Albert, H. Bateman, W. N. Birchby, J. R. Campbell, Mrs. T. Clark, M. Collier, M. E. Conn, P. H. Daus, I. B. Ernsberger, H. E. Glazier, F. C. Hall, E. R. Hedrick, H. C. Hicks, G. H. Hunt, G. James, M. N. Keith, G. R. Livingston, D. Lodwick, W. E. Mason, W. A. Newlin, B. Podolsky, L. E. Reynolds, W. P. Russell, D. V. Steed, F. C. Touton, H. C. Van Buskirk, L. E. Wear, M. G. Whiting, H. C. Willett, Clyde Wolfe, E. R. Worthington.

The following papers were presented. Short abstracts appear below.

1. "An integration method of summing series," by Professor GLENN JAMES, University of California, Southern Branch.

2. "Note on the solution of a functional equation," by Professor H. C. WILLETT, University of Southern California.

3. "An improved harmonic analyzer," by Mr. BORIS PODOLSKY, University of Southern California.

4. "A general theorem relating to a sphere touching the faces of a tetrahedron," by Professor HARRY BATEMAN, California Institute of Technology.

5. "Note on a series for π derived from the harmonic series," by Professor W. N. BIRCHBY, California Institute of Technology.

6. "Modern methods in machine mathematics," by Professor CLYDE WOLFE, California Institute of Technology.

1. Professor James described an integration method of summing series with a finite number of terms and illustrated the method by summing the geometrical progression.

2. Professor Willett gave a solution of the functional equation $\phi(x+y)+\phi(x-y)=2\phi(x)\cdot\phi(y)$. When this equation was encountered in a course in non-Euclidean geometry, he found it desirable to present a solution which was not long nor too difficult to understand. By assuming that ϕ could be expanded into a Taylor series, the solution was readily obtained.

3. The instrument described in this paper was designed by the author for the use of the bureau of power and light of the city of Los Angeles. Its distinguishing feature was its adaptability for the analysis of curves with so-called hidden periodicities.

4. If a circle rolls along a line, then the intersection of the tangents drawn from two fixed points on the line to the circle, describes a cubic curve. Professor Bateman discussed the extension of this problem to a sphere rolling on a plane, and in particular to the case when the point of contact of the sphere and the plane lies on the circumcircle of the three fixed points.

5. Professor Birchby discussed various series formed from the harmonic series by making the signs $+$ or $-$ according to a certain law. The arrangement of signs corresponded to the taking off and putting on of rings in the well known Chinese ring puzzle.

6. Professor Wolfe discussed the use of the calculating machine and suggested improvements that were desirable for certain types of calculations.

P. H. DAUS, *Secretary-Treasurer.*

THE SPRING MEETING OF THE KENTUCKY SECTION

The spring meeting of the Kentucky Section was held at the University of Kentucky in the Civil Engineering and Physics Building, on Saturday, May 2, 1925. Those who attended the meeting were entertained at luncheon in the University Cafeteria. Professor J. M. Davis, the chairman of the section, presided at both sessions. During the course of his welcoming address, he gave the same admonition that he gave to the section as its chairman fourteen

years ago, namely, not to give all of our time to presenting truths which we have discovered, but rather to present mathematical truths so that people are inspired to think clearly for themselves.

There were thirty-five persons in attendance, of whom the following are members of the Association: Thurman Andrew, P. P. Boyd, J. L. Clayton, Mary E. Clarke, C. G. Crooks, J. M. Davis, H. H. Downing, A. R. Fehn, W. R. Hutcherson, Katherine Kienzle, Elizabeth Le Stourgeon, C. A. Maney, T. A. Martin, R. I. Pepper, E. L. Rees, C. H. Richardson, F. A. Scott.

The program follows, accompanied by abstracts of the papers:

(1) "l'Hospital's solutions of equations by means of algebraic curves," by Professor P. P. BOYD, University of Kentucky.

(2) "Probabilities as applied to life insurance," by Professor T. A. MARTIN, Berea College.

(3) "The distribution of means, a problem in sampling," by Professor C. H. RICHARDSON, Georgetown College.

(4) "Order of signs in a series," by Professor H. H. DOWNING, University of Kentucky.

(5) "How much mathematics should be given in the high school," by Professor C. E. CALDWELL, Eastern Kentucky Teachers College, (by invitation).

(6) "The freshman course in mathematical analysis," by Professor C. A. MANEY, Transylvania College.

1. Dean Boyd reviewed the work of Marquis de l'Hospital, and especially the posthumous book of 1707 on Conic Sections. Various examples were given from this work, showing l'Hospital's methods for solving higher degree equations, and giving also some of his solutions of "determinate problems" by means of these geometric solutions of equations.

2. Two essential elements in life insurance are general average and probability. These elements are essential in order to build on a sound business basis. The difference between insurance built on actuarial science and "fraternal insurance" is that fraternal insurance has ignored to a large degree the principles of probability and general average that are so essential to sound business. There is a large number of different kinds of policies suitable to different classes of people. Life insurance, well founded, is a necessity to the welfare of society, a great blessing to mankind.

3. In his paper, Mr. Richardson derived the relationships that exist between the first four moments of the distribution of means and corresponding moments of the parent distribution, both for the case when the parent population is finite as well as infinite.

4. The speaker referred to the usual way of caring for the singly alternating plus and minus terms in a series by the use of $(-1)^{n-1}$ as a factor in the coeffi-

cient of the nth or general term. If the series involves terms alternating doubly, or triply, or etc., plus and minus the coefficient of the nth term may be properly cared for by the use of -1 with an exponent $[(n-1)/r]$, where n is the number of the term of the series, r is the number of terms in the sequence of positives or negatives, and $[(n-1)/r]$ is the largest integer in $(n-1)/r$.

6. Fifty-one questionnaire letters were sent out to a number of institutions on the approved list of the American Association of Universities where it appeared (from catalog statements) that such a course was being taught. Thirty-three replies were received. In twenty-eight colleges and universities this course was the basic introductory course in the department. This course is one in which the fundamental viewpoints of the calculus are introduced in the freshman year. Topics in algebra, trigonometry, analytic geometry, and the calculus are included and treated in a unified manner generally in connection with typical problems of exact science. The evolution of this course is thought of in connection with the tendency in the modern junior college program to revise all courses so as to contribute to the general aim of a comprehension of modern civilization. Unified mathematics, both as a tool for all exact science and as a preparation for advanced mathematics is considered deserving of a place as a required subject in the Junior College program. Where the course has been thoroughly tried out it has proved to be a success, and in many of the best colleges and universities of the country it occupies a permanent position in the curriculum.

At the close of the business session Professor T. A. MARTIN, Berea College, was elected chairman and Professor A. R. FEHN, Centre College, secretary-treasurer for the coming year.

A. R. FEHN, *Secretary-Treasurer.*

ROBERT ADRAIN, AND THE BEGINNINGS OF AMERICAN MATHEMATICS[1]

By JULIAN L. COOLIDGE, Harvard University

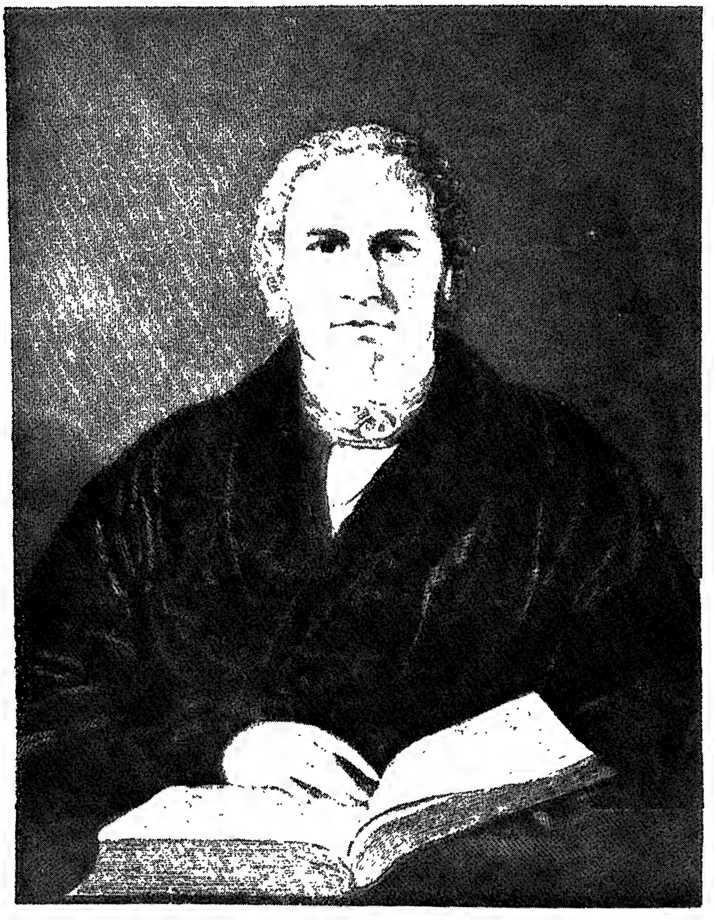
Robert Adrain was born in Carrickfergus, Ireland, on September 30, 1775.[2] He seems to have been of Huguenot ancestry, on his father's side at least, or such is the testimony of his grandson Elbert Adrain Brinkerhoff.[3] His anonymous biographer,[4] whose account has been accepted by almost all who

[1] Retiring presidential address delivered at Kansas City, Dec. 30, 1925.

[2] By a very curious coincidence I inadvertently began the preparation of the present address on the hundred fiftieth anniversary of Adrain's birth.

[3] Biographical sketch of Robert Adrain, L.L.D. E.A. Brinkerhoff. *Mathematical Magazine*, vol. II No. 4, 1891, pp. 56-58.

[4] Robert Adrain L.L.D. *United States Magazine and Democratic Review*, vol. XIV, June 1844, pp. 646 to 652. It is said to have been written by the son Garnett Bowditch Adrain.



Robt Adrain

cient of the nth or general term. If the series involves terms alternating doubly, or triply, or etc., plus and minus the coefficient of the nth term may be properly cared for by the use of -1 with an exponent $[(n-1)/r]$, where n is the number of the term of the series, r is the number of terms in the sequence of positives or negatives, and $[(n-1)/r]$ is the largest integer in $(n-1)/r$.

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have followed, goes further and says that Robert's father, who died in 1790, left France after the revocation of the Edict of Nantes. If he left France, it probably was after this regrettable event which took place in 1685! In any case this gentleman appreciated Robert's talents and gave him the best available education, so that when the boy was thrown on his own resources at the ripe age of fifteen, he was able to support himself by opening a school. Moreover his school was a success, for presently we find him employed as private tutor in the house of an officer of the crown named Mortimer. The connection terminated suddenly with the Irish uprising of 1798 when Mortimer took up arms for his king, while Adrain was an officer in the insurgent forces, with a price on his head. This awkward situation was terminated by Mortimer's death in battle. Adrain was so severely wounded by a shot in the back, fired by one of his own men, that he was left on the field for dead, and the report of his death was generally believed. Kind friends nursed him back to health and he managed with great difficulty to escape to the United States with his wife and infant daughter. The cholera was raging in New York, but he found ready asylum in Princeton in the house of Mrs. Theobald Tone, widow of his commanding officer in Ireland.[1]

Adrain had been a teacher in Ireland, a teacher he became in America, serving as master in the academy at Princeton. Report has it that he taught well, and was not a stranger to the use of the rod.[2] After two or three years he moved to York, Pa., to take the principalship of the local academy, and in 1805 he became principal of the academy at Reading, Pa. The mathematical work which he published while holding this last position attracted favorable notice, and in 1809 he was called to the professorship of mathematics in Rutgers, then Queen's College, New Brunswick, N. J. The college gave him the honorary degree of Master of Arts in 1810 and probably expected that he would serve them indefinitely, but this was not to be, for in 1813 he accepted the call to a professorship in Columbia University, New York City. I am unable to say what was the title of his chair. According to the Columbia general catalogue he was professor of mathematics and natural history till 1820, when he became professor of mathematics and astronomy. But he signs some of his published writings with the title of professor of mathematics and natural philosophy and this more plausible designation is used by all his biographers. Columbia granted him an L.L.D. in 1818. In 1812 he was elected a Fellow of the American Philosophical Society, Philadelphia, and a year later of the American Academy of Arts and Sciences, Boston. He certainly could not complain of lack of appreciation for his scientific work at this period. Whether he obtained equal success as a teacher is open to question; Dr. Benjamin

[1] Hageman, *Princeton and Its Institutions*, vol. I, Philadelphia, 1879, p. 197.
[2] *Ibid.*, p. 212.

Haight of the class of 1828 said of him in 1874[1] "If one was thoroughly prepared in his recitation, all was well, but if a student was in doubt or needed a word of explanation in a difficult problem, he not only did not get assistance, but was sent down with some remark of the sort 'If you cannot understand Euclid, dearie, (a term he frequently used when out of temper) I cannot explain it to you.' The consequence was that only a small portion of the class could keep up with his course; those who had entered college thoroughly versed in the elements of mathematics and who studied very diligently after they had entered his lecture room, in my class not more than one-fifth of the number. I ought to add, however, that those who went to him in private always found him kind in manner and ready to answer their questions, and help them out of their difficulties."

At this time his portrait, now in the possession of Columbia University, was painted, apparently by the Irish artist, Ingham. This is the portrait reproduced elsewhere in the present issue.

Two other items connected with this period of his life, I mention with reserve. The anonymous biographer says[2] that soon after his election to the learned societies mentioned above, he was elected to several philosophical societies in Europe. I have been utterly unable to verify this, merely noting that Adrain gave no mention of such in signing his name, although he did append LL.D., F.A.P.S., F.A.A.S., etc. I have equally failed in my attempts to verify the statement of Brinkerhoff that about 1817 he commenced a correspondence with Laplace.[3]

Adrain's connection with Columbia was severed in 1826; why, we do not know. The statement that the change to the country was necessitated by the state of his wife's health must be taken with reserve for reasons which will appear presently. He returned to New Brunswick and to his former professorship, but only held it for a year, as he then accepted the professorship of mathematics in the University of Pennsylvania. There was a great deal of correspondence between Adrain and the University trustees before this could be finally arranged.[4] He had just bought a new farm and his family absolutely refused to go, also Adrain had some feeling about his duty to Rutgers. Eventually all was straightened out, and to Philadelphia he went leaving his family or part of it behind to enjoy the farm. His salary was $2500.

Adrain, who had become Vice Provost in 1828, stayed in Philadelphia till 1834. The anonymous biographer tells us that he left on account of his

[1] I have not seen the original of this, but am following W. B. Campbell in an address he delivered May 17, 1923, before the New Brunswick Historical Club. A typewritten copy of part of this address was kindly sent me by G. A. Osborn, Esq., Librarian of Rutgers University.

[2] *Democratic Review*, p. 648.

[3] *Loc. cit.*, p. 457.

[4] Copies of this were kindly sent me by Mr. Osborn. See also Campbell, *loc. cit.*

wife's health, but the sad fact is that the records tell a different tale.[1] On April 10, Adrain complained to "the Honourable the Trustees" that the members of the Junior class had become noisy and disorderly. He did not know why this had happened, and seemed to have no idea how to meet the situation. The board appointed a committee of three to confer with the faculty, the upshot of which was a report to the effect that the disorders in Adrain's classes had continued for some time, that he seemed unable to suppress them, that the faculty were not in a position to help him, that the contagion might spread to other classes, and that the situation was intolerable. The conclusion was inevitable; Adrain resigned, May 2, 1834, and the board passed very handsome resolutions expressing great sorrow over the loss of such a distinguished professor.

Perhaps it will be of interest to give a list of the subjects which Adrain taught at the University of Pennsylvania in 1829:[2]

Freshman year,

Arithmetic reviewed, Algebra to quadratic equations inclusive, Euclid.

Sophomore year,

Algebra and geometry completed, application of algebra to geometry, plane and spherical trigonometry, surveying and mensuration.

Junior year,

Analytic geometry through conic sections, differential calculus, perspective geography, use of globes and charts.

Senior year,

Integral calculus, analytical dynamics, physical astronomy.

Truly a wonderful amount to require of all the students!

Adrain's next move after leaving Philadelphia was curious. Leaving his family in New Brunswick he went to New York for the period 1836-1840 to teach in the Columbia College grammar school. Considering his eminence as a mathematician, and his recent failures as a disciplinarian, this is indeed a passing strange episode. It did not last long. He returned to New Brunswick in 1840, and died there August 10, 1843, in the 68th year of his age.

Adrain was married in Ireland to Ann Pollock, and had by her seven children: Margaret, born in Ireland, never married; Mary Moore, married her father's pupil J. N. Brinkerhoff; John, removed to Ohio; Sarah, married E. P. Stinson; Robert, graduated from Rutgers in 1827 and continued to live in New Brunswick; Elisa, married Peter Williamson; and Garnett Bowditch, Rutgers, 1833, member of Congress, 1856-1860.[3]

It is time to turn from Adrain's personal history to his mathematical production.[4] This began about five years after his arrival at the very first

[1] Copies of these also were sent to me by Mr. Osborn.

[2] This list was kindly given me by Professor M. J. Babb of the University.

[3] For further particulars see Campbell, *loc. cit.*

[4] An incomplete list of Adrain's work, prepared by Artemas Martin, is appended to the biography by his grandson *cit.*

moment possible, for he was one of the earliest contributors to the first mathematical journal published in this country, the *Mathematical Correspondent*, George Baron, editor, New York, 1804. The editor's motto, as stated in the announcement, was to "inspire youth with the love of mathematical knowledge by alluring their attention to the solution of pleasant and curious questions, and to promote the mathematics by opening a channel for the ready conveyance of discoveries and improvements. . . . ."

The *Correspondent* was, as might be expected, mainly a problem magazine but this was no particular distinction. One hundred years earlier the *Ladies' Diary* began publishing questions and answers of a mathematical nature, and I doubt whether any exclusively mathematical periodical before Baron's had essentially any other character. It is well to remember that the austere Crelle's *Journal*, published "Aufgaben und Lehrsätze" as late as 1838.[1]

Baron starts his first number, as we might expect, with a long didactic article of his own. It deals with proportion in arithmetic. One or two short articles follow, and then a number of questions, of which the fourth was to find the capacity of a hemispherical shell of given measurement. The solution of this problem in the following number seems to have been Adrain's first published work in mathematics. He subsequently answered every question proposed, with the exception of one number. The early questions were trivial enough, although occasionally amusing. On p. 186 we have one beginning hopefully "Five political vagabonds, A, B, C, D, and E are transported from New York." The quality improves thereafter and other material appears. On p. 103 Adrain publishes a "Disquisition concerning the Motion of a Ship which is steered to a certain Point of the Compass." He was a diligent reader of Laplace, and, to the end of his days, was interested in the earth considered as a rotating fluid spheroid in equilibrium. If a ship be steered due North or South, or due East or West, the balance between the centrifugal force and the earth's attraction is destroyed, and she will drift in a direction perpendicular to the course steered. In Article XIX he gave what is really a splendid problem concerning a curve which he calls the "Catenaria volvens." This is the form taken by a homogeneous flexible, non-elastic string rotating about two fixed points with a constant angular velocity, the attraction of gravity being disregarded. This problem is now quite a favorite in books on elliptic functions. I can not find that anyone ever considered it before Adrain, and after he had explained it and showed how it led to elliptic integrals, which, naturally he was not able to handle, it was forgotten again till 1860 when it was taken up by Clebsch, who had never heard either of Adrain or the *Correspondent*.[2]

[1] A bibliography by D. S. Hart of early American mathematical periodicals will be found in *The Analyst*, vol. II, Des Moines, Iowa, 1875, pp. 131-138.

[2] Crelle, vol. 57, 1860, pp. 93 ff. See also Marcolongo, Napoli, *Rendiconti*, (2) vol. VI, 1892.

Adrain's best known contribution to the *Correspondent* was his "View of Diophantine Algebra" which he somewhat inaptly defines as the "method of finding such rational values of one or more indeterminate quantities, that any proposed function of those quantities may become a rational square."

This essay covers some fifty pages. The first thirty explain the general methods, which are not novel. In the last twenty he solves specific problems. Here Adrain published the first article dealing with this branch of arithmetic that ever appeared in America.

But the *Correspondent* was doomed. Some of the contributors were flippant in their remarks, to put the matter mildly. The worst offender in this respect was one signing himself "A. Rabbit" and giving as his domicile, first Harlaem (sic) near New York, and then Uncle Sam's Downbelow. More serious was the disinclination of many subscribers to pay what they owed, a difficulty not confined to mathematical publications. Even a belated engraving of Baron's own melancholy countenance failed to save the situation.

At this point Adrain threw himself into the breach, and brought out in 1807 the first sheets of a second volume, which included the last twenty pages of the Diophantine algebra. For some reason the trial was not satisfactory, for all the material was used afresh in his own new journal, *The Analyst, or Mathematical Companion*, Philadelphia, 1808.[1] This publication was in many respects similar to its predecessor, but flippant and poetical contributions were excluded, and the standard of excellence was visibly improved. The first number contained, besides the close of the Diophantine algebra, a short and unimportant essay by the editor on the utility of mathematics, and a first batch of problems. These problems were solved in the second number, which included as well an article by Adrain on the use of logarithms in solving exponential equations of a complicated sort and certain new problems, including the following rather attractive one "On what day in the year does the sun's apparent diameter increase the fastest?"

In the third number Adrain published a discussion of what he called "isotomous" curves. Given a set of plane curves through a common point and with the same tangent there. The locus of points cutting them at equal arcual distances is called an "isotomous" curve for the set. In particular he took the case of a pencil of tangent circles and discussed the spiral whose polar equation is

$$r = k(\sin \theta/\theta).$$

He looked rather closely at the series developments involved in the problem, quoting Bernoulli and Landen, and then showed by ingenious geometrical reasoning that the area is finite, but the length infinite.

[1] This must be the explanation of the statement of the anonymous biographer, *loc. cit.* p. 647 that the first number of the *Analyst* had so many misprints, that Adrain incurred the loss of republishing in Philadelphia.

Adrain contributed to the fourth number in a different way, generalizing a rather trivial question. What is the shortest curve, terminated by two opposite sides of a rectangle, which passes through a specified point within and divides it into two parts of given area? His reasoning here is decidedly ingenious, and he shows a striking knowledge of the isoperimetric problem. But this number of the *Analyst* will always be notable for another reason.

In No. II Robert Patterson had set the prize question how to correct the measurement of a polygon, whose successive sides are given in length and direction, but which, when plotted, does not close up. In No. IV Bowditch gave a solution based on certain specific, and not too convincing, assumptions. Adrain attacked it on a far higher ground.

What general laws, he asks, will errors in measurement follow? What is the probability that we shall make an error ξ in measuring a length X?[1]

Suppose that we measure two quantities whose true values are X and Y. What is the probability that we shall make the respective errors ξ and η where $\xi+\eta=C$? Adrain took it as evident that it is most likely that these errors shall be proportional to the quantities measured, i. e.,

$$\frac{\xi}{x}=\frac{\eta}{y}.$$

Now let $\phi(\xi, X)$ be the probability of making the error ξ in the first measurement. Assuming the two events are independent, which is contrary to this previous assumption, their compound probability is the product of their individual probabilities, so that the function we would maximize is $\phi(\xi, X)\phi(\eta, Y)$. Equating the logarithmic derivative to zero,

$$\frac{\phi'(\xi\ ,\ x)}{\phi(\xi\ ,\ x)}d\xi+\frac{\phi'(\eta\ ,\ y)}{\phi(\eta\ ,\ y)}d\eta=0$$

$$d\xi+d\eta=0$$

$$\frac{\phi'(\xi\ ,\ x)}{\phi(\xi\ ,\ x)}=\frac{\phi'(\eta\ ,\ y)}{\phi(\eta\ ,\ y)}\quad\text{when}\quad\frac{\xi}{x}=\frac{\eta}{y}.$$

The *simplest* solution is

$$\frac{\phi'(\xi\ ,\ x)}{\phi(\xi\ ,\ x)}=C\,\frac{\xi}{x},$$

$$\phi(\xi\ ,\ x)=re^{-d\xi^2/x}.$$

Here we have the first known demonstration of the exponential law of error, published a year later by Gauss, and usually associated with his name.[2]

[1] In this discussion, I shall use my own notation, as that is more familiar to the modern reader.

[2] This proof is reproduced by Abbe, "A historical note on the method of least squares," *American Journal of Science*, 3rd. series. vol. I, New Haven, 1871, pp. 41 ff.

The number of so-called proofs of the Gaussian law is very large. Not one of them is absolutely convincing, all rest on one or more rather arbitrary assumptions. Undoubtedly the law is not strictly true, and there is no such thing as a general law of accidental errors. We may go further and say that Adrain's proof is by no means the best we have. That is not in the least astonishing. On the other hand, I am not certain that the objections that have been made to it are always well founded. There is the example of Glaisher[1] who said that the assumption that it is most likely that the error should be directly proportional to the distance measured is inadmissible. "In whatever manner the measure is effected, the error ought certainly to be relatively less."

I can not agree with this view. The probability of any error is a pure number, and should not be altered if the quantity measured, and the corresponding error, are altered proportionately in scale. This also appears from the exponential law itself, when rightly stated. This law does not tell us that the probability of an error ξ is $ke^{-k^2\xi^2}/\sqrt{\pi}$, such a statement is absurd. What it tells us is that the probability of an error in the infinitesimal region $\xi \pm \frac{1}{2}d\xi$ differs by an infinitesimal of higher order from $ke^{-k^2\xi^2}d\xi/\sqrt{\pi}$. If we multiply ξ and $d\xi$ each by r, we must divide k by r at the same time; the probability is unaltered. Glaisher's other objection is that

$$\frac{\phi'(\xi\ ,\ x)}{\phi(\xi\ ,\ x)} = \frac{\phi'(\eta\ ,\ y)}{\phi(\eta\ ,\ y)}\ ,\quad \text{where } \frac{\xi}{x} = \frac{\eta}{y}\ ,$$

merely involves

$$\phi = \left[x\left(\frac{\xi}{x}\right)\right]^x .$$

This is certainly true; Adrain expressly stated that he took the *simplest* solution. Why the simplest solution should be *the* solution is a mystery.

It is a curious fact that some readers of Adrain's proof failed to turn the page, and see that he gave another immediately following. Such was the case with Abbe, but the second proof was later reproduced by Merriman.[2] Using our previous notation, the probability of making errors ξ and η in taking the measurements X and Y, is assumed to be $\phi(\xi, X)\ \phi(\eta, Y)$. There will be a curve of like probability in the XY plane, and if we assume that positive and negative errors are equally likely, and that the measures in abscissa are entirely comparable to those in ordinate, this curve will be symmetric with regard to both axes, and meet a vertical or horizontal line but twice. Finally, Adrain

[1] "On the law of the facility of errors of observation" *Memoirs of the Royal Astronomical Society*, part I, 1871, especially p. 78.

[2] "On the history of the method of least squares", *The Analyst*, vol. IV, Des Moines, 1876, pp. 33.

tells us, the curve must be the simplest that fulfills these conditions, namely a circle. We thus get $\phi(\xi, X)\phi(\eta, Y)$ is a maximum when

$$\xi^2+\eta^2=\rho^2 \;,$$

$$\frac{\phi'(\xi\;,\,x)}{\phi(\xi\;,\,x)}d\xi+\frac{\phi'(\eta\;,\,y)}{\phi(\eta\;,\,y)}d\eta=0\;,\quad \frac{\xi}{\rho}d\xi+\frac{\eta}{\rho}d\eta=0\;,$$

$$\phi=re^{-d\xi^2/\rho}\;.$$

It seems to me that this proof is decidedly weaker than the other. The statement that the curve *must* be the simplest is unfortunate. If there be a law of error, if we are hunting for anything that really exists, there is no reason why it should give the simplest curve. It is the previous question of why the simplest solution should be the right one in an aggravated form.

No, Adrain's proofs are far from perfect, perhaps they are the weakest that have been offered. While no published proof is perfect, there is no excuse for publishing any new proof unless it be in some sense an improvement on a previous one. Nothing can take away from Adrain the credit of having been the first to face squarely the problem of deducing a general law for the distribution of errors, and of carrying it through to the point of finding a formula which, while not perfect, and not always applicable, is still the best that has been devised. Is it too much to say also that this formula was the first broad principle of pure mathematics discovered in America?

After developing his error formula, Adrain proceeded to deduce therefrom the classical process of least squares, much as it was published, without demonstration two years previously by Legendre.[1] Legendre's memoir was in Adrain's library and doubtless he was guided in developing the law of error by a desire to justify the least-square method. In the *Analyst* he makes four applications of this method, namely, to determine a point on a line from inconsistent observations, to do the same for a point in space, to correct a dead reckoning at sea, and to solve Patterson's problem of the non-closing polygon. The numerical corrections which he reaches in this case agree closely with those suggested by Bowditch. He makes a fifth application in a subsequent memoir which we shall notice in due time.

There is little of interest in the remainder of the first volume of the *Analyst* except perhaps Adrain's closing prize question, to explain the tides on our earth solely by the earth's own motion and attraction, irrespective of the influence of the sun or moon. Naturally no solution was ever published, one wonders whether he ever regretted having proposed the question.

The *Analyst*, like the *Correspondent*, died after one volume. We do not know the reason, we only regret the fact, noting that there was no new serial

[1] *Nouvelle méthode pour la détermination des orbites des comètes*, Paris, 1806.

devoted to mathematical publication in America till 1814. Adrain had to content his mathematical craving in other ways, and he chose the editing of an American edition of the *Course in Mathematics* by Charles Hutton, LL.D. F.R.S., Professor of Mathematics in the Royal Military Academy. One can not wonder at the bravery of the British officers at Waterloo if they had mastered Hutton. Yet the book enjoyed an extraordinary popularity, passing through thirteen English, and four American editions. It is curious that a man who had read Lagrange should choose to edit Hutton. Adrain's own contributions to the book he edited consist, in the first edition,[1] in a large number of really helpful footnotes, and two or three specific points which he mentions in the preface. One of these is his definiton of a surd:[2] "An irrational quantity or surd is that of which the value can not be accurately expressed in numbers, as the square roots of 2, 3, 5." Adrain never doubted that these numbers had square roots in some sense. He probably would have agreed that, under his definition, the number π was a surd, he probably believed that it was; Hermite and Lindemann were far in the future. He tells us in the preface that Hutton's own definition was that a surd was that which has not an exact root. This would seem to include, incidentally, all prime numbers. Another detail where he claims originality is in the finding of oscillations of a pendulum whose cord, passing over a freely turning pulley is attached to an object lighter than the bob.[3] With regard to his footnotes as mentioned above, they are generally helpful, and mark an improvement over the original text. It is a pity that they are more frequent in the first volume, than in the second which covers more advanced material. It is conceivable that some of this lay outside the circle of his natural interests. We are not sure how deeply he was concerned with such topics as "geodesic operations" or "the spouting of fluids."

These remarks apply to the first American edition. In the third, Adrain introduces a really important contribution to mathematical teaching in America, an essay on descriptive geometry. In appraising this, let us bear in mind that Monge's *Géometrie Descriptive* was only published in 1800. Adrain's essay saw the light in 1822, there was no article in German on the subject[4] till 1828. He deserves considerable credit for having so early discerned the importance of this topic, and having made a clear and concise exposition of it. Unfortunately, there is a question of priority that must be examined. Adrain never introduced this subject into his various editions of Hutton till 1822. The natural sources from which we should expect him to draw his material would be Monge and Lacroix. There is not the faintest resemblance. In no case are two successive problems in Adrain successive in Monge, in no

[1] New York, 1812.

[2] P. 17. [3] Vol. II, pp. 556-557.

[4] Loria, in Cantor's *Geschichte der Mathematik*, vol. IV, Leipzig, 1908, p. 626.

case are three successive problems in Adrain successive in Lacroix, and, in fact there are only two cases of like sequence of two problems. But a year before, in 1821 Croizet, professor at West Point, published a descriptive geometry. We have these coincidences. Adrain 3, 4, 5, 6 are Croizet 1, 2, 3, 4. Adrain 8, 9, 10 are Croizet 5, 7 and 8. Adrain 14, 15, 16, 18 are Croizet 9, 11, 12, and 13. Later, dealing with the rather unsuitable question of spherical triangles, we have Adrain 1, 2, 3, 4, 5, 6 identical with Croizet 1, 2, 3, 6, 5, 4. Neither Monge nor Lacroix touch spherical triangles. There is no great similarity in the proofs. Sometimes Adrain's proofs are like Croizet's, more often they are not. Adrain often gives more than one proof which Croizet never does. Perhaps both had access to some source which I have not seen. My general impression, however, is that Adrain saw Croizet's book, that he got therefrom the idea of adding this essay to his Hutton, and that he thought through the whole subject in his own way, and gave such proofs as he liked. But he should have made some mention of Croizet. I think there may have been some jealousy between the two men.

In 1814 Adrain had another try at serial publication, the attempt being to continue the *Analyst*. The main part of the one short number was an algebraic explanation of Euclid's theory of proportion, afterwards copied in James Ryan's *Algebra* (1824). There follow sixteen questions including the prize one to determine whether the mouth of the Mississippi is farther from the centre of the earth than the source; and then the *Analyst* sinks quietly to rest.

In 1817 Adrain contributed to the third volume of *Portico*, a literary and historical magazine of which five volumes were published in Baltimore. Only the third contained mathematical matter, and there were not many mathematical correspondents. Two of Adrain's contributions to this are of interest. One consisted in finding the proportions of a cylinder of revolution which will revolve indefinitely about any axis through its centre. The question had been raised by Playfair,[1] who had come to the erroneous conclusion that no such cylinder was possible. The other had to deal with the differential equation which Adrain writes in fluxional notation

$$\frac{a\dot{x}+y\dot{x}}{\dot{y}}=x+y-\frac{x\dot{y}}{\dot{x}}$$

adding "Now it is required to investigate the general relation of fluents (integrals) involving one arbitrary quantity, and, besides, the particular solution of which the proposed equation is susceptible, and which is not comprehended in the fluent involving an arbitrary quantity."[2]

[1] *System of Natural Philosophy*, Edinburgh, 1812.
[2] *Portico*, vol. III, Baltimore, 1817, pp. 77 and 246.

This problem is taken from Emerson's *Fluxions*, and Adrain first gave with the problem the incorrect solution there offered, without mentioning that it was wrong. After a while Croizet solved it neatly and correctly,[1] using differential notation, whereupon Adrain published his own solution[2] in fluxional notation. He gave both the complete and the singular solutions, and showed how the latter could be obtained by equating to 0 the discriminant of the former, looked on as a function of the constant of integration; standard practice for which he referred to Lagrange. Why did he publish this after Lagrange had given the general theory for all such equations, and Croizet has solved the particular equation in question? It lends color to the theory that there was jealousy between the two men.

In 1818 Adrain left the *Portico* to contribute to the *Scientific Journal* published in New York by William Marat, only nine numbers appearing, February 1818 to October 1819. This storehouse of learning contains a large amount of popular and, supposedly, useful science; the mathematical part is also popular. Adrain's contributions were unimportant. He does not seem to have been very proud of them, as he did not use his own name but signed himself Analyticus, New York.[3]

In 1818 appeared two other papers of Adrain's of a more serious nature. These are published in the *Transactions of the American Philosophical Society*.[4] The first, called an "Investigation of the Figure of the Earth, and the Gravity in Different Latitudes" sounds very deep indeed and was so esteemed by his contemporaries. Says the anonymous biographer[5] "A neat, elegant, ingenious and profound production, exhibiting mind of the first order, ... which gave him great celebrity not only in this country, but abroad." Adrain deserved greater celebrity than he ever acquired, either here or abroad, but not for this particular paper which is interesting and painstaking, but not at all striking. Laplace and Clairaut showed that if x be the length of a seconds pendulum at the equator, and r its length in latitude λ, while y is a coefficient to be determined, then $r=x+y\sin^2\lambda$. Laplace combined fifteen observations for different latitudes by the best methods he knew, and deduced 1/336 as the ellipticity of the earth. Adrain combined them by the method of least squares and found 1/319. He then proceeded to show that the great difference in result did not arise from the difference in methods but from two errors in calculation made by Laplace. When these are rectified, Laplace's method gives 2/633 as opposed to the 2/638 of Adrain.

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[1] *Ibid.*, p. 406. [2] *Ibid.*, p. 499.

[3] The copy I have seen belonged to Bowditch. The name Adrain is written in ink next to the word Analyticus.

[4] New series, vol. I, 1818, pp. 119 ff. and 353 ff.

[5] *Loc. cit.* p. 648.

Adrain's other contribution to this journal has the title "Research concerning the Mean Diameter of the Earth." It consisted in giving six different definitions of a sphere to be called the "mean" of a given spheroid and in showing that they lead to the same result, which result he uses to find the mean of our earth.

In 1819 Adrain found a new mouthpiece. M. Nash published in New York the *Ladies' and Gentlemen's Diary*. The publisher announced that his periodical was "intended for an annual magazine including a variety of matter chiefly original on subjects of general utility in the arts, sciences, agriculture, manufactures, etc. etc." The plan was endorsed by six mathematicians including Adrain and Marat of the late *Scientific Journal*. The bulk of it is what will be found in any almanac, with special reference to astronomical data, but there is in addition useful and entertaining information, poetry, and mathematics. Adrain's contribution consisted chiefly in two problems which he answered in subsequent numbers himself. The first, No. I, p. 62, is to find the law of stretching of a uniform thread which is considered weightless, but to which a weight is attached at the lower end, the upper being fast. The other is the old *Analyst* question of whether the mouth of the Mississippi is farther from the centre of the earth than is the source. This he answers in the affirmative, No. III, 53, on the hypothesis that the source is not more than a mile above sea level.[1]

The extinction of the *Ladies' and Gentlemen's Diary* in 1822 led Adrain into his last editorial venture *The Mathematical Diary*. Thirteen numbers were published in New York from 1825 to 1833. Adrain was editor of those numbered I to VI (there was no apparent distinction between III and IV) his successor being James Ryan.[2] The first number begins with an apology of the usual type, and an essay by Adrain on the rectification and quadrature of the circle which is not remarkable. More unusual is the inclusion of a review of the English translation of a volume on mechanics by Venturoli of Bologna. The volume closes with the usual sort of problems. Our interest is aroused by Adrain's prize question in No. 2. The speed of the current of a certain river is a given function of the distance from the bank. If a boat can be propelled through the water with a given velocity by what path will she make the quickest crossing? Here is a perfectly straightforward question in the calculus of variations, and it was solved by Strong of Hamilton College in No. 3 by the classical methods of Euler and Lagrange. Presumably Adrain knew what he was doing when he asked the question, and so was in possession of the elements of this calculus.

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[1] Professor Babb wrote me, October 28, 1925, that Adrain had also computed all the dates in the magazine.

[2] This journal was regularly, if briefly, reviewed in the *Bulletin des Sciences Mathématiques*, etc. of Terusac.

Adrain's interest in the *Diary* did not cease when he resigned the editorship. In No. VI he showed that he had not lost his early interest in arithmetic by proving that the two quantities x^2+xy+y^2, x^2-xy+y^2 can not both be perfect squares. He asked another attractive question to which Bowditch and Nulty, besides himself, contributed answers in No. VII, to find the time of oscillation of a short bar which is balanced horizontally on a sphere, and then slightly displaced. He answered another oscillation question in No. VIII, but the great interest of that number consists in the fact that a junior in Harvard contributed a discussion of the motion of a particle rolling down a quadrant of an ellipse under gravity; that junior was Benjamin Peirce. The number closed with a prize question by Adrain on the rolling of a disc on a horizontal plane, which question he answered in No. IX. In No. X he answered one or two questions, the most important dealing with a compound pendulum. Another pendulum question appeared in No. XII, p. 175, and this, so far as I have been able to discover, was his last piece of original published mathematical work, although he was only fifty-seven at the time. In the field, not of original publication, but of editorship, Adrain did one more serious piece of work, namely, he published a revised edition of *A New Treatise on the Use of Globes, or a Philosophical View of the Earth and the Heavens* by Thomas Keith. The first English edition of this curious work appeared in 1805, the second in 1808. The first American edition appeared in New York in 1811. It is a strange compound. The earth and the heavens are both conceived as spheres, and both represented by globes, so Keith writes a book which is a cross between a treatise on popular astronomy and one on physical geography, with much of the sort of information that one finds in an almanac, and a concluding part with one hundred two problems and some three hundred short astronomical questions. Adrain completed his labor on the new edition in 1826, although it was not published till 1832. He added a large number of new footnotes, cuts, and other explanatory material. Sometimes the alterations are small, as when he corrects Keith's statement that the Andes are the highest mountains in the world with a footnote to the effect that perhaps there are higher mountains in the Himalayas; in other cases the new material is more serious like the footnotes on the reason for the ellipticity of the earth's orbit, the nature of refraction, etc. A fourteen page chapter dealing with different proposed explanations of Noah's flood is replaced by short chapters on sidereal astronomy. The astronomical tables are altered and improved. While we may regret that a man of Adrain's original capacity should waste his time with this sort of thing, it must be confessed that here, as in other cases, he made evident improvement in the book he chose to edit.

It is a great pleasure to be able to say that Adrain's mathematical activity was not covered by his published work. He left behind a considerable amount

of manuscript, which has never yet been described, but of which we shall one day learn the nature. Professor Babb of the University of Pennsylvania has been studying this and all other material dealing with Adrain for years; when the results of his researches are finally published we shall have at last a proper basis for estimating his importance to mathematics.[1]

What shall we say of Adrain in the light of such information as is now available? He was the contemporary of Cauchy, Abel and Gauss whose biographers have given them no more enthusiastic praise than has been bestowed on him, but it would be blasphemy to compare him with any one of them. While he was displaying skill and ingenuity in solving problems, they were enriching mathematical science with long memoirs of unperishable value. These men were among the greatest mathematicians, not only of their time, but of all time. Adrain makes a very poor showing when compared with them. But is that the right way to appraise him after all?

He enjoyed most limited library facilities, according to our modern standards, but he made the very best use of what was available. He steeped himself in the works of the best writers on mathematics who had preceded him, there was hardly a branch of mathematical science which did not interest him and to which he did not make some small contribution. He lacked contact with his great European contemporaries, and his judgment of the standing of certain mathematical writers was occasionally at fault as when he wrote in the preface to the *Analyst* "The greatest mathematicians as Pascal, Leibnitz, the Bernoullis, ... Emerson, Simpson, Hutton, and Vince." But he cultivated assiduously the best scientific contacts that were available to him in the United States. There can be no question as to his outranking every American mathematician who was really his contemporary. Bowditch's *Navigator* may have been one of the most useful books ever written, and his translation of Laplace was of splendid service to higher education in America, but he never evolved anything which could be called an original mathematical theory. Others may have given better demonstrations of the exponential law of error, Adrain gave the first. His methods in Diophantine analysis were not novel, but he did obtain some new theorems. He may have frittered away in problem solving a talent that might have produced noteworthy contributions to our science, but his problems were far ahead of those suggested by others. It was indeed no small accomplishment in his day, and under his conditions, to ask and answer questions involving the calculus of variations and

[1] Professor Babb, who has courteously answered a number of questions which I have raised in the preparation of the present article, kindly offered to acquaint me with the contents of a number of Adrain's unpublished manuscripts. I could not feel that I should be justified in taking advantage of this generous offer. Had I been aware, when I started writing, that he had given the subject a far deeper study than was possible for me in the time available, I should have hesitated to speak of Adrain at all.

elliptic integrals. What he might have accomplished under more favorable circumstances must always remain a subject of conjecture. What he did accomplish entitles him to the glory of a pioneer in the development of American mathematics.

FORMULAS FOR THE ERROR IN SIMPSON'S RULE

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1. Introduction. If $y=f(x)$ is a continuous function of x in the interval $a \leqq x \leqq b$, Simpson's one-third rule is

$$\int_a^b y\,dx = \frac{h}{3}[y_0+y_n+4(y_1+y_3+\cdots\cdots+y_{n-1}) \\ +2(y_2+y_4+\cdots\cdots+y_{n-2})] \qquad (1)$$

where n is an *even* integer and $h=(b-a)/n$.

This is probably the most widely used formula for numerical integration, and its extensive use is doubtless due to its simplicity, ease of application, and relatively high accuracy. In many problems where Simpson's rule is employed the given data are less accurate than the formula, and in such cases there is no need of considering the error due to the rule. Problems do arise, however, in which the values of the given function can be obtained to a higher degree of accuracy than Simpson's rule is capable of yielding with a convenient number of ordinates, and in such cases it is desirable to have a means of determining just how many figures in the computed result are correct.

The earliest expression for the error inherent in Simpson's rule was given by James Stirling[1] in 1730 for the special case of three ordinates. The expression which is most often met in the literature is

$$E=-\frac{h^4}{180}(b-a)f^{\text{iv}}(\xi)\,, \qquad a \leqq \xi \leqq b\,. \qquad (2)$$

A more useful formula, due to Chevilliet,[2] is the following:

$$E=-\frac{h^4}{180}[f'''(b)-f'''(a)]\,, \qquad (3)$$

from which (2) follows by the law of the mean. Formula (3) is not exact, but it gives the principal part of the error.

It is obvious that (2) and (3) apply only when the analytic form of the function to be integrated is known and that (2) can give only the limits between

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[1] *Methodus Differentialis* (1730), p. 146.

[2] *Comptes Rendus* 78(1874), p. 1841.

which the actual error lies. The object of the present paper is to develop definite formulas which will give the principal part of the error inherent in Simpson's rule in all cases, whether the analytic form of the function is known or unknown. The first fundamental formula, (4), to be derived gives an exact expression for the error, but applies only to functions of known form; the second, (13), gives only the principal part of the error, but is applicable in any problem where Simpson's rule may be used. Neither of these formulas, so far as the writer is aware, has hitherto been published. Formula (20) is also believed to be new.

2. Case 1. Analytic form of the function known.

THEOREM I. *When the analytic form of the function is known, the error inherent in Simpson's rule is given by the formula*

$$E=-\frac{h^5}{90}\Big\{f^{\mathrm{iv}}(x_1)+f^{\mathrm{iv}}(x_3)+\cdots+f^{\mathrm{iv}}(x_{n-1})$$

$$+\frac{h^2}{21}[f^{\mathrm{vi}}(x_1)+f^{\mathrm{vi}}(x_3)+\cdots+f^{\mathrm{vi}}(x_{n-1})]$$

$$+\frac{h^4}{1008}[f^{\mathrm{viii}}(x_1)+f^{\mathrm{viii}}(x_3)+\cdots+f^{\mathrm{viii}}(x_{n-1})]+\cdots\cdots\Big\}. \tag{4}$$

PROOF. Let $y=f(x)$ be continuous in the interval $a\leqq x\leqq b$ and let it have continuous derivatives of all orders within this interval. Also let

$$\int_a^x f(x)dx=F(x)+C\ .$$

Then for the interval $k-h\leqq x\leqq k+h$, where $a+h\leqq k\leqq b-h$, we have

$$\int_{k-h}^{k+h} f(x)dx=F(k+h)-F(k-h)\ . \tag{5}$$

Expanding the terms on the right by Taylor's theorem and remembering that

$$F'(x)=f(x)\ ,\quad F''(x)=f'(x)\ ,\text{ etc., we have}$$

$$F(k+h)=F(k)+hf(k)+\frac{h^2}{2!}f'(k)+\frac{h^3}{3!}f''(k)+\cdots$$

$$F(k-h)=F(k)-hf(k)+\frac{h^2}{2!}f'(k)-\frac{h^3}{3!}f''(k)+\cdots.$$

Hence

$$I=\int_{k-h}^{k+h} f(x)dx = 2\left(hf(k)+\frac{h^3}{3!}f''(k)+\frac{h^5}{5!}f^{\mathrm{iv}}(k)+\cdots\right). \tag{6}$$

The value of this integral by Simpson's rule is

$$S=\int_{k-h}^{k+h} f(x)dx = \frac{h}{3}[f(k-h)+4f(k)+f(k+h)]. \tag{7}$$

Expanding $f(k-h)$ and $f(k+h)$ by Taylor's theorem, we can write (7) in the equivalent form

$$S=\frac{h}{3}\left[6f(k)+h^2f''(k)+\frac{2h^4}{4!}f^{\mathrm{iv}}(k)+\frac{2h^6}{6!}f^{\mathrm{vi}}(k)+\cdots\right]. \tag{8}$$

Hence the *error* due to Simpson's rule is

$$E=I-S=-\frac{h^5}{90}\left[f^{\mathrm{iv}}(k)+\frac{h^2}{21}f^{\mathrm{vi}}(k)+\frac{h^4}{1008}f^{\mathrm{viii}}(k)+\cdots\right]. \tag{9}$$

This is the error for any interval from $x=k-h$ to $x=k+h$. Let the whole interval (a, b) be divided into an even number n of subintervals, each of length h, by the points $x_0, x_1, x_2, \cdots, x_n$, where $x_0=a$, $x_1=a+h, \cdots, x_n=b$; then on putting $k=x_1, x_3, \cdots, x_{n-1}$ successively in (9) and adding the results, we get

$$\begin{aligned} E=-\frac{h^5}{90}\Big\{ & f^{\mathrm{iv}}(x_1)+f^{\mathrm{iv}}(x_3)+\cdots+f^{\mathrm{iv}}(x_{n-1}) \\ & +\frac{h^2}{21}\left[f^{\mathrm{vi}}(x_1)+f^{\mathrm{vi}}(x_3)+\cdots+f^{\mathrm{vi}}(x_{n-1})\right] \\ & +\frac{h^4}{1008}\left[f^{\mathrm{viii}}(x_1)+f^{\mathrm{viii}}(x_3)+\cdots+f^{\mathrm{viii}}(x_{n-1})\right]+\cdots\Big\}. \end{aligned} \tag{10}$$

as the error for the whole interval.

3. **Case 2. Analytic form of the function unknown.** If the form of the function $f(x)$ is unknown and we are given only a graph or a set of tabular values of the function and argument, we can still find an expression for the principal part of the error in Simpson's rule by assuming that the given function is continuous and has continuous derivatives throughout the interval (a, b). To do this we replace the derivatives in (9) and (10) by their values in terms of *differences*. This procedure really amounts to replacing the given function by a polynomial of the nth degree through the $n+1$ points (x_0, y_0),

$(x_1, y_1), \cdots, (x_n, y_n)$. The most convenient form of polynomial for our purpose is given by Stirling's formula of interpolation. If we change the independent variable from x to u by means of the substitution

$$x = k + hu, \quad \text{or} \quad u = (x-k)/h, \tag{11}$$

we can write Stirling's formula in the form

$$\begin{aligned} y = f(x) = f(k+hu) = y_k + u\frac{\Delta y_k + \Delta y_{k-h}}{2} + \frac{u^2}{2}\Delta^2 y_{k-h} \\ + \frac{u(u^2-1^2)}{3!}\,\frac{\Delta^3 y_{k-h} + \Delta^3 y_{k-2h}}{2} + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{k-2h} \\ + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\,\frac{\Delta^5 y_{k-2h} + \Delta^5 y_{k-3h}}{2} \\ + \frac{u^2(u^2-1^2)(u^2-2^2)}{6!}\Delta^6 y_{k-3h} + \cdots, \end{aligned} \tag{12}$$

where the differences are given by the following scheme:

x	u	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
$k-3h$	-3	y_{k-3h}						
			Δy_{k-3h}					
$k-2h$	-2	y_{k-2h}		$\Delta^2 y_{k-3h}$				
			Δy_{k-2h}		$\Delta^3 y_{k-3h}$			
$k-h$	-1	y_{k-h}		$\Delta^2 y_{k-2h}$		$\Delta^4 y_{k-3h}$		
			Δy_{k-h}		$\Delta^3 y_{k-2h}$		$\Delta^5 y_{k-3h}$	
k	0	y_k		$\Delta^2 y_{k-h}$		$\Delta^4 y_{k-2h}$		$\Delta^6 y_{k-3h}$
			Δy_k		$\Delta^3 y_{k-h}$		$\Delta^5 y_{k-2h}$	
$k+h$	1	y_{k+h}		$\Delta^2 y_k$		$\Delta^4 y_{k-h}$		
			Δy_{k+h}		$\Delta^3 y_k$			
$k+2h$	2	y_{k+2h}		$\Delta^2 y_{k+h}$				
			Δy_{k+2h}					
$k+3h$	3	y_{k+3h}						

THEOREM II. *The principal part of the error inherent in Simpson's rule is given by the formula*

$$E=-\frac{h}{90}[y_{-1}+y_{n+1}-4(y_0+y_n)+7(y_1+y_{n-1})$$
$$-8(y_2+y_4+\cdot\cdot\cdot+y_{n-2})+8(y_3+y_5\cdot\cdot\cdot+y_{n-3})] \tag{13}$$

when $n\geqq 6$, and by the formulas

$$E_2=-\frac{h}{90}[y_{-1}+y_3-4(y_0+y_2)+6y_1], \tag{13a}$$

$$E_4=-\frac{h}{90}[y_{-1}+y_5-4(y_0+y_4)+7(y_1+y_3)-8y_2] \tag{13b}$$

when $n=2$ and $n=4$, respectively.

PROOF. Differentiating (12) with respect to x by means of (11) and then putting $u=0$ in each derivative, we get the following results:

$$f^{\text{iv}}(k)=\frac{1}{h^4}\left(\Delta^4 y_{k-2h}-\frac{1}{6}\Delta^6 y_{k-3h}+\frac{7}{240}\Delta^8 y_{k-4h}\cdot\cdot\cdot\right),$$

$$f^{\text{vi}}(k)=\frac{1}{h^6}\left(\Delta^6 y_{k-3h}-\frac{1}{4}\Delta^8 y_{k-4h}\cdot\cdot\cdot\right),$$

$$f^{\text{viii}}(k)=\frac{1}{h^8}\left(\Delta^8 y_{k-4h}+\cdot\cdot\cdot\right).$$

Substituting in (9) these values for the derivatives and then dropping (for reasons to be given later) all terms containing sixth, eighth, and higher differences, we get

$$E=-\frac{h}{90}\cdot\Delta^4 y_{k-2h}. \tag{14}$$

Replacing $\Delta^4 y_{k-2h}$ by its value in terms of the y's, we have

$$E=-\frac{h}{90}(y_{k-2h}-4y_{k-h}+6y_k-4y_{k+h}+y_{k+2h}) \tag{15}$$

as the error for any interval of width $2h$ and mid-point $x=k$.

Putting $k=x_1$, we have[1]

$$E_2=-\frac{h}{90}(y_{-1}-4y_0+6y_1-4y_2+y_3)$$

$$=-\frac{h}{90}[y_{-1}+y_3-4(y_0+y_2)+6y_1] \,. \tag{16}$$

Likewise, on putting $k=x_3, x_5, \cdots, x_{n-1}$ in succession and adding the results, we get

$$E_4=-\frac{h}{90}[y_{-1}+y_5-4(y_0+y_4)+7(y_1+y_3)-8y_2] \ , \tag{17}$$

$$E_6=-\frac{h}{90}[y_{-1}+y_7-4(y_0+y_6)+7(y_1+y_5)-8(y_2+y_4)+8y_3] \ , \tag{18}$$

$$E_n=-\frac{h}{90}[y_{-1}+y_{n+1}-4(y_0+y_n)+7(y_1+y_{n-1})$$
$$-8(y_2+y_4+\cdots+y_{n-2})+8(y_3+y_5+\cdots+y_{n-3})] \ , n\geqq 6 \,. \tag{19}$$

REMARKS. 1. The practical advantages of formulas (13), (13*a*), (13*b*) are that they are definite in magnitude and sign, they apply to any function to which Simpson's rule can be applied, and they involve only the quantities used in the rule itself—with the exception of the two extreme ordinates y_{-1} and y_{n+1}. The only disadvantage is that they involve the two extra-interval ordinates just mentioned, but this disadvantage is not serious.

2. It is worth noting that these formulas (13), (13*a*), (13*b*) can be obtained by integrating Stirling's interpolation formula (12). Remembering that $dx=hdu$ and integrating (12) between the limits $x=k-h$ to $x=k+h$ or $u=-1$ to $u=1$, we get

$$\int_{k-h}^{k+h} f(x)dx=h\int_{-1}^{1} f(k+hu)du=2h\left(y_k+\frac{\Delta^2 y_{k-h}}{6}-\frac{\Delta^4 y_{k-2h}}{180}+\frac{\Delta^6 y_{k-3h}}{1512}-\cdots\right).$$

[1] This is equivalent to the expression which Stirling obtained in 1730 for three ordinates. Stirling wrote (for two subintervals each of width h)

$$\text{Area }=\frac{A+4B}{6}\times \text{ base}$$

$$\text{Correction term (error) }=\frac{P-4A+6B}{180}\times \text{ base,}$$

where, in the notation of the present article, $A=y_0+y_2$, $B=y_1$, $P=y_{-1}+y_3$, and stated that the correction term was to be added when negative and subtracted when positive. See his *Methodus Differentialis* (1730), p. 146.

The first two terms on the right give Simpson's one-third rule; the succeeding terms are correction terms. The first of the correction terms, $-(2h/180)\Delta^4 y_{k-2h}$, is the same as (14) above.

3. Since the quantities given by (3), (4), and (13) are definite in sign and numerical value, they may be applied as corrections to the results found by Simpson's rule. The application of these corrections will usually give a final result as accurate as could be obtained with Simpson's rule alone with twice as many ordinates.

4. **Reasons for dropping sixth and higher differences.** Sixth and eighth differences were discarded in the derivation of (13) because of two facts of practical importance.

(a) When the values of the y's used in Simpson's rule are the results of measurements of any kind or of computations from a formula, they are liable to be affected with errors of measurement or of computation. The effect of such errors is cumulative in the process of taking differences, so that what might appear to be a sixth difference of considerable magnitude would really be an accumulation of errors. For example, if each of the y's were affected with an error of magnitude ϵ, the error in the sixth difference would be

$$\epsilon_{k-3h}-6\epsilon_{k-2h}+15\epsilon_{k-h}-20\epsilon_k+15\epsilon_{k+h}-6\epsilon_{k+2h}+\epsilon_{k+3h}\ ;$$

and since these errors (ϵ) might not all be of the same sign, their combined maximum effect might possibly be as great as $2^6\epsilon$ or 64ϵ. Obviously nothing would be gained by retaining the higher differences in problems of this sort.

(b) The second reason for dropping the higher differences is this: It will be observed that formulas (13), (13a), (13b) contain two ordinates, y_{-1} and y_{n+1}, outside the limits of integration. If sixth, eighth, and higher differences were retained, the resulting formulas for the error would contain four extra-interval ordinates (two at each end) in the case of sixth differences, six (three at each end) in the case of eighth differences, and so on. The necessity for bringing in these extra ordinates would decrease the usefulness of the error formulas.

5. **To find the value of h for a stipulated degree of accuracy.** If we should wish to know the value of h corresponding to a stipulated error in the final result, we could find it as follows:

(a) If the analytic form of the function is known and the third derivative is easily calculated, substitute in (3) the given E and the calculated values of $f'''(b)$ and $f'''(a)$; then solve for h.

(b) If the form of the function is not known, or if known but the third derivative is not easily found, assume a convenient value, h_1, for h, find the

corresponding E_1 by means of (13), (13a), or (13b) and then use the relation $\frac{|E_p|}{|E_1|}=\frac{h^4}{h_1^4}$, (obtained from (3)), from which $h=h_1\sqrt[4]{(|E_p|/|E_1|)}$, (20)

where E_p is the prescribed error in the final result.

6. Examples.

(*a*) $\int_1^2 \log x\,dx$ $=0.38629432$ by integration
$=0.38629338$ by Simpson's rule, taking $h = 0.1$.

Hence

Actual error $=+0.00000094$, Corrected result $= 0.38629435$,
Error by (3) $=+0.00000097$, Final error $=-0.00000003$.
Error by (13) $=+0.00000097$,

Value of h corresponding to this final error is $h=0.042$.

(*b*) $\int_{0.2}^{1.4} (\sin x-\log x-e^x)dx$ $= 4.05095$ by integration
$= 4.05107$ by Simpson's rule, for $h=0.1$.

Actual error $=-0.00012$, Corrected result $= 4.05092$,
Error by (3) $=-0.00014$, Final error $= 0.00003$.
Error by (13) $=-0.00015$,

DEFINITIONS AND POSTULATES FOR RELATIVITY[1]

By H. P. MANNING, Providence, R. I.

The object of this paper is to offer some definitions and postulates that may serve as a basis for relativity, at least in a space of one dimension. Some writers have given postulates and theorems with proofs[2] but I know of no complete formal system such as we have for several branches of mathematics, and, in particular, for geometry. Perhaps the nearest approach to such a treatment is to be found in the presentation of A. A. Robb based on what he calls "conical order." His undefined elements are "instants" and his undefined relation (he calls it "fundamental") is the relation "after," but his twenty-one postulates and many technical terms look somewhat forbidding. We propose to use familiar terms and in a way that is consistent with their ordinary

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[1] Presented to the American Mathematical Society, October 31, 1925.

[2] For example, E. V. Huntington in the paper referred to in this MONTHLY (*1925*, 187, footnote). (This paper was reprinted in the *Philosophical Magazine* for April, 1912, pages 494-513); R. D. Carmichael, *The Theory of Relativity*, New York, 1920; A. A. Robb, *Theory of Time and Space*, 1914, *The Absolute Relations of Time and Space*, 1921, both published at Cambridge, England. See also article by C. Carathéodory, "Zur Axiomatik der speziellen Relativitätstheorie," *Sitzungsber. der Preuss. Akad., Phys.-Math. Klasse*, 1925, I-VII, pp. 12-27.

Birkhoff in his *Relativity and Modern Physics* (reviewed in this MONTHLY, 1925, pp. 185-201) lays considerable stress on postulational treatment of time and space, but he seems to be using the term in a physical sense.

meanings, although it will require great care to hold ourselves strictly to our definitions.

In the explanation usually given of the determination by light signals of the time and position of an event in a space of one dimension or line, an observer at a point A sends a flash of light as a signal to a second point B at time t_1 and receives the reflected flash at time t_2. The reflection at B is an event whose time according to the observer is $t=\frac{1}{2}(t_2+t_1)$, and whose position is denoted by the abscissa $x=\frac{1}{2}c(t_2-t_1)$, c a constant, which we can make equal to 1 if we wish.

Instead of a signal reflected at B we can think of the flash as a light particle meeting at B a second light particle that is coming towards A, and then we can imagine a continuous series of light particles passing along our line in the positive direction, each denoted by the time t_1 when it is at A, and a second continuous series passing along our line in the negative direction, each denoted by the time t_2 when it is at A. Any event will be at the meeting of two light particles of the two series, and we can think of the meeting itself as the event. These considerations have led to the conceptions and definitions that follow.

We will make use of ordered series that would be called by Huntington[1] *linear continuous series* of the type that has no first element and no last element. An example is the series of all real numbers, and it will be convenient to speak of any such series as a series of the type of the series of all real numbers. We will assume that we have two such series with no element in common. We will call the elements of one series *positive light particles* and the elements of the other *negative light particles*. Each of these terms is to be regarded as a single

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[1] E. V. Huntington, *The Continuum and other types of Serial Order*, second edition, Cambridge, Mass., 1917.

According to Huntington, a linear continuous series is a class K with a relation *precedes*. The relation is denoted by the sign $<$ and the class satisfies the following postulates:

Postulate 0. The class K is not an empty class nor a class containing only one element.

Postulate 1. If a and b are distinct elements of K, then $a<b$ or $b<a$.

Postulate 2. If $a<b$, then a and b are distinct.

Postulate 3. If $a<b$ and $b<c$, then $a<c$.

Postulate C1. If K_1 and K_2 are two non-empty parts of K such that every element of K belongs either to K_1 or K_2 and every element of K_1 precedes every element of K_2, then there is at least one element X in K such that

(1) Any element that precedes X belongs to K_1, and

(2) Any element that follows X belongs to K_2.

("b follows a" is the same as "a precedes b.")

Postulate C3. The class K contains a denumerable sub-class R such that between any two elements of K there is an element of R.

In the series of all real numbers the rational numbers form such a sub-class.

These postulates are taken from §§ 12 and 54 of *The Continuum.*

To determine the particular type of series that is like the series of all real numbers, we add a *Final postulate.* The class K contains no first element and no last element.

expression, with no reference to the ordinary meanings of the three words that make it up, and such expressions as "light particles" and "series of light particles" only as abbreviations for the full terms.

If a and b are two elements of a series of any kind, then a precedes b in one direction and b precedes a in the other direction in the series. In each of our two series we will choose one of the two directions and use the word "precedes" in only one sense. The series of all real numbers will be taken in the ascending order, and any series of this type that may arise will be defined with one particular order as the order which alone is to be considered.

Two light particles, one from each series, constitute an *event.*

We can establish a one-to-one correspondence in an infinite number of ways between the elements taken in order of two linear continuous series of the type of the series of all real numbers. This theorem is proved by Huntington.[1] If we establish such a correspondence between the positive light particles and the negative light particles we shall be selecting an infinite number of events in a definite order that will themselves form a series of the same type. We will call such a series a *particle.* It is to be understood that there is no connection between the word particle so used and as used in the expression "light particle."

Let A be a given particle and in some particular way assign to the events of A in order in a one-to-one correspondence the series of all real numbers. The number assigned to any event of A we will call *A's local time for that event,* but the word "local" may often be omitted when we do not need to emphasize the fact that it is time at an event of A. Then assign to each light particle the number that indicates A's local time for the event of A to which it belongs. Let t_1 denote this number for a positive light particle and t_2 for a negative

[1] Let R and R' be denumerable sub-classes such as are required by Postulate $C3$. Let their elements when arranged as sequences be denoted by

$$a_1\ a_2\ a_3\ —\ —\ —\ —\ —$$

and

$$b_1\ b_2\ b_3\ —\ —\ —\ —\ —.$$

First take a_1 and b_1 as corresponding elements. These divide the rest of the elements of R and R' each into two sub-classes, the two sub-classes in R corresponding in order to the two sub-classes in R'. Find to which sub-class of R a_2 belongs and let it correspond to the first b that belongs to the corresponding sub-class of R'. The elements of R and R' that have not been used now are divided into three sub-classes, those in R corresponding in order to those in R'. Next take the first b that has not been used, note to which of the three sub-classes of R' it belongs, and let it correspond to the first a that belongs to the corresponding sub-class of R. If we continue in this way, taking a's and b's alternately, any given element of R or R' will be taken at some stage and every step of the process will be possible.

Any elements of the two classes that do not belong to R or R' will be defined by cuts in R and R', and the correspondence of R and R' establishes a correspondence of these cuts, and a complete correspondence of the two given series with corresponding elements in the same order.

There will be an infinite number of sub-classes R and R', and the elements of any such sub-class can be arranged as a sequence in an infinite number of ways.

This proof is taken with slight modifications from §§ 61, 59 and 45 of *The Continuum.*

light particle. The light particles of the two series will be represented by the values of t_1 and t_2, respectively, and any event by the numbers t_1 and t_2 of the two light particles that constitute it.

If we make the series of all real numbers correspond to itself in a one-to-one correspondence of its elements taken in order, we shall be establishing a functional relation in which each variable is an increasing continuous function of the other.[1] In any functional relation of this kind corresponding increments of the two variables always have the same sign and neither is ever zero. When the equation can be differentiated the two inverse derivatives may take the values zero and infinity for a particular pair of values of the variables,[2] but they are never negative. When we define a particle by establishing a one-to-one correspondence of the positive and negative light particles, this correspondence carries with it a correspondence of the numbers t_1 and t_2 assigned to them, that is, a correspondence of the series of all real numbers with itself. Therefore the particle is represented by an equation (or functional relation) satisfied by the values of t_1 and t_2 for any event of the particle. By this equation t_1 and t_2 are made increasing continuous functions, each of the other, corresponding increments always having the same sign, and neither ever zero, and when the derivatives exist and are not zero nor infinite the differentials dt_1 and dt_2 have the same sign. If we plot events, using t_1 and t_2 as coördinates, the graph of a particle will be a line or curve with everywhere a positive slope, or at least with all chords sloping positively.

For an event, t_1, t_2, we define the *time according to A* and the *position according to A* by means of the equations

$$t=\tfrac{1}{2}(t_2+t_1)\ ,\quad x=\tfrac{1}{2}c(t_2-t_1)\ , \tag{1}$$

c a constant. Thus an event is represented also by its time and position and a particle by an equation in t and x.

For any two events of a particle the increments of time and position are given by the equations

$$\Delta t=\tfrac{1}{2}(\Delta t_2+\Delta t_1)\ ,\qquad \Delta x=\tfrac{1}{2}c(\Delta t_2-\Delta t_1)\ ,$$

and since Δt_1 and Δt_2 always have the same sign, Δx is always numerically less than $c\Delta t$.

When the equation of a particle can be differentiated

$$dt=\tfrac{1}{2}(dt_2+dt_1)\ ,\qquad dx=\tfrac{1}{2}c(dt_2-dt_1)\ ,$$

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[1] The equation $au-bv=c$, where a and b are positive numbers, is an example of such a relation but the statement is intended to include all possible relations of the given type.

[2] As, for example, for $u=v=0$ when the relation is $v=u^3$, u and v real.

and if the derivatives of t_1 and t_2 with respect to each other are never zero nor infinite we shall always have dx numerically less than cdt.

The solution for t_1 and t_2 of the equations that define t and x gives us

$$t_1 = t - (x/c) \ , \quad t_2 = t + (x/c) \ , \tag{2}$$

and a similar solution can be written down for the increments, and for the differentials when they exist.

When the differentials exist we can define the *velocity according to A* of a particle at any one of its events as equal to dx/dt, and so also as equal to

$$c\,\frac{dt_2 - dt_1}{dt_2 + dt_1}\ , \tag{3}$$

and when the derivatives of t_1 and t_2 with respect to each other are never zero nor infinite the velocity will always be numerically less than c.

If one of these derivatives becomes zero for a particular event the velocity at that event becomes numerically equal to c. If at some event these derivatives do not exist, we might consider the ratio of increments $\Delta x/\Delta t$. But to avoid too many details we will omit these cases and consider only particles that have at each of their events a velocity numerically less than c.

Let u denote the velocity. Then

$$\frac{dt_1}{1-(u/c)} = \frac{dt_2}{1+(u/c)} = dt \ . \tag{4}$$

Now let us establish a second system of reckoning time and position. Taking a second particle B, we assign to the events of B in order in some way the series of all real numbers, each number as B's local time for the event to which it is assigned; and then we assign this number to each of the two light particles that constitute the event. Using[1] t_1' and t_2' as before we used t_1 and t_2, we base upon these numbers a system of reckoning the time and position of any event, say t' and x', defined in the same way as t and x.

The assignment of t_1 and t_1' to the positive light particles establishes a correspondence of the series of all real numbers to itself. Thus t_1 and t_1' are

[1] Instead of starting with a particle A we might build up our first system by assigning the series of all real numbers directly to the positive light particles and also to the negative light particles, in each case in a one-to-one correspondence of the elements taken in order, letting t_1 and t_2, respectively, denote the numbers so assigned. Then, if we wish, we could consider the particle A for which $t_1 = t_2$, or we could let A denote any particle for which $dt_1 = dt_2$, so that t_1 and t_2 would differ only by a constant. These particles would be described as stationary, and we might think of their aggregate as constituting a material body that is stationary (according to this system). For the second system we should assign the series of real numbers in some other way to the two series of light particles, calling them now t_1' and t_2'. Time, position and velocity would be defined as before, but defined as according to one or the other of the two systems rather than as according to A or B.

functions of each other, increasing continuous functions. Likewise t_2 and t_2' are increasing continuous functions of each other. Here again, to avoid too many details, we will consider only cases in which the derivatives of all of these functions exist and do not become zero nor infinite, and so the differentials of t_1 and t_1' have the same sign, and the differentials of t_2 and t_2' have the same sign.

In general, our differentials are taken "for some particle," or "at some particle," but the ratio of dt_1' to dt_1 will be the same for all particles at a given event and for all events that contain the same positive light particle. Likewise the ratio of dt_2' to dt_2 will be the same for all events that contain the same negative light particle.

From the equations connecting these differentials we can obtain the equations that connect dt' and dx' with dt and dx for any particle P, and then the equation for the velocities of P according to the two systems and the equation for the differentials of t and t' at P.

Take, in particular, the particle B itself. For this particle $dt_1'=dt_2'=dt'$, and, if v is the velocity of B according to A, the differentials of the time at B according to the two systems will be equal, respectively, to $dt_1/(1-v/c)$ by (4) and dt_1'. Then if a is the time-rate of B according to A, that is, the ratio of the differentials of the two kinds of time at B, we shall have

$$dt_1'=a\,dt_1/(1-v/c)\ . \tag{5}$$

The values of v and a are determined for each event of B and each event of B is determined by the value of t_1 for it. Thus in the above equation we may regard v and a as functions of t_1, and this equation will then be the differential of the relation already referred to between t_1 and t_1'.

The same calculations with dt_2 and dt_2' give the equation

$$dt_2'=a\,dt_2/(1+v/c)\ , \tag{6}$$

where v and a may be regarded as functions of t_2, so that this equation is the differential of the relation between t_2 and t_2'.

Equations (5) and (6) may be applied to any particle at any of its events provided that it is understood that in (5) v and a are the velocity and time-rate of B at the event of B for which t_1 has the same value as for the given event, and that in (6) v and a are the velocity and time-rate of B at the event of B for which t_2 has the same value as for the given event. Thus, in general, these quantities will not have the same values in the two equations and the equations will not be simultaneous in them.

They will be simultaneous:

(1) if v and a are constants, the particle B having a constant velocity and a constant time-rate according to A,

(2) at the events of the particle B itself.

We are particularly interested in applying these equations to the particle A and determining the mutual relation of the two systems of reckoning time and position. At A, $dt_1 = dt_2 = dt$. If B is so related to A that we can take v and a the same in the two equations when we apply them to A, we shall have

$$dt_2'/dt_1' = (1-v/c)/(1+v/c) \ , \tag{7}$$

and therefore, by (4), the velocity of A according to B will be equal to minus the velocity of B according to A.

As to the time-rate of A according to B, that is, the ratio of the differential of A's local time to the differential of B's time at A, the addition of (5) and (6) with division by 2 will give $dt' = adt/(1-v^2/c^2)$, or

$$dt = (1/ak^2)dt' \ , \tag{8}$$

where $k = 1/\sqrt{(1-v^2/c^2)}$.

The relation of the two systems is characterized by the nature of the time-rate a. Two cases are usually considered:

(1) $a=1$, B's local time the same (except perhaps for a constant) as A's time at B. If every particle is made to have a local time the same as the time at it of a given system, the space-time is said to be *aeolotropic.*

(2) $a=1/k$. In this case A and B have each a time-rate according to the other that is equal to $1/k$ or $\sqrt{(1-v^2/c^2)}$. Even when v is not the same in (5) and (6) we may take $a=1/k$ in these equations. They will then become

$$dt_1' = dt_1\sqrt{(1+v/c)}/\sqrt{(1-v/c)} \ , \quad dt_2' = dt_2\sqrt{(1-v/c)}/\sqrt{(1+v/c)} \ . \tag{9}$$

This may be called the case of relativity, but when v is not the same in the two equations the relations of A and B will not be symmetrical.[1]

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[1] Suppose, for example, that B leaves A, goes a certain distance and immediately returns, going all the time with a constant velocity v. If his time-rate is $1/k$ he will find that his time has been going more slowly than A's time. But according to B it is A that has gone a certain distance in the other direction and returned, and during all of his motion his velocity is numerically equal to v and his time is going more slowly than B's time with him. But A, instead of returning immediately, appears to B to be stationary for a certain middle period, and during this time his local time goes more rapidly than B's time with him, enough more rapidly to make up for all the time lost during the other periods, and to account for the amount that B's time does actually lose as compared with A's time. This can be shown in various ways. With our equations it results from the fact that in the middle period the v in one equation is the negative of the v in the other.

If we had set up our two systems without reference to any particles A or B in the way explained the footnote on page 87 and then considered the two sets of particles stationary with respect to the two systems, we should have found in the example above that the relations of the two systems are symmetrical.

DETERMINATION OF THE REDUCIBLE CASES OF THE FIXED CENTRODE OF THREE-BAR MOTION

By EMMA WHITON McDONALD, Berkeley, California

The problem of the fixed centrode of three-bar motion has been studied by Roberts,[1] Cayley,[2] Darboux,[3] Clifford[4] and others. In particular Clifford has pointed out that the centrode may reduce to an ellipse and has proposed the problem of finding all cases in which the curve which is in general of order eight, reduces. This problem is solved in the following way.

Suppose in a plane a quadrilateral $ABCD$ with sides of definite length is so formed that AD is fixed in the plane and AB and CD are allowed to rotate about A and D. Then B and C will be hinged points. The problem under consideration is a study of the locus of intersection of AB and CD as the figure is rotated about A and D as pivots. This is the locus of the instantaneous center in the fixed plane, or the fixed centrode of the three-bar motion.

For each position of AB there are four points in which it may meet CD, and similarly for each position of CD there are four points in which it meets AB. Then since the correspondence is four to four any line meets the curve in eight points. Hence it is a curve of the eighth order. But since the line AB can meet the curve only in four points and the point A, A, and similarly B, must be a quadruple point. Since each multiple point of order k is equivalent to $\frac{1}{2}k(k-1)$ double points, A and B are equivalent to twelve double points. To each of the four branches of the curve at A and B will correspond two coincident tangents and, at least in the case of continuous rotation of both AB and CD, no more. Therefore the class of the curve appears to be eight. Then $m=8$, $n=8$, $d=12$. Substituting these values in Plücker's formulae $T=12$, $K=8$ and $I=8$. Hence the deficiency of the curve is one. These conclusions agree with the work of Darboux in which he shows that the curve is elliptic.

In a rectangular system of coordinates let A be at the origin and $AD=d$ lie along the positive x-axis. Then if $AB=a$, $BC=b$, $CD=c$, the coördinates of B and C are $(a \cos \theta_1, a \sin \theta_1)$ and $(c \cos \theta_2+d, c \sin \theta_2)$ respectively. Then

$$b^2=(c\cos\theta_2+d-a\cos\theta_1)^2+(c\sin\theta_2-a\sin\theta_1)^2 \tag{1}$$

or

$$a^2-b^2+c^2+d^2+2cd\cos\theta_2-2ac\cos\theta_1\cos\theta_2-2ad\cos\theta_1-2ac\sin\theta_1\sin\theta_2=0\ . \tag{2}$$

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[1] *Proceedings of the London Math. Society*, vol. 7, p. 14.

[2] *Proceedings of the London Math. Society*, vol. 7, p. 136.

[3] *Bulletin des Sciences Mathématiques*, 1879.

[4] *Elements of Dynamic*, Clifford.

Placing

$$\sin\theta_1 = \frac{2t}{1+t^2}, \qquad \sin\theta_2 = \frac{2T}{1+T^2}, \tag{A}$$

equation (2) becomes

$$\begin{aligned} T^2\{(1+t^2)(a^2-b^2+c^2+d^2-2cd)-(1-t^2)(2ad-2ac)\}-8actT \\ +(1+t^2)(a^2-b^2+c^2+d^2+2cd)-(1-t^2)(2ad+2ac)=0 \, . \end{aligned} \tag{3}$$

Solving this equation for T

$$T = \frac{4act \pm \sqrt{\begin{aligned}16a^2c^2t^2 - \{(1+t^2)^2[(a^2-b^2+c^2+d^2)^2-4c^2d^2] \\ -4ad(1-t^4)(a^2-b^2-c^2+d^2)-4a^2(1-t^2)^2(c^2-d^2)\}\end{aligned}}}{(1+t^2)(a^2-b^2+c^2+d^2-2cd)+2a(1-t^2)(c-d)} \, . \tag{4}$$

The expression under the radical is a quadratic in t^2 in the form

$$\begin{aligned} &-t^4\{(a^2-b^2+c^2+d^2)^2-4c^2d^2+4ad(a^2-b^2-c^2+d^2)-4a^2(c^2-d^2)\} \\ &+t^2\{16a^2c^2-[2(a^2-b^2+c^2+d^2)^2-4c^2d^2+8a^2(c^2-d^2)]\} \\ &-\{(a^2-b^2+c^2+d^2)^2-4c^2d^2-4ad(a^2-b^2-c^2+d^2)-4a^2(c^2-d^2)\} \, . \end{aligned} \tag{5}$$

If

$$\begin{aligned} p_1 &= a+b+c+d \, , & q_1 &= -a+b+c+d \, , \\ p_2 &= a+b-c-d \, , & q_2 &= a-b+c+d \, , \\ p_3 &= a-b-c+d \, , & q_3 &= a+b-c+d \, , \\ p_4 &= a-b+c-d \, , & q_4 &= a+b+c-d \, , \end{aligned} \tag{6}$$

(5) becomes

$$-t^4(p_1p_3q_2q_3)-t^2(p_1p_2p_3p_4-q_1q_2q_3q_4)+p_2p_4q_1q_4 \tag{7}$$

which factors into

$$(-p_1p_3t^2+q_1q_4)(q_2q_3t^2+p_2p_4) \, .$$

Then equation (3) becomes

$$T^2(t^2p_3q_3+p_4q_4)-8actT+t^2p_1q_2-p_2q_1=0 \tag{8}$$

and

$$T = \frac{4\,act \pm \sqrt{(-p_1p_3t^2+q_1q_4)(q_2q_3t^2+p_2p_4)}}{p_4q_4+t^2p_3q_3} \, . \tag{9}$$

Considering the coördinates of the point P,

$$y = \tan\theta_1 \cdot x = \tan\theta_2 \cdot x - \tan\theta_2 \cdot d \, ,$$

Then

$$x=\frac{d\tan\theta_2}{\tan\theta_2-\tan\theta_1}, \qquad y=\frac{d\tan\theta_1\tan\theta_2}{\tan\theta_2-\tan\theta_1}.$$

Making the substitution of (A) and expressing the equation of the curve in homogeneous coördinates,

$$x=dT(1-t^2)\ , \quad y=2dtT, \quad z=T(1-t^2)-t(1-T^2)\ . \tag{B}$$

To determine the order of the curve represented by (B), substitute the values for x, y and z in the equation of a straight line, $Ax+By+Cz=0$. Then

$$T^2(Ct)+T[Ad(1-t^2)+2Bdt+C(1-t^2)]-Ct=0\ . \tag{10}$$

The intersections of the line with the curve are found by solving equations (10) and (8) simultaneously. The eliminant of these equations is

$$\begin{aligned}&p_1p_3q_2q_3(Ad+C)^2t^8-4Bdp_1p_3q_2q_3(Ad+C)t^7+\{p_1p_3q_2q_3[4B^2d^2-2(Ad+C)^2]\\&+(Ad+C)^2(p_1p_4q_2q_4-p_2p_3q_1q_3)+8acC(Ad+C)(p_3q_3-p_1q_2)\\&+C^2(p_1q_2+p_3q_3)^2\}t^6+\{4Bd(Ad+C)(p_1p_3q_2q_3-p_1p_4q_2q_4+p_2p_3q_1q_3)\\&+16acdBC(p_1q_2-p_3q_3)\}t^5+\{(p_1p_4q_2q_4-p_2p_3q_1q_3)[2C^2+4B^2d^2\\&-2(Ad+C)^2]+8acC(Ad+C)(p_1q_2+p_2q_1-p_3q_3+p_4q_4)+2C^2(p_3p_4q_3q_4\\&-p_1p_2q_1q_2)+(Ad+C)^2(p_1p_3q_2q_3-p_2p_4q_1q_4)-64a^2c^2C^2\}t^4\\&+\{4Bd(Ad+C)(p_1p_4q_2q_4-p_2p_3q_1q_3+p_2p_4q_1q_4)-16acdCB(p_2q_1+p_4q_4)\}t^3\\&+\{(Ad+C)^2(p_1p_4q_2q_4-p_2p_3q_1q_3)-[4B^2d^2-2(Ad+C)^2]p_2p_4q_1q_4\\&+C^2(p_2q_1-p_4q_4)^2-8acC(Ad+C)(p_2q_1+p_4q_4)\}t^2\\&-4Bdp_2p_4q_1q_4(Ad+C)t-p_2p_4q_1q_4(Ad+C)^2=0\end{aligned} \tag{11}$$

Since this equation has at most eight solutions the straight line meets the curve in not more than eight points. Hence in general the curve is of order eight.

The problem for solution is to determine all of the cases in which the deficiency or order of the curve reduces. In order that the radical reduce from the fourth to the second degree, and consequently that the deficiency of the curve become zero instead of one, it is evident from equation (9) that one or more of the p's or q's must equal zero. Considered geometrically the q's and p_1 cannot equal zero. Hence the three cases of interest are those in which p_2, p_3, and p_4 may equal zero. If p_2 or p_4 equals zero the last two terms of equation (11) vanish, and when $p_3=0$ the first two vanish. Hence in each case the curve reduces from order eight to order six. When two of the p's are simul-

taneously equal to zero, the curve further reduces. Place $p_2=0$ and $p_3=0$. Then $a=c$ and $b=d$, and equation (11) becomes

$$A(a+b)(Aa-Ab-2C)t^4-4B(a+b)(Aa-Ab-C)t^3$$
$$+2(2a^2B^2-a^2A^2+A^2b^2+2AbC-2B^2b^2)t^2$$
$$+4B(a-b)(Aa+Ab+C)t+A(a-b)(Aa+Ab+2C)=0\,.$$

From this eliminant it is evident that $z=0$, which indicates that the line at infinity is part of the curve. The residual curve is an ellipse with foci A and D to which reference has already been made. In case p_2 and p_4 equal zero or p_3 and p_4 equal zero, the two pairs of adjacent sides of the quadrilateral are equal. Then the centrode is a curve of order four. It may be of interest to form the equation of the curve under these conditions.

If $AB=BC=a$, $CD=DA=b$ and CD, DA are in any position, let AB, BC, AB', $B'C$ be the two possible positions of the other sides. If BB' and AC meet in E put $AE=EC=p$, $BE=EB'=q$, $B'D=t$. Extend AB' to meet CD in F and let $B'F=r$ and $FD=u$; then

$$b^2=(t+q)^2+p^2\,,\qquad a^2=p^2+q^2\,,\qquad b^2-a^2=t(t+2q)=g^2\,,$$

$$r=\frac{at^2}{g^2}\,,\qquad u=\frac{bt^2}{g^2}\,,\quad\text{and}\quad \sqrt{x^2+y^2}=a+\frac{at^2}{g^2}\,,\qquad \sqrt{(b-x)^2+y^2}=\frac{bt^2}{g^2}\,,$$

or

$$b\sqrt{x^2+y^2}-a\sqrt{(b-x)^2+y^2}=ab\,.$$

Then the centrode is the Cartesian

$$(a^2-b^2)(x^2+y^2)\left[(x^2+y^2)(a^2-b^2)-4a^2bx\right]+4a^2b^2\left[(a^2-b^2)x^2-b^2y^2\right]=0\,.$$

QUESTIONS AND DISCUSSIONS

EDITED BY TOMLINSON FORT, Hunter College, Park Ave. and 68th St., New York, N.Y. and H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. A CRITERION THAT A CUBIC EQUATION HAS AN INTEGRAL ROOT

By H. S. VANDIVER, University of Texas.

If the quadratic equation $x^2+bx+c=0$, where a and b are rational integers, has a rational integer as a root, then it is obviously necessary and sufficient that the discriminant b^2-4c is a perfect square. The object of the present note is to derive an analogon of this result for a cubic equation. The analogy is more apparent if the result is stated in terms of the theory of algebraic numbers as here given. However, as I intend to make the discussion as elementary in character as possible, I have avoided the use of anything belonging to said theory except the definitions of quadratic numbers and quadratic integers.

If x, p, and q are rational integers and

$$x^3+px+q=0\ , \tag{1}$$

assume that $x=y+z$ and $yz=-p/3$. Then from well known relations

$$y^2+yz+z^2=x^2+p/3\ , \qquad (y-z)(y^2+yz+z^2)=2\sqrt{R}$$

where $R=q^2/4+p^3/27$, whence $y-z=2\sqrt{R}/x^2+p/3$, and, using $y+z=x$, we have $y=x/2+3\sqrt{R}/3x^2+p$. From this, since x is an integer it follows that

$$\frac{m+n\sqrt{-3\Delta}}{k}=y\ , \tag{2}$$

where m, n, and k are integers, that is, y is a number belonging to the algebraic field, defined by $\sqrt{-3\Delta}$, (if -3Δ is a perfect square the field is the rational field, where Δ is the discriminant of (1)). Now also $y^3=(-q/2)+\sqrt{R}$, whence

$$27y^3=\frac{-27q+3\sqrt{-3\Delta}}{2}\ . \tag{3}$$

We shall now show that $3y$ is an *integer* in the field defined by $\sqrt{-3\Delta}$. The value of $(3y)$ in (2) satisfies an equation of the form

$$\alpha w^2+\beta w+\gamma=0\,.$$

An equation having as roots the cubes of the roots of this equation is

$$u^2+\left(\frac{\beta^3}{\alpha^3}-\frac{3\beta\gamma}{\alpha^2}\right)u+\frac{\gamma^3}{\alpha^3}=0\ ;$$

by (3), $(3y)^3$ is an integer in the field $\sqrt{-3\Delta}$, so that the coefficients of the last equation are all rational integers. Let β/α be reduced to its lowest terms and suppose then $\beta/\alpha=\beta_1/\alpha_1$ where β_1 is prime to α_1. Now if γ^3/α^3 is an integer it follows that $\gamma=\alpha t$, t an integer. Hence from the other coefficient we have $\beta_1^3-3\beta_1\alpha_1^2t\equiv 0 \pmod{\alpha_1^3}$ and since β_1 is prime to α_1, $\beta_1^2\equiv 3\alpha_1^2 \pmod{\alpha_1^3}$ $\equiv 0 \pmod{\alpha_1^2}$, $\beta_1\equiv 0 \pmod{\alpha_1}$, whence $\alpha_1=1$, and β is divisible by α, which proves that $3y$ is an integer in the field. We may then state the result: *The necessary and sufficient condition that* (1) *has a rational integral root where p and q are rational integers, is that*

$$\frac{-27q+3\sqrt{-3\Delta}}{2}$$

is the cube of an integer in the algebraic field defined by $\sqrt{-3\Delta}$, where Δ is the discriminant of (1).

II. Al-Bîrûnî's Method of Approximation of Chord 40°.

By Dr. Carl Schoy, University of Frankfort a.M.

In his *Qânûm Mas'ûdî*, written about 1000 A. D., al-Bîrûnî[1] gives a method for finding the approximate value of chord 40°, or s_9 (the side of a regular nonagon inscribed in a circle of unit radius). He proceeds from the known approximations

$$s_{12}=\text{chord } 30^\circ=0^p\ 31'\ 3''\ 29'''\ 49^{\text{iv}}\ 36^{\text{v}}$$
$$s_{30}=\text{chord } 12^\circ=0^p\ 12'\ 32''\ 37'''\ 17^{\text{iv}}\ 46^{\text{v}},$$

where 1^p represents the radius. He also makes use of a known rule for finding chord $\frac{1}{2}\alpha$ and chord $\frac{1}{8}\alpha$ when chord α is known, and the rule for chord $(\alpha+\beta)$.

By these means he finds that

$$\text{chord }(30^\circ+12^\circ)=\text{chord } 42^\circ$$
$$=0^p43'0''14'''57^{\text{iv}}15^{\text{v}},$$

and that

$$\text{chord } \tfrac{1}{4}\cdot 42^\circ=\text{chord } 10^\circ30'$$
$$=0^p11'58''48'''41^{\text{iv}}56^{\text{v}}.$$

Then

$$\text{chord }(30^\circ+10^\circ30')=\text{chord } 40^\circ30'$$
$$=0^p41'32''2'''34^{\text{iv}}6^{\text{v}}.$$

Proceeding as before,

$$\tfrac{1}{4}\cdot 40^\circ30'=10^\circ7'30'',$$

[1] Mohamed ibn Ahmed, Abû'l Rîhân, al-Bîrûnî (or Bêrûnî).

and

$$\begin{aligned}\text{chord } \tfrac{1}{4}\cdot 40^\circ 30' &= \text{chord } 10^\circ 7' 30'' \\ &= 0^{p}10'35''20'''42^{\mathrm{iv}}53^{\mathrm{v}},\end{aligned}$$

whence

$$\begin{aligned}\text{chord } (30^\circ + 10^\circ 7'30'') &= \text{chord } 40^\circ 7'30'' \\ &= 0^{p}41'9''15'''26^{\mathrm{iv}}.\end{aligned}$$

Repeating the process,

$$\begin{aligned}\tfrac{1}{4}\cdot 40^\circ 7'30'' &= 10^\circ 1'52''30''', \\ \text{chord } \tfrac{1}{4}\cdot 40^\circ 7'30'' &= \text{chord } 10^\circ 1'52''30''' \\ &= 0^{p}10'29''58'''38^{\mathrm{iv}}26^{\mathrm{v}},\end{aligned}$$

whence

$$\begin{aligned}\text{chord } (30^\circ + 10^\circ 1'52''30''') &= \text{chord } 40^\circ 1'52''30'' \\ &= 0^{p}41' \text{ (with an error in the MS).}\end{aligned}$$

Then

$$\tfrac{1}{4}\cdot 40^\circ 1'52''30'' = 10^\circ 0'28''7'''30^{\mathrm{iv}},$$

and

$$\begin{aligned}\text{chord } \tfrac{1}{4}\cdot 40^\circ 1'52''30'' &= \text{chord } 10^\circ 0'28''7'''30^{\mathrm{iv}} \\ &= 0^{p}10'28''0'''37^{\mathrm{iv}}15^{\mathrm{v}},\end{aligned}$$

and

$$\begin{aligned}\text{chord } (30^\circ + 10^\circ 0'28''7'''30^{\mathrm{iv}}) &= \text{chord } 40^\circ 0'28''7'''30^{\mathrm{iv}} \\ &= 0^{p}41' \text{ (with an error in the MS).}\end{aligned}$$

Then

$$\begin{aligned}\tfrac{1}{4}\cdot 40^\circ 0'28''7'''30^{\mathrm{iv}} &= 10^\circ 0'7''1'''52^{\mathrm{iv}}, \\ \text{chord } 10^\circ 0'7''1'''52^{\mathrm{iv}} &= 0^{p}10'27''38'''37^{\mathrm{iv}}14^{\mathrm{v}},\end{aligned}$$

and

$$\begin{aligned}\text{chord } 40^\circ 0'7''1'''52^{\mathrm{iv}} &= 0^{p}41'2''39'''37^{\mathrm{iv}}15^{\mathrm{v}}6^{\mathrm{vi}}. \\ \tfrac{1}{4}\cdot 40^\circ 0'7''1'''52^{\mathrm{iv}} &= 10^\circ 0'1''45'''28^{\mathrm{iv}}7^{\mathrm{v}}30^{\mathrm{vi}} \\ \text{chord } 10^\circ 0'1'' \ldots &= 0^{p}10'27''33'''6^{\mathrm{iv}}11^{\mathrm{v}} \\ \text{chord } 40^\circ 0'1''45''' \ldots &= 0^{p}41'2''37'''25^{\mathrm{iv}}53^{\mathrm{v}} \\ \tfrac{1}{4}\cdot 40^\circ 0'1''45''' \ldots &= 10^\circ 0'0''26'''22^{\mathrm{iv}}1^{\mathrm{v}}52^{\mathrm{vi}}30^{\mathrm{vii}} \\ \text{chord } 10^\circ 0'0''26''' \ldots &= 0^{p}10'27''31'''44^{\mathrm{iv}}26^{\mathrm{v}} \\ \text{chord } 40^\circ 0'0''26''' \ldots &= 0^{p}41'2''33'''8^{\mathrm{iv}}2^{\mathrm{v}} \\ \tfrac{1}{4}\cdot 40^\circ 0'0''26''' \ldots &= 10^\circ 0'0''6'''35^{\mathrm{iv}}30^{\mathrm{v}}28^{\mathrm{vi}}7^{\mathrm{vii}}30^{\mathrm{viii}} \\ \text{chord } 10^\circ 0'0''6'''35^{\mathrm{iv}} \ldots &= 0^{p}10'27''31'''23^{\mathrm{iv}}42^{\mathrm{v}} \\ \text{chord } 40^\circ 0'0''6'''35^{\mathrm{iv}} \ldots &= 0^{p}41'2''32'''48^{\mathrm{iv}}35^{\mathrm{v}} \\ \tfrac{1}{4}\cdot 40^\circ 0'0''6'''35^{\mathrm{iv}} \ldots &= 10^\circ 0'0''1'''38^{\mathrm{iv}}12^{\mathrm{v}}37^{\mathrm{vi}}1^{\mathrm{vii}}52^{\mathrm{viii}}30^{\mathrm{ix}} \\ \text{chord } 10^\circ 0'0''1'''38^{\mathrm{iv}} \ldots &= 0^{p}10'27''31'''18^{\mathrm{iv}}33^{\mathrm{v}} \\ \text{chord } 40^\circ 0'0''1'''38^{\mathrm{iv}} \ldots &= 0^{p}41'2''32'''43^{\mathrm{iv}}43^{\mathrm{v}} \\ \tfrac{1}{4}\cdot 40^\circ 0'0''1'''38^{\mathrm{iv}} \ldots &= 10^\circ 0'0''0'''24^{\mathrm{iv}}43^{\mathrm{v}}9^{\mathrm{vi}}15^{\mathrm{vii}}28^{\mathrm{viii}}7^{\mathrm{ix}}30^{\mathrm{x}} \\ \text{chord } 10^\circ 0'0''0'''24^{\mathrm{iv}} \ldots &= 0^{p}10'27''31'''17^{\mathrm{iv}}15^{\mathrm{v}} \\ \text{chord } 40^\circ 0'0''0'''24^{\mathrm{iv}} \ldots &= 0^{p}41'2''32'''42^{\mathrm{iv}}29^{\mathrm{v}}.\end{aligned}$$

This last value approximates the value of chord $40^\circ = s_9$, which we know from the method of cubic equations to be

$$0^{p}\ 41'\ 2''\ 32'''\ 41^{\mathrm{iv}}\ 55^{\mathrm{v}}.$$

The slight difference between the values shows the high degree of the approximation.[1]

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[1] Translated by Professor David Eugene Smith. The method will later appear in Dr. Schoy's forthcoming work, *Die trigonometrische Lehren des al-Bîrûnî*, Hannover, Heinz Lafaire.

III. An Illustration of the Usefulness of the Method of Isolation in Statics

By Hymen Diamond, Student, Harvard University

A marked tendency in analytical mechanics is the elimination of internal reactions. However, there is in statics a method which, although it brings in internal forces, has the distinct advantage of being elementary in character, and it sometimes leads in a most simple way to the desired results. This is the method of isolation, or of the "freezing" of parts. The following problem furnishes a striking illustration of the efficacy of this method. In this case the method of isolation yields an expression, which gives the radius of curvature in terms of the known constants of the system. For other problems of the same general nature the radius of curvature may be found in terms of the constants and the coördinates. Incidentally the problem itself will doubtless be welcome in courses in mechanics as a companion to the old standbys, the freely hanging chain, and the suspension bridge cable, particularly because of the elementary character of the results.

A hole is punched in the center line, slightly below the top of each sheet of a ream of theme paper, and the paper is suspended from a weightless string passing through this hole. The string is smooth; more precisely, its contact with the paper is kept frictionless by some device. For example, the paper might be suspended in a freight car in motion, the vibration of which would jar the sheets slightly and thus eliminate any friction between the paper and the string, and between the sheets themselves. The paper is also constrained to remain in a vertical position by two vertical plates, one at each end of the system. Between these plates however, the sheets are free to move vertically without friction.

Now, since the paper is thin, the string will hang in a polygon having a large number of small sides, and as the thickness of the sheets is decreased the polygon approaches a smooth curve. Let us pass to the limit, decreasing the thickness indefinitely, and consider the string as hanging in a smooth curve. It is the form of this curve which we wish to find.

Let us isolate the system consisting of an arc PP' of the string. (Fig. 1.) The forces acting on this system are:

(a) S, the force due to the paper;

(b) T and T', the tensions in the string at P and P'.

Since the string is smooth,

$$T = T' = T_0 \ .$$

Further, it will be convenient to introduce, instead of S, a quantity R such that

$$S = \bar{R}\Delta s \ ,$$

where Δs is the arc PP'. Here R may be thought of as the mean specific force for the arc PP'.

Finally, let $\Delta\varphi$ be the increment in the angle, φ, which the normal to the string at a point P makes with the normal to the string at some initial point, *e. g.*, the lowest point of the string; and let ϵ be the angle which S makes with the normal at P.

We are now ready to resolve the forces in the direction of the normal to the string at P.

$$\bar{R}\Delta s \cos\epsilon = T_0 \sin \Delta\varphi \ ;$$

$$\bar{R} = \frac{T_0 \sin \Delta\varphi}{\Delta s \cos\epsilon} = T_0 \frac{\sin\Delta\varphi}{\Delta\varphi} \cdot \frac{\Delta\varphi}{\Delta s} \cdot \frac{1}{\cos\epsilon} \, .$$

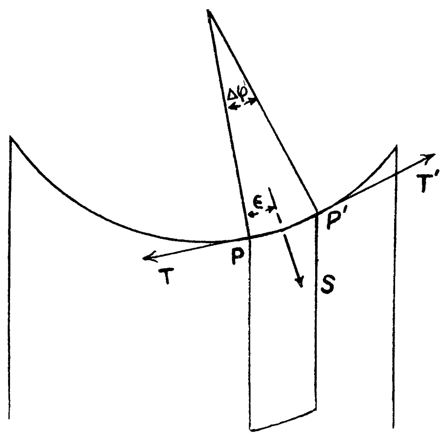

FIG. 1

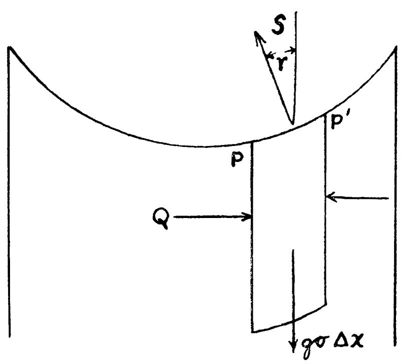

FIG. 2

Hence,

$$\lim_{\Delta\varphi \doteq 0} \bar{R} = \lim_{\Delta\varphi \doteq 0} T_0 \frac{\sin\Delta\varphi}{\Delta\varphi} \cdot \frac{\Delta\varphi}{\Delta s} \cdot \frac{1}{\cos\epsilon} \, . \tag{1}$$

Now,

$$\lim_{\Delta\varphi \doteq 0} \cos\epsilon = 1 \ , \qquad \lim_{\Delta\varphi \doteq 0} \frac{\sin\Delta\varphi}{\Delta\varphi} = 1 \ , \qquad \lim_{\Delta\varphi \doteq 0} \frac{\Delta\varphi}{\Delta s} = \frac{1}{\rho} \ ,$$

where ρ is the radius of curvature. Hence,

$$\lim_{\Delta\varphi \doteq 0} \bar{R} = R = \frac{T_0}{\rho} \, .$$

We now proceed to isolate a second system consisting of the paper in a section PP', and to find R in terms of the forces acting on this system. (Fig. 2.)

The forces acting on the system are,

(a) S, the reaction of the string,

(b) $g\sigma\Delta x$, the weight of the paper, which is distributed uniformly along the horizontal, and which has a mass, σ, per unit length.

(c) Q and Q', the forces exerted by the paper outside of the system PP', which are normal to the paper in the system PP', and so along the horizontal.

Let $\bar{\tau}$ be the angle which S makes with the vertical direction. Resolving along the vertical, we have:

$$S \cos \bar{\tau} = g\sigma\Delta x \ ,$$

or

$$\bar{R}\frac{\Delta s}{\Delta x} \cos \bar{\tau} = g\sigma \ , \qquad \lim_{\Delta x \doteq 0} \bar{R}\frac{\Delta s}{\Delta x} \cos \bar{\tau} = g\sigma \ .$$

We have already seen that the limit which $\bar{R}$ approaches is R, and

$$\lim_{\Delta x \doteq 0} \frac{\Delta s}{\Delta x} = \frac{ds}{dx} = \sec \tau \ , \qquad \lim_{\Delta x \doteq 0} \cos \bar{\tau} = \cos \tau \ .$$

Hence, $R = g\sigma$.

But it has been shown above that $R = T_0/\rho$. Hence, $T_0/\rho = g\sigma$, $\rho = T_0/g\sigma$, and since T_0, g, and σ are constant, the curve is a circle of radius $T_0/g\sigma$.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Altägyptische Zeitmessung. By LUDWIG BORCHARDT. Band I, Lieferung B, of "Die Geschichte der Zeitmessung und der Uhren." Berlin and Leipzig, Walter de Gruyter & Co., 1920. 70 pages, 18 plates + 25 other illustrations. Price 10 gold marks.

It is not merely an idle curiosity that perennially leads men to study the various systems of chronology that have been devised and the plans for measuring time to meet their practical needs. No scientific problem that people of all degrees of intelligence encounter has given more trouble than that of finding a simple method of arranging a calendar and of recording with a sufficiently high degree of accuracy the passing of the hours and days,—the years and the centuries. The entire question is complicated by the fact that it naturally fell to the priestly caste to regulate the early calendars, its members being the only ones possessed of sufficient learning to carry on such scientific labors. The result, however, has been to bind the calendar so closely to the various types of religion that any suggestion of reform is looked upon by uneducated

people as an attack upon their creeds, and their churches. Thus while we have a very awkward calendar in general use today, and while a much better one could easily be devised, the world is still compelled, or compels itself, to endure what it has rather than to adopt a perfectly simple plan that would save much trouble and expense.

It is for such reasons that a standard and authoritative study like that which this series gives to us is of particular value in a century which will probably see a reform even more far-reaching than that which the Church of Rome suggested in 1582.

As to the Egyptian system of time measure, Dr. Borchardt is the world's best authority. He has kept pace with recent discoveries and his monographs and other works upon the subject are well known. He is open-minded upon the general problem, and frankly states that the data thus far available are not sufficient to allow for a definitive history of the chronology of the peoples of the Nile valley. He has been able, however, to give us a fairly satisfactory statement relating to the calendar, but without adding materially, in this feature, to what was already known. His essential contributions to the subject appear in the part devoted to "Die altägyptischen Zeitmesser." This contains a description of most, if not all, of the important specimens of water clocks, sun dials, and stellar clocks thus far discovered. The most interesting feature of this part of the work consists in the conjectural restoration of the instruments and in the mathematical study of their accuracy. It appears that the oldest known specimen of that form of clepsydra in which the time is ascertained by the lowering of the surface of the water dates from *c.* 1400 B. C. It is in the form of an inverted frustum of a cone and was found at Karnak in 1904. The scales on which the hours were read changed with the months, thus allowing for the variation in their length. To show the difficulty of the investigation it may be mentioned that the next specimen of this type, in point of age, dates from the Alexandrian period, more than a thousand years later. The author gives photographic illustrations of several other specimens, with scale drawings of the interiors. He also lists all known specimens and gives a photographic copy of that part of the well-known Oxyrhynchus Papyrus, with his own restoration, describing the method of construction of this type of clock and showing the incorrect rule then in use for the volume of a frustum of a cone, viz., $V=\frac{1}{3}h[\frac{1}{2}(R+r)]^2$.

The unusual type in which water flowed into instead of out of the vessel is also described, a specimen dating from *c.* 100 A. D. having been found at Edfu in 1901.

The oldest actual sundial to be found dates from *c.* 1500 B. C. and is illustrated photographically in this work. The possible explanation of the use of the early types is given in the Tanis Papyrus already known through the

study made by Griffith and Flinders Petrie, and this is also illustrated in the text. Other specimens of dials of later date are also shown and described, as is also the stellar clock.

An important feature of the work is a tabular summary of the types of dials, their dates, and the sources of information, thus giving in synoptic form the story of the progress of hour reckoning among the Egyptians from very early times to the period of the Roman conquest.

The book is admirably printed and illustrated and serves as a fitting culmination to the author's extended researches and monographs upon this important branch of human knowledge.

DAVID EUGENE SMITH.

Plane Trigonometry with Tables, 3rd edition. By C. I. PALMER and C. W. LEIGH. New York, The McGraw-Hill Book Company, 1925. xiv+221 +136 pages. Price $2.50.

The first edition of this text appeared in 1914, and was reviewed in the MONTHLY for April, 1915, by Professor C. F. Craig. In the third edition certain parts have been revised and rearranged, and much new material added, including new problems, a more complete development of the fundamental operations with complex numbers, new work on series, and a chapter on spherical trigonometry. The contents are so arranged that they may be adapted for either a short course covering only the essentials, or a longer one involving line values, graphs of inverse functions, complex numbers, De Moivre's theorem, computation of functions, theory of logarithms, hyperbolic functions, and spherical trigonometry.

It may be of interest to note some of the places where the authors wander from the conventional path. Unlike the previous reviewer, the present writer commends the practice of defining the trigonometric functions, first for the general angle, and then specializing for the acute angle. The latter definitions are used in deducing the addition formulas for $(A+B)<90°$, the extension of these being obtained by analytical methods. In transforming one expression involving trigonometric functions to another, the authors write the word "to" between the expressions instead of the usual sign of equality which to many students suggests cross multiplying or transposing.

Among the commendable features of the book is the early introduction and constant use of inverse functions. The problems are abundant and interesting, having been brought up to date by making use of the aeroplane, automobile, and the motorist who true to life, has a "tendency to cut the corners." Mention should be made of the excellent historical notes, and the very interesting explanations of the applications of some of the theory.

Considering the many admirable qualities of the book, one wonders what led the authors in this new edition to add after certain results and formulas the questionable advice "This should be carefully memorized."

In general it may be said that the text seems scholarly and usable, rich in material for the classroom and for reference.

L. P. COPELAND.

Mathematics of Life Insurance. By L. WAYLAND DOWLING. New York, McGraw-Hill Book Co., 1925. x+121 pages. Price $1.75.

This book is intended primarily as a first course for those young men and women who wish to become trained actuaries or as a final course for other students who desire an elementary knowledge of the fundamental principles underlying life insurance.

It is well designed as to scope and the execution is on the whole admirable, the explanations being notable for lucidity. This makes it all the more regrettable that it at the same time presents a few rather serious defects. These will be mentioned in the order in which they appear in the book rather than in order of importance.

The subject of probability is taken up from the *a priori* or subjective standpoint and while the author has some good company in this respect it has always seemed to the present writer a peculiarly inappropriate method of presentation as an introduction to life insurance where the probabilities are based upon experience.

In Article 21 it is said, with respect to the "life curve" which is a graphic representation of the "number living" column of the life table, that it should be regarded as "the lower limit of a fluctuating curve representing actual mortality." I suppose the author has in mind the fact that, due to the improvement in mortality rates during the past half century the mortality tables in general use have come to represent a considerably higher rate of mortality than the average experience of recent years. That is, however, an accidental condition and even then does not justify the statement quoted. If the mortality table represents the actual experience its "life curve" corresponds to the mean position of the fluctuating curve rather than a lower limit.

Probably the gravest defect is however the use of the official international symbol for the annual premium for an endowment insurance as the symbol for the annual premium for a term insurance. This requires the invention of a new and inconvenient symbol for the former when it is needed. The accepted notation provides symbols for both these functions and no new ones were necessary.

It only remains to point out that on page 109 there is a clerical error apparently due to some confusion between mean *algebraic* deviation and mean

absolute deviation. The integral there given is the proper one for the former, which is equal to zero, but it is equated to the expression for the latter.

ROBERT HENDERSON.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

American Journal of Mathematics, volume 47, no. 4, October 1925: "Third Paper on Tensor Analysis" by G. Y. Rainich, 225-248; "On Certain Symmetric Sums of Determinants" by L. L. Dines, 249-256; "A Direct Solution of Systems of Linear Differential Equations having Constant Coefficients" by J. A. Nyswander, 257-276; "On Generalizations of the Bernoulli Functions and Numbers" by E. T. Bell, 277-288; "The Construction of Certain Periodic Orbits of the Three Body Problem" by H. E. Buchanan, 289-301.

Education, volume 46, no. 3, November 1925: "A Plea for Arithmetic" by C. H. Cordell, 170-178.

Journal of the Franklin Institute, volume 200, no. 5, November 1925: "A Mathematical Theory of the Drying of Wood" by Fordyce Tuttle, 609-614.

The Messenger of Mathematics, volume 55, no. 3, July 1925: "A new method for calculating the Bernoulli numbers" by F. J. Feinler, 40-43.

Proceedings of the National Academy of Sciences, volume 11, no. 10, October 1925: "Announcement of a Projective Theory of Affinely Connected Manifolds" by T. Y. Thomas, 588-589; "Transformations of Einstein Spaces" by H. P. Robertson, 590-591; "On the Equi-Projective Geometry of Paths" by T. Y. Thomas, 592-594. Volume 11, no. 11, November 1925: "Postulates for Reversible Order on a Closed Line (Separation of Point Pairs)" by E. V. Huntington, 687-688.

School Science and Mathematics, volume 25, no. 8, whole no. 217, November 1925: "An Investigation in the Teaching of the Skills of Ninth Grade Algebra" by Raleigh Schorling and Selma A. Lindell, 813-816; "A Study of Mathematical Abilities, Powers and Skills as Shown by Certain Classes in Physical Science" by V. C. Lohr, 834-843

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION.

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3164. Proposed by R. H. Sciobereti, University of California.

Study qualitatively the variations of the function

$$y=2xe^{(1/x)}+4x^3-15x^2+18x,$$

(the numerical values of the extrema are not required)
Find a polynomial $F(x)$ such that the difference $y-F(x)$ approaches zero when x becomes infinite.

3165. Proposed by Samuel Beatty, University of Toronto.

Prove that the upper and lower Jordan measure of a set S is the Lebesgue measure of $M\ (S, S')$, $D\{S, C(CS)'\}$, respectively, where $C(T)$ denotes the complement of T.

3166. Proposed by A. A. Bennett, Lehigh University.

Given an isosceles triangle ABC, in which $AC=BC$, and a circle with center at C. Find a point, P, on the circle such that the tangent to the circle at P bisects the angle APB.

3167. Proposed by H. Betz, University of Missouri.

Consider a particle moving in a straight line in the plane of an ellipse and inside the ellipse in such a manner that whenever it strikes the boundary of the ellipse it is "reflected" just as a ray of light would be, striking a mirror. The particle will, therefore, travel indefinitely often back and forth across the ellipse.

Let its path be referred to as its orbit. Now if, initially, the orbit passes through one focus of the ellipse, it will, in accordance with an elementary property of the ellipse, pass through the other focus also, and so on, indefinitely. Show that the orbit will converge, in the limit, to the major axis of the ellipse.

3168. Proposed by H. T. Davis, Indiana University.

Making the abbreviation

$$D_n(a_1, a_2, \ldots, a_n)=\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_1 \end{vmatrix}$$

prove the following:

(*a*) If $\delta_n = D_n(\frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \cdots, \frac{1}{(2n)!})$, then $\lim\limits_{n\to\infty} \left|\frac{\delta_{n-1}}{\delta_n}\right| = \frac{\pi^2}{4}$.

(*b*) The Bernoulli numbers, B_n, are given by the determinant

$$B_n = (2n)!\, D_{2n}\left(\frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \cdots, \frac{1}{(2n+1)!}\right).$$

(*c*) If D_i denotes $D_i(a_1, a_2, \ldots, a_i)$, $(i=1, 2, \ldots, n)$, then we have

$$D_n(-D_1, D_2, -D_3, \ldots, (-1)^n D_n)=(-1)^n a_n.$$

3169. Proposed by C. C. Camp, University of Illinois.

Two parallel vertical walls stand upon horizontal ground. A ladder of length a has its foot at the bottom of the first wall and leans against the second. A ladder of length b has its foot at the bottom of the second wall and leans against the first. What must be the distance between the walls so that the ladders will cross at a height h? When is a solution possible?

3170. Proposed by R. H. Sciobereti, University of California.

Given the base BC of a spherical triangle in position and magnitude and given the magnitude of the angle A which is opposite to BC, find the locus of A.

3171. Proposed by Philip Fitch, Denver, Colorado.

A right circular cylinder with a diameter d is composed of wood and metal. The wooden part is l cm long and has a density s. The metal part is l' cm long and has a density s'. If it is allowed to float in still water, what angle will the axis of the cylinder make with the surface of the water?

3172. Proposed by J. B. Reynolds, Lehigh University.

A linkage consisting of four equal uniform rods each of length $2a$ and weight w, loosely jointed in the form of a rhombus in a vertical plane, carries a weight P at the lowest vertex and is supported by the two upper rods resting against a smooth horizontal circular cylinder of radius r; find the time of a small vibration of the system in the vertical plane.

SOLUTIONS

2688 [1918, 119]. Proposed by Frank Irwin, University of California.

With four quantities, a_1, a_2, a_3, a_4, we may, without changing their order, form the following complex fractions:

$$\frac{a_1}{\dfrac{a_2}{\dfrac{a_3}{a_4}}} \qquad \frac{a_1}{\dfrac{\dfrac{a_2}{a_3}}{a_4}} \qquad \frac{\dfrac{a_1}{a_2}}{\dfrac{a_3}{a_4}} \qquad \frac{\dfrac{a_1}{\dfrac{a_2}{a_3}}}{a_4} \qquad \frac{\dfrac{\dfrac{a_1}{a_2}}{a_3}}{a_4}$$

But these have not all different values; the first and fourth are equal. Determine how many different rational functions of the quantities $a_1, a_2, \ldots, a_n$ may be obtained in this way, and which can be represented in more than one way as a complex fraction of the above kind, and which in only one way.

Solution by the Proposer

Let S represent the simple fraction equal to the given complex fraction, N and D its numerator and denominator respectively; and the a_i are to be thought of as arranged in N in the order of their subscripts, and so for D.

Then N begins with a_1, D with a_2; this, if not self-evident, may be proved at once by mathematical induction from $n-1$ to n. No S then can be represented as one of our complex fractions unless it is of the form $a_1\ a_{k+1} \ldots / a_2 \ldots a_k \ \ldots$, $2 \leqq k \leqq n$.

But it is further true that every such S can be so represented. For assume this last proposition to be true in the case $n-1$. Then each of the fractions $a_1/a_2 \ldots a_{k-1}$ and the reciprocal of $a_{k+1} \ldots /a_k \ldots$ can be represented as complex fractions; and if we put the first of these complex fractions in the numerator, the other in the denominator of a fraction, we get a complex fraction equal to our given fraction, S.

This enables us to determine at once the number of different S that may be represented as complex fractions. For starting with a_1, in N, a_2 in D, we may next place a_3 in either N or D, similarly for a_4, etc.; the number sought is then 2^{n-2}. For instance, for $n=5$, S may have any of the following $8=2^3$ values (wherein I have indicated the a_i by their subscripts, i, merely); 1/2345, 15/234, 145/23, 14/235, 1345/2, 134/25, 135/24, 13/245.

Next the a, say a_i, that comes immediately below the main cross-stroke (so I call the *longest* line that divides the complex fraction into a numerator and a denominator, themselves in general complex fractions) will, by the first proposition stated above, be in D, while a_{i+1} will be in N. It follows that given S, for instance 1467/23589, we may place the main cross-stroke just above the last of any of successive numbers (subscripts) in D (in the example, just above 3 or 5 or 9), nowhere else. The answer to our second question is, then, that any S that contains in D more than a single group of successive subscripts can be represented in two ways (at least) as a complex fraction, and no other S can. For instance, 1346/25 is equal to each of the three complex fractions:

$$\frac{\dfrac{1}{\dfrac{\dfrac{2}{3}}{4}}}{\dfrac{5}{6}}, \qquad \frac{1}{\dfrac{\dfrac{2}{3}}{\dfrac{4}{\dfrac{5}{6}}}}, \qquad \frac{1}{\dfrac{\dfrac{\dfrac{2}{3}}{\dfrac{4}{5}}}{6}}$$

2706 [1918, 216]. **Proposed by H. F. MacNeish, New York City.**

Through a given point draw a straight line cutting a given straight line and a given circle, such that the part of the line between the given point and the given line may be equal to the part within the given circle.

Discussion by A. A. Bennett, Lehigh University

This problem does not come under the most familiar cases, and in fact no Euclidean construction is possible. Analytic methods are of course required to establish this latter statement. Let the circle be taken as $x^2+y^2=1$, the fixed point, P, as (a, b), the fixed line, L, as $x=a+2c$, and let $y=mx+(b-ma)$ be a variable line through P. The chord of this variable line intercepted by the circle will have as the square of its length, $4\{1-[(b-ma)^2/(m^2+1)]\}$. The square of the length of the segment along the variable line from P to its intersection with L is $4c^2(m^2+1)$. Equating these, we obtain the following quartic equation for m,

$$c^2m^4+m^2(2c^2+a^2-1)+m(-2ab)+(c^2+b^2-1)=0.$$

This has as resolvent cubic, the following:

$$c^4u^3-c^2u^2(2c^2+a^2-1)-4c^2u(c^2+b^2-1)+4(2c^2+a^2-1)(c^2+b^2-1)-4a^2b^2=0.$$

Writing $cu=2(v+c)$, we have

$$2cv^3+v^2(4c^2-a^2+1)+2cv(2-a^2-b^2)+(1-a^2-b^2)=0.$$

A trial of possible rational divisors shows that this cubic is irreducible in the domain, $R(a, b, c)$, and therefore the proposed construction is impossible.

Also solved by W. J. Patterson, and C. A. Rupp.

3122 [1925, 138] **Proposed by A. A. Bennett, Lehigh University.**

Let $P(n)/Q(n)$ denote the fraction reduced to its lowest terms which represents the minimum value of $\varphi(m)/m$, for $0<m<n$, where $\varphi(m)$ is the indicator or totient of m. Show that $P(n)$ is of the form $2^a \cdot 3^b$ for every $n \leq 2 \cdot 10^{11}$.

Solution by Constance R. Ballantine, New York University.

Since

$$\varphi(p^a)=p^a-p^{a-1},$$
$$\varphi(p^a)/p^a=(p-1)/p.$$

Thus, if $m=p_1^{a1}\, p_2^{a2}\, p_3^{a3} \;.\;.\;.\;.\;.\; p_k^{ak}$, where the p_i are all distinct and no $a_i=0$,

$$\varphi(m)/m=\prod_{i=1}^{k}(p_i-1)/p_i.$$

Hence, if m is restricted to values of the form $p_1\cdot p_2\cdot p_3 \;.\;.\;.\;.\;. p_k$, where the p_i are all distinct, we obtain the same range of values for $\varphi(m)/m$ as if we allow the p_i to take on any positive (integral) exponents.

Further, it is obvious that $\varphi(m)/m$ has a smaller value for $m=2\cdot 3\cdot 5 \;.\;.\;.\;.\;. p$, where all primes $\leq p$ are included as factors, than for an m from which any of these factors are omitted, no others being added.

Again, if the product of several primes $p_1p_2 \;.\;.\;. p_k$ be replaced in m by a single prime $\bar{p}$ greater than at least one of them, the factor $\prod_{i=1}^{k}(p-1)/p$ will be replaced by the greater factor $(\bar{p}-1)/\bar{p}$. Hence $\varphi(m)/m$ will have its smallest value for $m<n$ if m is a product of consecutive primes, beginning with 2

Now, if $m_1=2\cdot 3\cdot 5 \;.\;.\;.\;.\;.\; p_1$,

$m_2=2\cdot 3\cdot 5 \;.\;.\;.\;.\;.\; p_1p_2$,

where p_1 and p_2 are successive primes,

$$\frac{\varphi(m_2)}{m_2}=\frac{\varphi(m_1)}{m_1}\cdot\frac{p_2-1}{p_2}<\frac{\varphi(m_2)}{m_1}.$$

Hence, if the value of m be restricted to numbers of the given form, $\varphi(m)/m$ is a monotonically decreasing function, and its minimum value in the interval from 0 to n is attained for the largest such m in the interval.

The numerator $P(n)$ of this minimum reduced to its lowest terms will be of the form $2^a \cdot 3^b$ so long as no factor except 2 and 3 is repeated in the product $2 \cdot 4 \cdot 6 \ldots\ldots (p-1)$. The first repetition is that of 5 for $p=31$. Thus the limiting value of n for which $P(n)$ is of the given form is

$$n=2 \cdot 3 \cdot 5 \ldots\ldots 29 \cdot 31-1<2 \cdot 10^{11}.$$

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Dr. H. A. SIMMONS has been appointed to an assistant professorship at Northwestern University. He was formerly at the University of Pittsburgh.

Dr. V. G. GROVE has returned to the Michigan State College as associate professor, after spending a year at the University of Chicago.

Dr. MARY SINCLAIR is on leave of absence from Oberlin College for the year 1925–26 and is studying in Italy.

Miss LESLIE GAYLORD, of Agnes Scott College, Decatur, Georgia, is absent on leave for the present year and is studying in Italy on appointment to a fellowship of the Italy-America Society.

Dr. V. A. TAN has been appointed to a professorship of mathematics at the University of the Philippines. He received the doctorate at the University of Chicago in June, 1925, after a prolonged period of study in America.

Dr. P. G. ROBINSON who has spent several years in study at the University of Chicago, partly also in teaching in the University High School, has accepted an appointment as instructor in mathematics at Iowa State College, Ames, Iowa.

Dr. D. L. HOLL, formerly instructor at Ohio Wesleyan University, is now assistant professor of mathematics at Iowa State College, Ames, Iowa.

Miss ECHO D. PEPPER, who held a fellowship in mathematics for two years at the University of Chicago, is now studying in Paris.

At Knox College, Galesburg, Illinois, Professor E. V. HUNTINGTON of Harvard University recently gave a course of lectures on "The foundations of elementary mathematics."

At Brown University, Professor R. G. D. RICHARDSON has been appointed dean of the graduate school. Professor R. C. ARCHIBALD is on leave for the second semester. Associate Professor MARSTON MORSE has been appointed

assistant professor at Harvard. Assistant Professor W. R. BURWELL has resigned to go into business. Further appointments and promotions are as follows: to be assistant professors, R. E. LANGER of Dartmouth College, M. H. INGRAHAM of University of Wisconsin; to be instructors, J. H. SIMESTER of the Carnegie Institute of Technology, H. S. THURSTON.

The publication of the second Carus Monograph was delayed unavoidably but is now definitely promised for the tenth of April. Nearly a thousand members of the Association have subscribed for the first Monograph and any who have neglected to do so may still subscribe by sending the order directly to Secretary W. D. CAIRNS. The special rate is available for institutional as well as for individual members.

The following reports of Summer Sessions to be held in 1926 have been received:

Columbia University, July 6 to August 13. In addition to courses in Trigonometry, Solid geometry, College algebra, Analytic geometry, and Calculus, and a series of courses for teachers of secondary mathematics, the following advanced courses are offered: By Professor E. R. HEDRICK: Theory of functions of a complex variable; Fundamental concepts of mathematics. By Professor W. B. FITE: Differential geometry. By Professor G. A. PFEIFFER: Calculus of variations. By Professor K. W. LAMSON: Differential equations.

Cornell University, July 5 to August 13. By Professor D. C. GILLESPIE: Analysis. By Professor VIRGIL SNYDER: Algebraic geometry. By Professor W. B. CARVER: Projective geometry. The following reading and research courses are also offered: By Professor C. F. CRAIG: Functions of a complex variable. By Professor SNYDER: Algebraic curves and surfaces. By Professor F. R. SHARPE: Hydrodynamics and elasticity. By Professors D. C. GILLESPIE and W. A. HURWITZ: Analysis. By Professors W. B. CARVER and F. W. OWENS: Projective geometry.

Johns Hopkins University, June 28 to August 6. In addition to courses in College algebra, Trigonometry, and Introductory calculus the following graduate course is offered: By Professor F. D. MURNAGHAN: Differential geometry, metric and non-metric, and Vector analysis.

Oberlin College, June 16 to August 4. By Professor W. D. CAIRNS: Teachers course in mathematics; The mathematics of finance, especially for teachers.

Ohio State University, first term, June 16 to July 24; second term, July 26 to August 28. In addition to courses in trigonometry, analytic geometry, calculus and the teaching of secondary mathematics, the following advanced courses are announced: By Professor H. BLUMBERG: Point sets; Introduction

to modern mathematics; Reading and research in analysis. By Professor C. L. ARNOLD: Advanced calculus; Advanced Euclidian geometry; Seminary in mathematics. By Professor R. L. WILDER: Differential equations; Fourier's series; Reading and research in analysis situs.

University of Chicago, first term, June 21 to July 28; second term, July 29 to September 3. In addition to the usual courses in trigonometry, college algebra, plane analytical geometry and differential and integral calculus, the following advanced courses are announced: By Professor L. E. DICKSON: Galois theory of equations; Division algebra; Thesis work in algebra and the theory of numbers. By Professor H. E. SLAUGHT: Differential equations; Elliptic integrals. By Professor E. T. BELL: Mathematical theory of relativity; Applications of elliptic functions to the theory of numbers; Reading and research in algebra and the theory of numbers. By Professor DUNHAM JACKSON: The theory of approximation by polynomials and trigonometric series; The mathematical theory of statistical correlation; Reading and research in analysis. By Professor A. C. LUNN: Development of optical theories; Vector analysis; Reading and research in applied mathematics. By Professor W. D. MACMILLAN: Analytic mechanics (dynamics); Theory of the potential. By Professor E. P. LANE: Synthetic projective geometry; Surfaces and congruences; Reading and research in geometry. By Mr. BARNARD: Theory of functions of a complex variable. By Professor WILLIAMS: Theory of equations. By Professor SMITH: Solid analytic geometry.

University of Illinois, June 21 to August 14. In addition to the usual courses in College algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses are offered: By Professor G. A. MILLER: The theory of numbers. By Professor A. B. COBLE: Algebraic geometry. By Professor A. R. CRATHORNE: Functions of a complex variable; and Mathematics of statistics. By Dr. C. C. CAMP: Differential equations. By Dr. H. R. BRAHANA, Advanced algebra. By Dr. T. BENNETT: Advanced analytic geometry.

University of Iowa, first term, June 14 to July 23. In addition to courses in Algebra, Trigonometry, Analytic geometry, and Calculus, the following courses are offered: By Professor C. C. WYLIE: Astronomy; Mathematics of finance. By Professor W. H. WILSON: Subject matter and teaching of mathematics. By Professor R. P. BAKER: Ordinary differential equations; Advanced algebra. By Professor E. W. CHITTENDEN: Advanced calculus; Abstract sets. Second term, July 26 to August 27. By Professor ROSCOE WOODS: Constructive geometry. By Professor J. F. REILLY: Differential equations; Fourier and periodogram analysis.

University of Michigan, June 21 to August 14. In addition to courses in Algebra, Plane and solid geometry, Trigonometry, Analytic geometry, Elementary calculus, Mathematical statistics, and the Mathematical theory of interest and insurance, the following advanced courses are offered: By Professor W. B. FORD: Advanced calculus; Determinants and the theory of equations. By Professor L. C. KARPINSKI: Teaching of algebra; History of mathematics. By Professor J. W. BRADSHAW: The Figures of solid geometry. By Professor PETER FIELD: Vector analysis. By Professor T. R. RUNNING: Graphical methods. By Professor H. C. CARVER: Theory of probability; Finite differences; Advanced mathematical theory of statistics. By Professor C. J. COE: Differential equations. By Professor L. A. HOPKINS: Differential equations. By Professor J. A. SHOHAT: Analytic mechanics; Theory of the potential. By Mr. M. F. JOHNSON: Solid analytic geometry.

University of Minnesota, first term, June 18 to July 31; second term, August 2 to September 4. The department of mathematics will offer an intensive course entitled: Selected topics in advanced mathematics. The topics for 1926 are: First term: By Professor R. W. BRINK: The mathematics of small vibrations. By Professor W. L. HART: Limits and series. By Professor A. L. UNDERHILL: Differential equations. Second term: By Professor A. L. UNDERHILL (topic to be announced later).

University of Oklahoma, June 7 to July 27. By Professor S. W. REAVES: Advanced calculus. By Professor J. O. HASSLER: Solid analytic geometry; Teachers' course in Mathematics. By Professor E. D. MEACHAM: Higher algebra. By Associate Professor N. ALTSHILLER-COURT: Modern Geometry. By Miss DORA MCFARLAND: Integral Calculus. By Miss FRANCES M. WRIGHT: Differential calculus.

University of Pennsylvania, July 6 to August 14. In addition to the usual courses in Solid geometry, Trigonometry, College algebra, Analytic geometry, and Calculus, the following courses are offered: By Professor J. D. ESHLEMAN: Elementary statistics. By Professor G. H. HALLETT: Advanced calculus. By Professor O. E. GLENN: Theory of invariants.

University of Texas, first term, by Dean H. Y. BENEDICT: Advanced calculus. By Professor E. L. DODD: Functions of real variables; Probability. By Professor R. L. MOORE: Theory of sets; Non-Euclidean geometry. By Associate Professor H. J. ETTLINGER: Boundary value problems; Ruler and compass constructions. By Associate Professor H. S. VANDIVER: Modern analytic geometry; Calculus. By Associate Professor L. L. SMAIL: Infinite processes. By Associate Professor C. D. RICE: Advanced calculus. By Adjunct Professor P. M. BATCHELDER: Theory of equations. By Adjunct Professor MARY DEC-

HERD: Solid analytic geometry. Second Term. By Associate Professor H. J. ETTLINGER: Boundary value problems; Definite integrals. By Associate Professor C. D. RICE: Differential geometry; Advanced calculus. By adjunct Professor C. M. CLEVELAND: Calculus. By Instructor R. G. LUBBEN: Calculus. By Lecturer E. W. CHITTENDEN (Iowa) Theory of sets; Professor CHITTENDEN'S second course is not yet determined. All freshman courses are given in both terms.

University of Wisconsin, June 28 to August 6. By Professor E. B. SKINNER: Theory of algebraic numbers; Irrational numbers. Professor A. DRESDEN: Analytic projective geometry; Elliptic functions. Professor H. W. MARCH: Theory of probability; Fourier series and integrals. Professor W. W. HART: Teaching of high school and junior high school mathematics; Teaching and supervision of Arithmetic. Mr. E. B. MILLER: Differential equations. Mathematical reading and research will be directed by Professors DRESDEN, MARCH and SKINNER.

Wyoming University, June 14 to July 21. By Professor H. C. GOSSARD: College algebra; Modern analytic geometry; The content of secondary mathematics; Seminar for advanced students. By Miss GRETA NEUBAUER: Calculus. July 22 to August 27. By Professor RECHARD: Plane trigonometry; Finite groups; Non-Euclidean geometry; Differential geometry.

St. Andrews University, Scotland, Aug. 3 to Aug. 13, a mathematical colloquium at which the following courses of lectures will be given: By Professor G. D. BIRKHOFF: The significance of dynamics for scientific thought. By Professor H. W. RICHMOND: Recent developments in algebraic geometry. By Professor S. BRODETSKY: Recent developments in applied mathematics. By Professor G. A. GIBSON: The history of mathematics in Scotland. By Professor H. W. TURNBULL: An introduction to the invariant theory. By Professor E. T. WHITTAKER and others: Informal talks.

Members of the colloquium may get board and lodging at the University Hall at a cost of 5 pounds per person. Applications for membership (enclosing a registration fee of 5 shillings) should be made before July 1 to *Professor* W. SADDLER, *The University, St. Andrews, Fifeshire.*

LIFE MEMBERSHIP IN THE MATHEMATICAL ASSOCIATION

This statement which was first published on page 281 of the MONTHLY for September, 1923, is here repeated for the convenience of new members. In accordance with the action of the Association at its Rochester meeting, September, 1922, members may obtain life membership in the Association by the payment, at the first of any calendar year, of an amount indicated in the accompanying table. In estimating one's age the birthday anniversary nearest to the first of January when payment is made should be taken.

Age	Fee	Age	Fee	Age	Fee
20	$77.84	40	$65.21	60	$43.54
21	77.40	41	64.33	61	42.28
22	76.94	42	63.44	62	41.02
23	76.47	43	62.52	63	39.76
24	75.97	44	61.57	64	38.49
25	75.46	45	60.60	65	37.22
26	74.93	46	59.61	66	35.96
27	74.38	47	58.59	67	34.69
28	73.81	48	57.55	68	33.44
29	73.22	49	56.48	69	32.20
30	72.60	50	55.40	70	30.96
31	71.97	51	54.29	71	29.74
32	71.31	52	53.16	72	28.54
33	70.63	53	52.01	73	27.35
34	69.93	54	50.85	74	26.18
35	69.20	55	49.66	75	25.04
36	68.45	56	48.47	76	23.92
37	67.68	57	47.25	77	22.82
38	66.88	58	46.03	78	21.76
39	66.05	59	44.78	79	20.72

W. D. CAIRNS, *Secretary-Treasurer.*

The Carus Mathematical Monographs

The CARUS MONOGRAPHS have already begun to fulfill their mission as intended by the generous donor, MRS. MARY HEGELER CARUS and her son DR. EDWARD H. CARUS.

The first edition of Monograph Number One, "Calculus of Variations," by Professor Bliss, will soon be exhausted.

Nearly 100 members of the Association have taken advantage of the distribution at cost. Those who neglected to do so at the start may still have the privilege by applying to the Secretary. This includes institutional as well as individual members.

It would be a great tribute to the donor and an honor to the Association if every member would subscribe for the complete series. Every dollar thus paid by members is a contribution to the permanent Publication Fund of the Association.

It is believed that the Association is rendering a great service to mathematics by this enterprise, and a one hundred per cent support from the membership would constitute an appropriate vote of confidence in the undertaking.

Monograph Number Two, "Analytic Functions of a Complex Variable," by Professor Curtiss, is ready for delivery and will be distributed at once to those who have sent in their subscriptions to the Secretary. All distribution to the general public will be made through

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.
Eleventh Annual Meeting, Philadelphia, Pa., December, 1926.
The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7-8.

IOWA, Cedar Rapids, April.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., May 8

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 7-8.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Fort Collins, Colo., April 16-17.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, Not yet determined.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

THE AMERICAN MATHEMATICAL MONTHLY

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THE MATHEMATICAL ASSOCIATION

The following twenty persons and one institution have been elected to membership in the Association:

To Individual Membership

O. E. Brown, M.S. (Iowa). Instr., Northwestern Univ., Evanston, Ill.

J. A. Carlson, B. S. in Math. (Washington). Instr. in Sc., High School, Davenport, Wash.

L. R. Chase, Instr., Rogers High School, Newport, R. I.

Teresa Cohen, Ph.D. (Johns Hopkins). Asst. Prof., Penna. State Coll., State College, Pa.

D. D. Driver, A.B. (Hesston). Instr., Hesston Coll., Hesston, Kans.

Beatrice A. Fenner, A.M. (Stanford). Reg. and Teacher, High School, Martinez, Calif.

C. H. Hale, A.B. (Tex. Chr. Univ.). Asso. Prof., Tarleton Agric. Coll., Stephenville, Tex.

Frances Harshbarger, A. M.(West Virginia). Instr., Potomac State School, Keyser, W. Va.

Ina E. Holroyd, B.S. (Kans. State Agric. Coll.; Kans. State Teachers Coll.) Instr., State Agric. Coll., Manhattan, Kans.

Ruth E. Leonard, M.S. (Chicago). Instr., Wesleyan Coll., Macon, Ga.

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M. E. Mullings, A.B. (Texas). Instr., Univ. of Texas, Austin, Tex.

J. E. Nelson, A.B. (Texas). Dir., Univ. Jr. Coll., San Antonio, Tex.

J. C. Nixon, A.M. in Educ. (Florida). Instr., Purdue Univ., W. LaFayette, Ind.

Mary J. Quigley, A.B. (Radcliffe). Instr., Teachers Coll. of the City of Boston, Boston, Mass.

G. Y. Rainich, Master of Pure Math. (Univ. of Kazan). Res. Scholar, Johns Hopkins Univ., Baltimore, Md.

Lulu L. Runge, A.M. (Wisconsin). Asst. Prof., Univ. of Nebraska, Lincoln, Nebr.

P. S. Wagner, A.M. (Johns Hopkins). Grad. Student, Johns Hopkins Univ., Baltimore, Md.

M. E. Westcott, B.S. (Northwestern). Instr., Northwestern Univ., Evanston, Ill.

Frances M. Wright, A.M. (Brown). Instr., Univ. of Oklahoma, Norman, Okla.

To Institutional Membership

The Tulane University of Louisiana, New Orleans, La.

W. D. Cairns, *Secretary-Treasurer.*

THE SECOND MEETING OF THE MICHIGAN SECTION

The second annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, on April 2, 1925. The meeting was called to order at 9:30 A.M. with Chairman E. R. Sleight presiding.

The attendance was sixty, including the following twenty-seven members of the Association: N. H. Anning, J. W. Baldwin, J. W. Bradshaw, S. E. Crowe, W. W. Denton, L. C. Emmons, J. P. Everett, Florence E. Field, S. E. Field, J. W. Glover, T. H. Hildebrandt, L. A. Hopkins, W. A. Jenkins, L. C. Karpinski, C. E. Love, Mrs. Selah W. Mullen, A. L. Nelson, H. L. Olson, W. H.

Pearce, V. C. Poor, Clair Reid, L. J. Rouse, T. R. Running, R. C. Shellenberger, E. R. Sleight, and G. G. Speeker.

At the business session the secretary reported a membership for the section of fifty-seven out of a Michigan membership of seventy-seven in the Association. The following were elected officers for 1925–1926: Chairman, E. R. SLEIGHT, Albion College; Secretary-treasurer, N. H. ANNING, University of Michigan; Member of executive committee, L. C. PLANT, Michigan State College (East Lansing).

The following five papers were read:

(1) "Some new projective differential covariants of a congruence" by H. L. Olson, University of Michigan, Ann Arbor.

(2) "On the origin of our number forms" by T. W. Fowle, College of the City of Detroit, Detroit (by invitation).

(3) "The asteroids" by L. A. Hopkins, University of Michigan, Ann Arbor.

(4) "A college course in geometry for prospective high school teachers" by G. G. Speeker, Michigan State College, East Lansing.

(5) "On a central difference summation formula" by W. A. Jenkins, University of Michigan, Ann Arbor.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Olson based his paper on one by Wilczynski entitled "Sur la théorie générale des congruences" published by the Royal Academy of Belgium. It treats of the loci determined by the two covariants

$$\eta = y_u + \frac{a'}{c'} y \, , \qquad \zeta = z_v + \frac{b}{d} z \, ,$$

$$y_v = mz \, , \quad z_u = ny \, , \quad y_{uu} = ay + bz + cy_u + dz_v \, , \quad z_{vv} = a'y + b'z + c'y_u + d'z_v \, ,$$

where a', c', b, and d are coefficients in the fundamental system of differential equations.

2. It seems a tenable hypothesis that our number forms are intimately related to the Chinese, which in turn came from picturing finger forms. Whether or not India and Arabia were necessary steps toward getting our numbers may be shown by the study of old records from China westward all across Asia to the Mediterranean. Mr. Fowle presented data relating at least seven of our numerals to the Chinese.

3. Professor Hopkins presented his discussion of the asteroids under three headings: (a) A resumé of the discoveries of the asteroids, the significance of these discoveries in the development of astronomy, and the present embarrassment due to their great number; (b) A review of some of the most important mathematical studies of asteroid motion, particularly in the application of

periodic orbits; (*c*) A brief statement of his own studies in this field, which have been published in the *Astronomical Journal*, and of the immediate tasks ahead.

4. Professor Speeker deplored the tendency in some quarters to dilute, as he sees it, the classical course in synthetic projective geometry by the introduction of a preponderance of metric theorems proved by the familiar Euclidean methods. He does not believe that the teacher of mathematics in the high school is so much in need of further drill in Euclidean theorems proved by Euclidean methods as he is in need of an enlargement of his mathematical outlook in the direction of geometry. And if this is the case, he is certain that synthetic projective geometry is the course best calculated to attain this end.

5. Mr. Jenkins called attention to a certain difference formula developed by Woolhouse, which interpolates a certain number of values between each of a given series of equidistant ordinates, then sums approximately the interpolated along with the given values in terms of the original ordinates and their differences. This formula was proved more convergent than Lubbock's formula, and an approximate test of accuracy was developed.

J. P. EVERETT, *Secretary-Treasurer.*

CONCERNING THE NET OF QUADRICS CIRCUMSCRIBING THE SPACE CUBIC

By J. H. NEELLEY, Yale University

1. Introduction. The "principle of duality" was one of the most importan discoveries in modern geometry during the past century, and was the culmin tion of the investigations of Poncelet, Gergonne, and others in the method of reciprocal polars. By means of this principle the mathematician has not only a theorem of an entirely different nature involving dual elements to correspond to each theorem he proves, but he is enabled to contemplate the properties of the dual elements of any locus and obtain a complete perspective of the configuration which would be otherwise impossible. An especially interesting type of locus is that which is self-dual. The conic is the simplest of these curves in the plane and affords the student a wealth of illustrations and exercises in the first course in projective geometry. The space cubic plays a similar role for three dimensions.

The present paper is concerned with the dual net of quadrics associated with the space cubic curve which is referred to by P. W. Wood.[1] To find the relations of these two nets let us derive the equations of the well known net of

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[1] *Cambridge Tracts in Mathematics and Mathematical Physics*, No. 14.

quadrics on which the space curve lies and obtain the dual net by the use both of analytic and of synthetic methods.[1]

2. The net of inscribed quadrics.[2] The space cubic curve ρ^3 may be represented by the binary cubic

$$(x\tau)^3 \equiv x_0\tau^3+3x_1\tau^2+3x_2\tau+x_3=0 \tag{1}$$

where τ is a parameter spread along the curve and $(x)\equiv(x_0, x_1, x_2, x_3)$ are the coordinates of a point in space. Then, for a given τ, (1) is an osculating plane of ρ^3; but, if (x) is given, (1) is a binary cubic with roots τ_i, $i=1, 2, 3$, the parameter values of the points of contact of the three osculating planes through (x).

If (x) is on ρ^3 then (1) is a perfect cube, $(\tau-t)^3=0$, from which we have the parametric equations

$$x_0=1\ , \qquad x_1=-t\ , \qquad x_2=t^2\ , \qquad x_3=-t^3\ . \tag{2}$$

Here $x_0=0$ and $x_3=0$ are osculating planes at $t=\infty$ and $t=0$ respectively, $x_1=0$ contains the point $t=0$ and the tangent line at $t=\infty$, and $x_2=0$ the line at $t=0$ and the point $t=\infty$.

The plane of the three points t_i, $i=1, 2, 3$, is

$$s_3x_0+s_2x_1+s_1x_2+x_3=0\ , \tag{3}$$

where s_i refers to the symmetric functions of t_1, t_2, t_3. If $t_2=t_3=t$ the plane contains the tangent line at t and the point t_1, whence

$$(t_1x_0+x_1)t^2+2(t_1x_1+x_2)t+(t_1x_2+x_3)=0\ . \tag{4}$$

The discriminant of (4) gives the osculant cone of ρ^3 which has its vertex at t_1. That is

$$(x_0x_2-x_1^2)t_1^2+(x_0x_3-x_1x_2)t_1+(x_1x_3-x_2^2)=0. \tag{5}$$

This is the locus of the bisecants of the curve which pass through t_1. On the other hand for a given point (x) in space, (5) represents the parameters of the points cut out by the unique bisecant through (x). When (x) is on ρ^3 one root of (5) is arbitrary and so we have

$$x_0x_2-x_1^2=0\ , \qquad x_0x_3-x_1x_2=0\ , \qquad x_1x_3-x_2^2=0\ . \tag{6}$$

These form the net of quadrics on ρ^3 which are satisfied by every point of the curve. The polarized form of (5) gives as the equation of the net of quadrics

$$2(x_0x_2-x_1^2)t_1t_2+(x_0x_3-x_1x_2)(t_1+t_2)+2(x_1x_3-x_2^2)=0\ . \tag{7}$$

[1] In this paper corresponding equations of dual forms are given the same numbers.

[2] The method used in this section is taken from notes by Professor A. B. Coble.

For any values assigned to t_1 and t_2 (7) is the quadric surface which contains the tangents at t_1 and t_2. Through each of the points t_1 and t_2 draw the null-secant, a line in the osculating plane through its contact point, which is coplanar with the tangent line from the other point. The pencil of planes on either of these null-secants determines, by their intersection with the curve, the same involution of pairs of points. The joins of these pairs are bisecants which form one system of rulings of the quadric. Hence the null-secants are paired in such a way that all bisecants which meet one null-secant meet another. The other system of rulings is made up of secant lines, one through each point of the curve, which meet both tangents drawn from t_1 and t_2. These two tangents belong to the first set of generators and the two null-secants are included in the latter set.

If the point (x) is on a tangent of ρ^3 we have

$$4(x_0x_2-x_1^2)(x_1x_3-x_2^2)-(x_0x_3-x_1x_2)^2=0 \tag{8}$$

which is the developable of the curve.

3. The net of circumscribed quadrics. The dual of this problem gives a net of quadrics equally well defined. Instead of considering the points of the curve and the bisecant lines joining them we investigate the osculating planes of ρ^3 and the bisecant axes in which they intersect.

The equation of the point t on the curve is

$$(\xi t)^3 \equiv \xi_0-\xi_1 t+\xi_2 t^2-\xi_3 t^3=0 \tag{1'}$$

since its coordinates are given by (2). On the other hand if a plane (ξ) is given, (1′) is a binary cubic whose roots are t_i the parameters of the points of the curve intersected by (ξ). If (ξ) is an osculating plane of the curve (1′) is a perfect cube and we have

$$\xi_0=\tau^3\ , \qquad \xi_1=3\tau^2\ , \qquad \xi_2=3\tau\ , \qquad \xi_3=1\ . \tag{2'}$$

The osculating planes of ρ^3 at the three points whose parameters are roots of (1) meet in the point

$$x_0=1\ , \qquad x_1=-\frac{1}{3}\sigma_1\ , \qquad x_2=\frac{1}{3}\sigma_2\ , \qquad x_3=-\sigma_3\ ,$$

where σ_i refers to the symmetric functions of τ_1, τ_2, τ_3. Hence the equation of the point is

$$\xi_0-\frac{1}{3}\sigma_1\xi_1+\frac{1}{3}\sigma_2\xi_2-\sigma_3\xi_3=0\ . \tag{3'}$$

Now set $\tau_2=\tau_3=\tau$ so that two of the planes of the curve coincide, then their intersection is a line of ρ^3 and the point where this tangent and the osculating plane at τ_1 meet is

$$(\xi_2-3\xi_3\tau_1)\tau^2+2(\xi_2\tau_1-\xi_1)\tau+(3\xi_0-\xi_1\tau_1)=0\,. \tag{4'}$$

The discriminant of (4′) for a given τ_1 is the equation of the conic, which is the envelope of the bisecant axes in which the planes of ρ^3 meet the plane at τ_1. That is

$$(3\xi_1\xi_3-\xi_2^2)\tau_1^2+(\xi_1\xi_2-9\xi_0\xi_3)\tau_1+(3\xi_0\xi_2-\xi_1^2)=0\ . \tag{5′}$$

For any plane (ξ) in space there are two values of τ_1 given by (5′). Hence two planes of ρ^3 have a common intersection with (ξ) or there is one bisecant axis of the curve in every plane of space. When (ξ) is made an osculating plane of ρ^3 one value of τ_1 is arbitrary, so that

$$3\xi_1\xi_3-\xi_2^2=0\ ,\qquad \xi_1\xi_2-9\xi_0\xi_3=0\ ,\qquad 3\xi_0\xi_2-\xi_1^2=0\ . \tag{6′}$$

These are the independent quadrics which form the net satisfied by every plane of ρ^3. We shall designate them ξ-quadrics in contradistinction to the x-quadrics of the dual problem. The polarized form of (5′) is the equation of the net of ξ-quadrics. It is

$$2(3\xi_1\xi_3-\xi_2^2)\tau_1\tau_2+(\xi_1\xi_2-9\xi_0\xi_3)(\tau_1+\tau_2)+2(3\xi_0\xi_2-\xi_1^2)=0\ . \tag{7′}$$

For a fixed τ_1 and τ_2, (7′) is the quadric which contains the tangents of ρ^3 at τ_1 and τ_2. So this quadric has the two tangents at τ_1 and τ_2 in common with the x-quadric which is determined by the same two points of ρ^3. In each of the osculating planes at τ_1 and τ_2 draw the null-secant which meets the tangent from the other point. The range of points on either of these null-secants determines, by contacts of osculating planes through each point, the same involution of pairs of planes. The intersections of these pairs are bisecant-axes which form one system of generators of the quadric. Hence all bisecant axes which meet one null-secant meet another. But these null-secants are the same two which are determined by the corresponding x-quadric when t_1 and t_2 are equal to τ_1 and τ_2 respectively. Whence

THEOREM: *The point quartic which is the intersection of two corresponding quadrics degenerates into four ranges of points, the tangents at the two points and the null-secant at each point which is coplanar with the tangent at the other point. And the quartic curve in planes common to the same quadrics becomes four pencils of planes whose axes are the same four lines.*

Obviously the second set of generators of the ξ-quadric consists of all the lines, one in each plane of ρ^3, which meet both of the tangents of the curve from τ_1 and τ_2. These two tangents belong to the first set of rulings, as a tangent is a bisecant axis whose determining planes are coincident, and the two null-secants belong to the second set of rulings.

When the plane (ξ) is on a line of ρ^3 that line is the bisecant axis of the plane. In that case we have

$$4(3\xi_1\xi_3-\xi_2^2)(3\xi_0\xi_2-\xi_1^2)-(\xi_1\xi_2-9\xi_0\xi_3)^2=0\,. \tag{8'}$$

This equation and (8) are the equations of the self-dual developable surface of the curve.

4. A cubic transformation. Since through any point (x) of space there passes one and only one bisecant, we will let (y) be the harmonic conjugate of (x) with respect to the two points of intersection of ρ^3 and the bisecant. The points (x) and (y) are apolar to any quadric of the x-net, so polarizing the net as to (y) and eliminating the y's from the resulting forms and $(y\tau)^3$ we have the cubicovariant of (1) in the form

$$(y\tau)^3\equiv\begin{vmatrix} x_2 & -2x_1 & x_0 & 0 \\ x_3 & -x_2 & -x_1 & x_0 \\ 0 & x_3 & -2x_2 & x_1 \\ \tau^3 & 3\tau^2 & 3\tau & 1 \end{vmatrix}\,. \tag{9}$$

If τ is given, (1) is the osculating plane at the point τ but (9) is a cubic surface which is the locus of (y). So the planes of ρ^3 are transformed into cubic surfaces.

In the same way we are led to a cubic transformation in the dual problem as in every plane (ξ) of space there is one bisecant axis of ρ^3. Let the plane (η) be the harmonic conjugate of the plane (ξ) as to the two planes of ρ^3 which determine the bisecant axis. Planes (ξ) and (η) are apolar to any quadric of the circumscribing net, so polarizing the quadrics (6′) as to (η) and eliminating the η's from these and $(\eta t)^3$ we have the cubicovariant of (1′) in the form

$$(\eta t)^3\equiv\begin{vmatrix} 0 & 3\xi_3 & -2\xi_2 & 3\xi_1 \\ -9\xi_3 & \xi_2 & \xi_1 & -9\xi_0 \\ 3\xi_2 & -2\xi_1 & 3\xi_0 & 0 \\ 1 & -t & t^2 & -t^3 \end{vmatrix}=0\,. \tag{9'}$$

This for a given t is a cubic surface which is the envelope of planes (η), the transform of the point of ρ^3 given by that value of t.

We note some dual properties of these surfaces given by (9) and (9′).

(a) If $\genfrac{}{}{0pt}{}{(x)}{(\xi)}$ is on a $\genfrac{}{}{0pt}{}{\text{plane}}{\text{point}}$ of ρ^3 at $\genfrac{}{}{0pt}{}{\tau,}{t,}$ $\genfrac{}{}{0pt}{}{(9)}{(9')}$ is the locus of all null-secants which meet the tangent line at $\genfrac{}{}{0pt}{}{\tau}{t}$. This is, therefore, a ruled cubic surface.

(b) Bisecant $\genfrac{}{}{0pt}{}{\text{lines}}{\text{axes}}$ are unaltered by this transformation but pairs of $\genfrac{}{}{0pt}{}{\text{points}}{\text{planes}}$ on them are interchanged.

(c) The two null-secants which any set of bisecant $\genfrac{}{}{0pt}{}{\text{lines}}{\text{axes}}$ meet are interchanged.

(*d*) The $\genfrac{}{}{0pt}{}{\text{points } \tau}{\text{planes } t}$ are sent into the lines of ρ^3 at $\genfrac{}{}{0pt}{}{\tau}{t}$.

(*e*) The three null-secants $\genfrac{}{}{0pt}{}{\text{on } (x)}{\text{in } (\xi)}$ determined by the three $\genfrac{}{}{0pt}{}{\text{planes}}{\text{points}}$ of ρ^3 $\genfrac{}{}{0pt}{}{\text{on } (x)}{\text{in } (\xi)}$ transform into the three null-secants $\genfrac{}{}{0pt}{}{\text{on } (y)}{\text{in } (\eta)}$ the transform of $\genfrac{}{}{0pt}{}{(x)}{(\xi)}$.

NOTE ON THE INTEGRABILITY OF THE DIFFERENTIAL EQUATION $Pdx+Qdy+Rdz=0$

By T. C. ESTY, Amherst College.

1. Introduction. In the differential equation

$$Pdx+Qdy+Rdz=0 \ , \tag{1}$$

the quantities P, Q, R are supposed to be functions of x, y, z, the rectangular coordinates of a point in space, and we shall restrict the discussion to cases in which these functions and their derivatives are finite, continuous and single-valued.

Equation (1) has an integral $u(x, y, z)=a$ if there is a function $u(x, y, z)$ whose total differential is equal to the left-hand member of (1), or to that expression multiplied by a factor, $f(x, y, z)$.

It is the purpose of this note to derive by the methods of vector analysis the necessary and sufficient condition that equation (1) is integrable. No originality is claimed, for the subject has been at least partially considered elsewhere in treatises on vector analysis or quaternions, but the present writer happens to know of only one text in which there is derived the sufficient condition that equation (1) may be made integrable by means of an integrating factor. This is done, in substantially the way presented here, by Professor James Byrnie Shaw in his *Vector Calculus*. It has therefore seemed that it might help some who are interested in simple applications of vector methods to present a complete treatment of the subject in an elementary form.

2. Case I. We shall first show that the necessary and sufficient condition that the left-hand member of (1) be an exact differential is

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\,, \qquad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}\,, \qquad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}\,. \tag{2}$$

Let us put $\sigma=Pi+Qj+Rk$, *and* $d\rho=idx+jdy+kdz$, so that (1) is

$$\sigma\cdot d\rho=0 \ . \tag{3}$$

THEOREM A. *If $\sigma\cdot d\rho$ is an exact differential, say $du(x, y, z)$, then $\nabla\times\sigma=0$, and equations (2) are satisfied.*

PROOF. If $\sigma\cdot d\rho$ is an exact differential, we may write $\sigma\cdot d\rho=du=d\rho\cdot\nabla u$, and since this holds for all values of $d\rho$, we have $\sigma=\nabla u$.
It follows that

$$\nabla\times\sigma=\nabla\times\nabla u=0\ , \tag{4}$$

or

$$\nabla\times\sigma=\begin{vmatrix} i & & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}=0\ ,$$

that is

$$\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)i+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)j+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)k=0\ .$$

Since this requires that the coefficients of i, j, k vanish separately, we see that equations (2) express the necessary condition that $Pdx+Qdy+Rdz$ be an exact differential.

THEOREM B. *Conversely, if equations* (2) *are satisfied, that is if* $\nabla\times\sigma=0$, *then* $\sigma\cdot d\rho$ *is an exact differential.*

We shall divide the proof of this theorem into three parts and shall prove that if $\nabla\times\sigma=0$, then

(*a*) The line integral $\int_0\sigma\cdot d\rho$ around any closed curve is equal to zero.

(*b*) The line integral $\int\sigma\cdot d\rho$ between any two points is independent of the path of integration.

(*c*) Assuming (*b*), $\sigma\cdot d\rho$ is an exact differential.

PROOF OF (*a*). By Stokes's theorem we know that the line integral of the vector function σ around any closed curve is equal to the surface integral of the curl of that function over any surface bounded by the curve. This is expressed by the equation

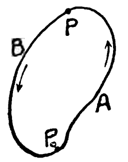
$$\int_0\sigma\cdot d\rho=\iint_S\nu\cdot\nabla\times\sigma\ dS\ , \tag{5}$$

in which ν is a unit vector drawn normal to the element dS of the surface, and toward that side of the surface from which the direction adopted in integrating around the curve is seen as counter-clockwise. The discussion is confined to a region in which singularities of σ and $\nabla\times\sigma$ do not occur.

Now since $\nabla\times\sigma=0$, it follows from (5) that

$$\int_0\sigma\cdot d\rho=0\ , \tag{6}$$

that is, the line integral of $\sigma \cdot d\rho$ around any closed curve is equal to zero.



PROOF OF (*b*). Let P_0APB be any path around which the integral $\int \sigma \cdot d\rho$ may be taken, and let I be the value of the integral from P_0 to P along P_0AP; also let II be its value from P_0 to P along P_0BP. Then $-II$ is the value of the integral from P to P_0 along PBP_0. Thus we have by (*a*)

$$\int_0 \sigma \cdot d\rho = \mathrm{I} - \mathrm{II} = 0 \ , \quad \text{whence} \quad \mathrm{I} = \mathrm{II} \ .$$

Hence, if ρ_0 and ρ denote the position vectors of P_0 and P, respectively, we see that the value of the integral

$$\int_{\rho_0}^{\rho} \sigma \cdot d\rho$$

is independent of the path of integration connecting P_0 and P.

PROOF OF (*c*). It follows from (*b*) that if P_0 is a fixed point, and P a variable point, the value of the integral is fully determined when ρ is given; that is, the integral is a scalar function of the position of P, say,

$$\int_{\rho_0}^{\rho} \sigma \cdot d\rho = u(x,y,z) \ .$$

If, therefore, P be taken infinitely near to P_0, we may write

$$\sigma \cdot d\rho = du \ ,$$

that is, $\sigma \cdot d\rho$ is an exact differential.

Thus it has been proved that equations (2) are the necessary and sufficient conditions that (1) be an exact differential equation.

3. Case II. We shall now show that if the left-hand member of equation (1) is not an exact differential, the necessary and sufficient condition that it may be made an exact differential by means of a factor $f(x, y, z)$ is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \ . \tag{7}$$

THEOREM C. *If $\sigma \cdot d\rho$ is not an exact differential, and there exists a function $f(x, y, z)$ such that the product of f and $\sigma \cdot d\rho$, namely $f\sigma \cdot d\rho$, is exact, then*

$$\sigma \cdot \nabla \times \sigma = 0 \ ,$$

and equation (7) *is satisfied.*

PROOF. Since $f\sigma \cdot d\rho$ is assumed to be an exact differential, we have by Case I,

$$\nabla \times (f\sigma) = 0 \ , \quad \text{or} \quad f \nabla \times \sigma + \nabla f \times \sigma = 0 \ . \tag{8}$$

Multiplying scalarly by σ, we get

$$f\sigma\cdot\nabla\times\sigma+\sigma\cdot\nabla f\times\sigma=0\ ,$$

or, since $\sigma\cdot\nabla f\times\sigma=0$, we find, after dropping the factor f,

$$\sigma\cdot\nabla\times\sigma=0\ , \tag{9}$$

which, when expanded, may be written in the form of equation (7). Thus it is seen that equation (7) is the necessary condition that (1) may be made an exact differential equation by means of an integrating factor.

THEOREM D. *Conversely, if equation* (7) *is satisfied, that is, if* $\sigma\cdot\nabla\times\sigma=0$, *then there exists a factor* $f(x, y, z)$ *such that* $f\sigma\cdot d\rho$ *is an exact differential.*

PROOF. Since it is assumed that $\sigma\cdot d\rho$ is not an exact differential, we must have by Case I, $\nabla\times\sigma\neq 0$. Also, since $\sigma\cdot\nabla\times\sigma=0$, we know that $\nabla\times\sigma$ is at every point perpendicular to σ. Now let us define a vector τ such that

$$\nabla\times\sigma=\sigma\times\tau\ . \tag{10}$$

Then τ is the position vector of points on the straight line represented by equation (10).

By a well known formula we may write

$$\nabla\cdot\sigma\times\tau\cdot\nabla\times\sigma-\sigma\cdot\nabla\times\tau\ . \tag{11}$$

But from (10)

$$\nabla\cdot\sigma\times\tau=\nabla\cdot\nabla\times\sigma=0\ , \quad \text{and} \quad \tau\cdot\nabla\times\sigma=\tau\cdot\sigma\cdot\times\tau=0\ .$$

Substituting the last two results in (11), we find that

$$\sigma\cdot\nabla\times\tau=0\ . \tag{12}$$

Hence, either

$$(a)\quad \sigma=0\ ,\quad \text{or}\quad (b)\quad \nabla\times\tau\perp\sigma\ ,\quad \text{or}\quad (c)\quad \nabla\times\tau=0\ .$$

But $\sigma=Pi+Qj+Rk$, and is not equal to zero. Furthermore, $\nabla\times\tau$ is not perpendicular to σ, for τ, as we have seen, is a variable vector, while σ has but one value at a given point.

Hence we conclude that

$$\nabla\times\tau=0\ .$$

It follows from Case I that $\tau\cdot d\rho$ is an exact differential, say, $df(x, y, z)$, and we may write

$$\tau\cdot d\rho=d\rho\cdot\nabla f\ , \quad \text{whence} \quad \tau=\nabla f\ .$$

Thus τ is equal to the gradient of a scalar function $f(x, y, z)$. For convenience, let us call this scalar function $\log f$ instead of f; then since

$$\nabla\log f=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\log f=i\frac{\frac{\partial f}{\partial x}}{f}+j\frac{\frac{\partial f}{\partial y}}{f}+k\frac{\frac{\partial f}{\partial z}}{f}=\frac{\nabla f}{f}\ ,$$

we have
$$\tau = \frac{\nabla f}{f} . \tag{13}$$

Substituting this value of τ in (10), we get

$$f \nabla \times \sigma = \sigma \times \nabla f , \quad \text{or} \quad f \nabla \times \sigma + \nabla f \times \sigma = 0 , \quad \text{or} \quad \nabla \times (f\sigma) = 0 .$$

Hence, by Case I, $f\sigma \cdot d\rho$ is an exact differential.

Thus it has been proved that $\sigma \cdot \nabla \times \sigma = 0$ is also the sufficient condition that $\sigma \cdot d\rho$ may be made an exact differential by means of a factor $f(x, y, z)$.

4. Conclusion. From the foregoing we see that when $\sigma \cdot \nabla \times \sigma = 0$ because $\nabla \times \sigma = 0$, equation (1) is an exact differential equation. On the other hand, when $\sigma \cdot \nabla \times \sigma = 0$, not because $\nabla \times \sigma = 0$, but because $\nabla \times \sigma$ is perpendicular to σ, there exists an integrating factor $f(x, y, z)$ which will make equation (1) an exact differential equation. In other words, the necessary and sufficient condition that equation (1) be integrable is $\sigma \cdot \nabla \times \sigma = 0$.

COROLLARY 1. *If*

$$\sigma \cdot d\rho = M dx + N dy ,$$

where M and N are scalar functions of x and y, the necessary and sufficient condition that $Mdx + Ndy = 0$ be an exact differential equation is

$$\nabla \times \sigma = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \right) \times (Mi + Nj) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) k = 0 ,$$

that is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} .$$

COROLLARY 2. *Since*

$$\sigma \cdot \nabla \times \sigma = (Mi + Nj) \cdot \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) k = 0 ,$$

and this is true for all values of σ, it follows that the equation $Mdx + Ndy = 0$ can always be made integrable by means of an integrating factor $f(x, y)$.

ON PARAMETRIC AND PSEUDOGRAPHIC TRANSFORMATIONS

By F. W. REED, Ohio University

1. Definition of the parametric transformation. If we apply a rotation θ to the rectangular axes, with reference to which the curve $x = f_1(\varphi)$, $y = f_2(\varphi)$ (φ, a parameter) is defined, under the condition that θ be a function of φ, then on eliminating φ we arrive at a new curve (or curve family) $x' = F_1(\theta)$, $y' = F_2(\theta)$.

Such a transformation we shall call a parametric rotation. Specializing by making θ constant we have an ordinary rotation of analytic geometry, the result being the same curve in new position. If instead of a rotation a translation $x'=x+f_3(\theta)$ $y'=y+f_4(\theta)$ is applied the operation is called a parametric translation. It is easy to see that functions $f_3(\theta)$ and $f_4(\theta)$ can be found that will carry any given curve (or point) $x=f_1(\varphi)$, $y=f_2(\varphi)$ over into any other given curve (or point) $x'=F_1(\theta)$, $y'=F_2(\theta)$ whatever the relation $\varphi=f_5(\theta)$. The arbitrary character of $f_5(\theta)$ permits restriction of the range of φ or of θ or of both. If $y=f(x)$ is to become $y'=F(x')$ then $f_1(\varphi)$ and $F_1(\theta)$ are rendered aribtrary.

Any curve that can be generated by a parametric rotation can also be generated from the same original curve by a parametric translation. The converse of this is not true.

We are not limited in the general transformation to combinations thus elementary. We may define more generally a parametric transformation of $x=f_1(\varphi)$, $y=f_2(\varphi)$ by means of $\varphi=f(\theta)$ into $x'=F_1(f_1, f_2, \theta)$, $y'=F_2(f_1, f_2, \theta)$.

2. The generation of certain trochoidal curves. The circle

$$x=1+r\cos\varphi\ , \qquad y=r\sin\varphi$$

referred to new axes with which the old axes make angles of θ becomes (omitting the customary primes)

$$\begin{aligned} x&=(1+r\cos\varphi)\cos\theta-r\sin\varphi\sin\theta=\cos\theta+r\cos(\theta+\varphi)\ ,\\ y&=(1+r\cos\varphi)\sin\theta+r\sin\varphi\cos\theta=\sin\theta+r\sin(\theta+\varphi)\ , \end{aligned} \tag{1}$$

or, on putting

$$\varphi=k\theta\ ,\quad x=\cos\theta+r\cos(1+k)\theta\ ,\quad y=\sin\theta+r\sin(1+k)\theta\ . \tag{2}$$

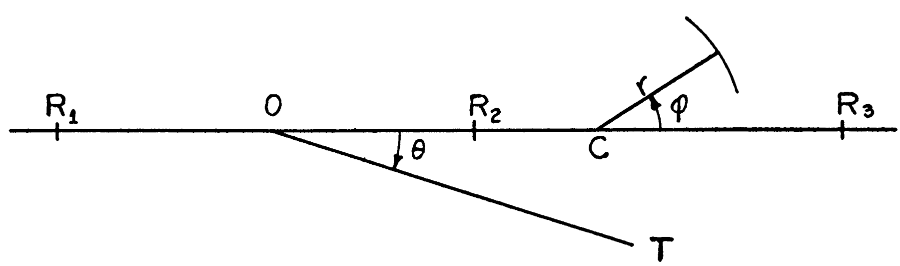


FIG. 1

Figure 1 shows the initial line OC, an arc of the circle referred to it, the rotating line OT which is the x-axis of the derived curves. φ is positive counterclockwise, θ positive clockwise. The line segment OC when divided in the ratio k to 1 gives left external, internal, right external points as R_1, R_2, R_3. The scheme be-

low in columns gives type of division of OC, definition of k, range of k, type of trochoidal curve:

$$R_1\,,\qquad k=\frac{OR_1}{R_1C}\,,\qquad -1<k<0\,,\qquad P\,,\,p\,;$$

$$R_2\,,\qquad k=\frac{OR_2}{R_2C}\,,\qquad 0<k<\infty\,,\qquad E\,,\,e\,;$$

$$R_3\,,\qquad k=\frac{OR_3}{R_3C}\,,\qquad -\infty<k<-1\,,\qquad H\,,\,h\,.$$

Generally then, omitting subscripts, $k=\frac{OR}{RC}$, $r=|CR|$, $OC=1$. The initials P, E, H, p, e, h refer to para-, epi-, and hypotrochoid and para-, epi-, and hypocycloid. The special cases enter thus:

$$P \text{ becomes } p \text{ when } k=-\left|\frac{r-1}{r}\right|=\frac{1-r}{r} \text{ since } r>1\,,$$

$$E \text{ becomes } e \text{ when } k=\frac{1-r}{r}\,, \text{ here } r<1\,.$$

$$H \text{ becomes } h \text{ when } k=-\frac{1+r}{r}\,.$$

In checking over these details it is necessary to have in mind that the tracing point is moving in a fixed circle and the paper in contact with the tracing point is being rotated about O.

3. Three theorems.[1] The curve given by equation (2) is (except as to size) identified with the curve

$$x'=\cos\theta+r'\cos(1+k')\theta\,,\quad y'=\sin\theta+r'\sin(1+k')\theta\,, \tag{2'}$$

where

$$r'=\frac{1}{r}\,;\qquad 1+k'=\frac{1}{1+k}\quad\text{or } k'=-\frac{k}{1+k}\,.$$

The corresponding ranges of k and k' together with the type of trochoid is thus tabulated:

$$-\infty<k<1\,,\ H \quad \text{goes into } -1>k'>-\infty\,,\ H\,;$$

$$-1<k<0\,,\ P \quad \text{goes into } \infty>k'>0\,,\ E\,;$$

$$0<k<\infty\,,\ E \quad \text{goes into } 0>k'>-1\,,\ P\,.$$

[1] Theorems I and III, stated here in the very language of the classical theory, are not new. See Loria, *Spezielle algebraische und transzendente ebene Kurven*, vol. II, p. 96. The articles "Cuspidal Rosettes," by W. F. Rigge, MONTHLY, (*1919*, 332–340), and "The General Theory of Cyclic-Harmonic Curves," by R. E. Moritz, *Annals of Mathematics*, vol. 23, 1921-22, may be consulted for content and for references.

So we can write

THEOREM I. *Every H can be generated in two ways; and every P also as an E, and conversely. The same holds, mutatis mutandis, for h, p, e.*

The related curves (2) and (2′) while always of the same shape differ usually as to size and rate of description. Changing units by multiplying the right members of (2′) by the factor r the resulting curve is of the same size as (2).

If we apply to the curve $x=f_1(\varphi)$, $y=f_2(\varphi)$ the parametric rotation θ_1 and to the resulting curve the parametric rotation θ_2 the final curve is the same as would have resulted from the application of the parametric rotation $\theta_1+\theta_2$ to the original curve. This states the group property of these rotations. As applied to the trochoids we have

THEOREM II. *The trochoids are a group of curves such that any one can generate any other by means of a parametric rotation.*

This property is seen best by imagining a stack of three sheets of paper, the upper two being carbon backed, on the upper and fixed sheet of which an arc φ of a circle is described while the lower sheets rotate through angles θ_1 and $\theta_1+\theta_2$ about a pin, the relations $\varphi=k_1\theta_1=k_2(\theta_1+\theta_2)$ being maintained.

The definition of k shows that R corresponds to the contact point between the fixed circle (radius OR) and the rolling circle (radius CR) used in the classic theory. If we change units from $OC=1$ to $OR=1$ the first equations in the pairs (2), (2′) become respectively

$$x=\frac{k+1}{k}\left(\cos\theta+r\cos(1+k)\theta\right),$$

$$x'=\frac{k'+1}{k'}\left(\cos\theta+r'\cos(1+k')\theta\right)=\frac{1}{rk}\left(r\cos\theta+\cos\frac{1}{1+k}\cdot\theta\right).$$

Here x and x' become identical in size when

$$\frac{k+1}{k}=-\frac{1}{rk},\quad\text{or}\quad k=-\frac{1+r}{r},$$

that is when (2) and (2′) are related hypocycloids. But since the relation defining k' can be written

$$\frac{1}{-k}+\frac{1}{-k'}=1,$$

where k and k' are both negative, and since $(1/k)$, $(1/k')$, are the radii of rolling circles when the radius of the fixed circle is unity, we see that the sum of the radii of the two circles which are used to draw the same hypocycloid is the radius of the fixed circle. In making the same argument for the related epitrochoid and paratrochoid the sign of x' should be changed by using $-(k'+1)/k'$ for $(k'+1)/k'$. Thus we shall have

THEOREM III. *If ρ_1, ρ_2, ρ_3 are radii of circles C_1, C_2, C_3 and $\rho_1+\rho_2=\rho_3$ then with C_3 fixed C_1 and C_2 can generate identical hypocycloids, and with C_1 fixed the epicycloid given by C_2 can be identical with the paracycloid given by C_3.*

4. The rosettes, case of r = 1. If $r=1$ equations (2) become

$$x=\cos\theta+\cos(1+k)\theta=2\cos\frac{2+k}{2}\cdot\theta\cos\frac{k\theta}{2},$$

$$y=\sin\theta+\sin(1+k)\theta=2\sin\frac{2+k}{2}\cdot\theta\cos\frac{k\theta}{2}.$$

Put

$$\frac{2+k}{2}\cdot\theta=\Theta,$$

then

$$x=2\cos\Theta\cos\frac{k\Theta}{2+k}=\rho\cos\Theta,\qquad y=2\sin\Theta\cos\frac{k\Theta}{2+k}=\rho\sin\Theta,$$

where

$$\rho=2\cos\frac{k\Theta}{2+k}.$$

Thus ρ, Θ are true polar coordinates and the relation last written is the equation of the rosettes in polar form. By choosing $n=k/(2+k)$ and again $-n=k'/(2+k')$ we can verify

THEOREM IV. *The equation of the rosettes $\rho=2\cos n\theta$ is derived from the equation of the circle $x=1+\cos\varphi$, $y=\sin\varphi$ by the parametric rotation defined by $k=2n/(1-n)$ or that defined by $k'=-2n/(1+n)$.*

5. Devices for studying the rosettes. Consider the equation $\rho=\cos p\theta/q$ (ρ, θ being polar coordinates, and p/q a fraction in lowest terms). Four methods will now be stated for setting up the curve or its equivalent.

(*a*) Pointwise construction is an obvious method. For stated or constructed values of θ values of ρ can be taken from tables or found by projections. Usually the nature of the curve is amply revealed by taking $\theta=kq\pi/(2p)$, $k=0, 1, 2, \cdots$.

(*b*) The idea of the parametric rotation just discussed can be carried out mechanically by simple gearing.

(*c*) A model can be made of paper leaves each having the outline of a complete loop, superposed and glued together in order. A ring of staples set radially at the vertices of the proposed rosette and a second ring set tangentially to a circle of its double points (case of $q>1$) would furnish a housing for a wire model.

(*d*) By using concentric circles of radii $1, 2, \cdots, q$ and radial lines dividing each quadrant into p equal angles, and connecting by a smooth curve the points (ρ, θ) as polar coordinates where $\theta=i\pi/2p$, $\rho=q-i$, $i=0, 1, \cdots, 2q$, we shall

have drawn the pseudograph (see section 8) of the rosette $\rho=\cos\ (p/q)\cdot\theta$ from first to second vertex, and further procedure is evident. By letting i vary continuously the ρ, θ definitions constitute the polar parametric equations of a simple spiral. It is evident that a figure giving the pseudograph of $\rho=\cos(p/q)\cdot\theta$ gives at the same time the nested pseudographs $\rho=\cos(p/j)\cdot\theta$, $j=1, 2, \cdots, q-1$. There is no requirement here that p/j be in lowest terms, or even that j be an integer.

6. Parametric rotations applied to polar curves. The polar curve

$$\rho=f(\varphi)\ , \qquad \varphi \text{ the polar angle,} \tag{3}$$

becomes in rectangular coordinates, parametric form,

$$x=f(\varphi)\cos\varphi\ , \qquad y=f(\varphi)\sin\varphi\ .$$

Applying the counterclockwise rotation θ to the polar axis we have

$$x=f(\varphi)\cos(\varphi+\theta)\ , \qquad y=f(\varphi)\sin(\varphi+\theta)\ .$$

Making the rotation parametric by putting $\varphi=k\theta$, then $(1+k)\theta=\Theta$, we have

$$\begin{aligned} x&=f(k\theta)\cos(1+k)\theta=f\left(\frac{k\Theta}{1+k}\right)\cos\Theta\ ,\\ y&=f(k\theta)\sin(1+k)\theta=f\left(\frac{k\Theta}{1+k}\right)\sin\Theta\ . \end{aligned} \tag{4}$$

Here Θ is a polar angle and the polar equation of the curve is written

$$\rho=f\left(\frac{k\Theta}{1+k}\right)\ . \tag{5}$$

Of course (5) can be derived directly from (3) using the geometrically evident relation $\Theta=\varphi+\theta$ and the equation $\varphi=k\theta$ defining the rotation. Thus (5) is the family of curves derived from (3) by parametric rotation. Equations (4) (in Θ) show the type of curves in rectangular coordinates that can be derived from a polar curve by parametric rotation.

The polar curve

$$\rho=f_1(\varphi)\ , \qquad \tau=f_2(\varphi)\ , \qquad \varphi=\text{a parameter}\ ,$$

becomes, under the parametric rotation $\varphi=k\theta$,

$$\rho'=f_1(k\theta)\ , \qquad \tau'=f_2(k\theta)+\theta\ .$$

7. Examples of parametric transformations. The rotation $\varphi=a\theta$ applied to the line $x=a$, $y=-\varphi$ gives the evolute of the circle $x=a\cos\theta+a\theta\sin\theta$, $y=a\sin\theta-a\theta\sin\theta$. The rotation $\varphi=a\theta$ applied to the line $x=\varphi$, $y=0$ gives the spiral $x=a\theta\cos\theta$, $y=a\theta\sin\theta$ or $\rho=a\theta$; or starting from the polar para-

meter form $\rho=\varphi$, $\tau=0$ the result is $\rho=a\theta$, $\tau=\theta$ or $\rho=a\tau$. The rotation $\varphi=k\theta$ applied to the straight line segment $x=a+b\cos\varphi\cos\alpha$, $y=b\cos\varphi\sin\alpha$, $\alpha=$ const., gives the curve $x=a\cos\theta+b\cos k\theta\cdot\cos(\alpha+\theta)$, $y=a\sin\theta+b\cos k\theta\cdot\sin(\alpha+\theta)$. For $\alpha=0$ the result, in polar coordinates, is $\rho=a+b\cos k\theta$.

The line segment $x=0$, $y=\sin\varphi$ translated by the relations $x'=x+\theta$, $y'=y$, $\varphi=k\theta$ gives the sine curve $x'=\theta$, $y'=\sin k\theta$ or $y'=\sin kx'$. The line segment $x=0$, $y=\sin\varphi$ translated by $x'=x+\sin(\theta-a)$, $y'=y$, $\varphi=k\theta$ gives the curve $x'=\sin(\theta-a)$, $y'=\sin k\theta$. The circle $x=-b\sin\varphi$, $y=-b\cos\varphi$ translated by $x'=x+\theta$, $y'=y$, $\varphi=k\theta$ gives the trochoid $x'=\theta-b\sin k\theta$, $y'=-b\cos k\theta$.

8. The pseudographic transformation. We have already defined the pseudograph of the rosette. The path curves used were segments of spirals and the passage from one to another was made by a reflection at the circle $\rho=q$. We shall now set up a device by means of which such specifications are carried out automatically.

Consider the two lineo-trigonometric functions *sinl* (read 'linear sine') and *cosl* defined by the relations

$$\begin{aligned}
\operatorname{sinl}(4k+\theta)&=0+\theta\,, & \operatorname{cosl}(4k+\theta)&=1-\theta\,,\\
\operatorname{sinl}(4k+1+\theta)&=1-\theta\,, & \operatorname{cosl}(4k+1+\theta)&=0-\theta\,,\\
\operatorname{sinl}(4k+2+\theta)&=0-\theta\,, & \operatorname{cosl}(4k+2+\theta)&=-1+\theta\,,\\
\operatorname{sinl}(4k+3+\theta)&=-1+\theta\,, & \operatorname{cosl}(4k+3+\theta)&=0+\theta\,,
\end{aligned}\tag{6}$$

where θ is a proper fraction or zero, k is an integer. The argument of these functions is the measure around the square with vertices at $(1,0),(0,1),(-1,0),(0,-1)$ in the order and from the point indicated, the side being taken as unit. Thus we set up a one-to-one correspondence between the values of $\sin\varphi$ and $\operatorname{sinl}\theta$ ($\cos\varphi$ and $\operatorname{cosl}\theta$) by means of the above definitions and the linear correspondence between their arguments $\varphi=\pi\theta/2$. The operation of changing from $\sin\varphi$, $\cos\varphi$ to $\operatorname{sinl}\theta$, $\operatorname{cosl}\theta$ we shall call a pseudographic transformation of the former function. Applied to these functions as they enter the equation or equations of a curve the operation will be called the pseudographic transformation of the curve and the resulting locus will be called the pseudograph of the curve.

Those formulas in trigonometry which depend only on the principle of equal triangles have here unique counterparts, for example $\operatorname{sinl}(-\theta)=-\operatorname{sinl}\theta$, $\operatorname{sinl}(1-\theta)=\operatorname{cosl}\theta$, $\operatorname{sinl}(2+\theta)=-\operatorname{sinl}\theta$. But $\sin^2\varphi+\cos^2\varphi=1$ has as its analogue $\pm\operatorname{sinl}\theta\pm\operatorname{cosl}\theta=1$ according to case, likewise the addition formulas fail to have unique counterparts.

9. Examples of the pseudograph. (*a*) The curve $y=\sin mx$ has for a pseudograph $y=\operatorname{sinl} mx$ and consists of the line segments connecting the successive maxima and minima of the original sine curve.

(b) The circle $x^2+y^2=1$ put in the particular parametric form $x=\cos\varphi$, $y=\sin\varphi$ has as its pseudograph the defining square used in the previous section.

(c) Consider now the Lissajous' curve

$$x=a\cos m\theta\ ,\qquad y=b\cos(n\theta-\alpha)\ , \tag{7}$$

m,n, integers relatively prime, (which includes the parabola and the ellipses). When θ is eliminated the degree of the resulting equation is given by the scheme: for $\alpha=0$, degree is n in x, m in y, and larger is total degree; for $\alpha\neq 0$, degree is $2n$ in x, $2m$ in y, and larger is total degree. The pseudograph of (7) is

$$x=a\ \mathrm{cosl}\ m\theta\ ,\qquad y=b\ \mathrm{cosl}\ (n\theta-\alpha)\ . \tag{7'}$$

The fact that θ is proportional to the distance passed over along the pseudograph (7′) makes it possible to pick out values of θ at vertices, double points, etc., with ease, and the correspondence between values of $\cos\theta$ and $\mathrm{cosl}\ \theta$ insures the existence of the same type of points on (7) identical as to number and order of distribution. The α-periodicity (change in α to give the same curve) can be read from the pseudograph.

By means of a dilatation on one of the coordinates in (7′) so that the new enclosing rectangle is similar to that included by the cushions of a billiard table a pseudograph is represented by the path of a billiard ball. This amounts to the simple expedient of choosing units so that the billiard table has dimensions $2a$ and $2b$, and is justified for the reason that the parameter θ (and not the x, y coordinates of (7′)) is used in computing the right members of (7).

With this justification we may change (7′) to

$$x=n\ \mathrm{cosl}\ m\theta\ ,\qquad y=m\ \mathrm{cosl}\ (n\theta-\alpha)\ .$$

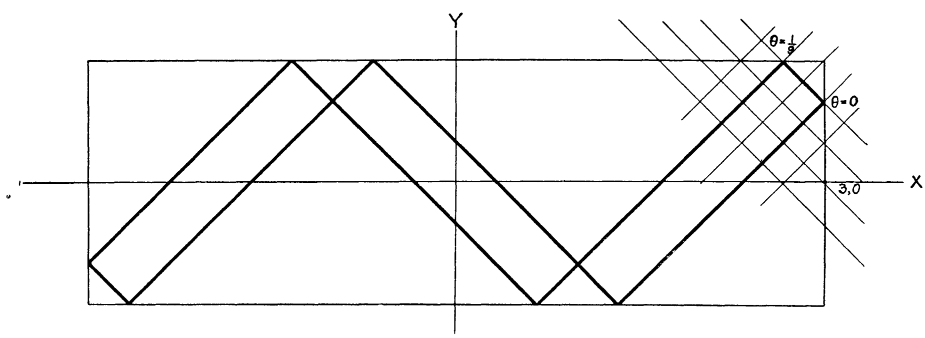


FIG. 2

We choose as origin a corner of a square on squared paper, axes running diagonally, and draw with unit diagonal as unit, the rectangle bounded by $x=\pm qn$, $y=\pm qm$ where q is the denominator of the fraction $m(1-\alpha)$ in lowest terms. Then (7′) is drawn by following the printed lines. If q is not conveniently small we change the plan and give it the value unity, but in this case the tracing

is done parallel to and not along the printed lines. Figure 2 is a pseudograph $x=3$ cosl θ, $y=$cosl $(3\theta-1/3)$ of the curve $x=a\cos\theta$, $y=b\cos(3\theta-\pi/6)$. The figure suggests how a ribbon of paper can be folded so that the pseudograph can be read around its edge.

10. Mechanical graphs. Without exception the curves met with in this discussion can be drawn mechanically.[1] The method is to select a motion for the tracing point and a motion for the paper from the elementary motions: (L) straight line, (R) rotary, (H) harmonic, any of which is easily taken from a master rotary motion. The two selected are to be geared. This gearing is the analogue of the relation $\varphi=k\theta$ in the theory of parametric transformations.

A CERTAIN QUARTIC SURFACE AND ITS REFLECTING PROPERTIES

By A. R. WILLIAMS, University of California

The quartic surface which is the subject of this paper was in effect discovered, and its principal properties described, without the use of equations, by Mrs. M. W. Caughlan of Oakland, California. She was at the time designing an automobile headlight.

1. Definition and equation. The surface in which she was particularly interested is most simply defined as the locus of points the sum of whose distances from a point F and a line l is a constant, say $2a$. Evidently the section of this surface by the plane through F perpendicular to l is an ellipse with major axis $2a$, one focus being at F and the other at F_1, where l meets the plane in question. It is easily seen also that the section of the surface by the plane determined by F and l consists of portions of two parabolas, both having F for focus and opening in opposite directions. Their vertices coincide with the vertices of the elliptical section just mentioned, and they both pass, of course, through the two points on l which are distant $2a$ from F. Using

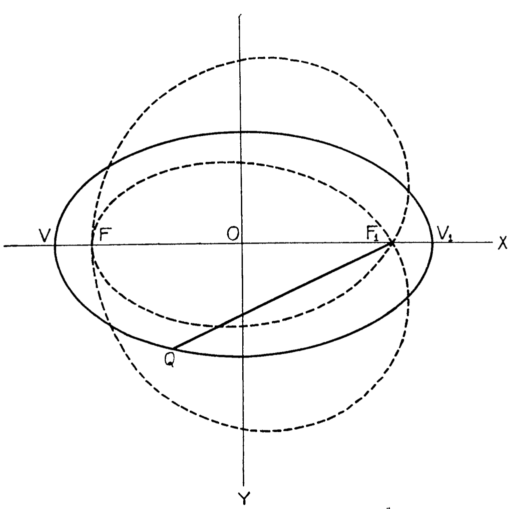


FIG. 1. Section by the plane $z=0$.

[1] The very comprehensive machine designed by W. F. Rigge is described in *Scientific American Supplement*, February 9 and 16, 1918.

3. Locus of the foci of the parabolic sections. These parabolic sections by planes through l have their vertices and foci in the central plane through F perpendicular to l, *i.e.* in the plane $z=0$. The locus of the foci is easily obtained and is an interesting curve shown by the dotted line in Figure 1. Thus in that figure the distance F_1Q is $a(1-e^2)/(1+e\cos\theta)$, where e is the eccentricity, d/a, of the ellipse, and θ is the angle V_1F_1Q. Since the parabolic section whose vertex is Q passes through M and N, where $NF_1=F_1M=2\sqrt{a^2-d^2}=2a\sqrt{1-e^2}$ (see Fig. 2), the distance, p, of the focus from the vertex is given by $4p(a(1-e^2)/(1+e\cos\theta))=4a^2(1-e^2)$. Hence $p=a(1+e\cos\theta)$, and the polar equation of the locus of the foci, the pole being at F_1, is given by

$$r=\frac{a(1-e^2)}{1+e\cos\theta}-a(1+e\cos\theta)$$

which in rectangular coordinates is

$$(x^2+y^2)^2[(1-e^2)x^2+y^2]+2aex(x^2+y^2)[(1-e^2)(2x^2+y^2)+y^2]$$
$$+[\sqrt{1-e^2}(2x^2+y^2)+y^2][\sqrt{1-e^2}(2x^2+y^2)-y^2]=0\ .$$

Here the origin is at F_1 and the Y axis is parallel to OY in Figure 1. This sextic is of course unicursal, since it is in 1:1 correspondence with the points of the ellipse. As is obvious from the equation, the quadruple point at the origin has four distinct tangents (two being imaginary), and so counts for six double points. There is a double point with distinct tangents at each of the circular points, and the tacnode at F gives two more double points, making ten in all. Moreover, since each parabolic section by a plane through l is a characteristic on our surface, and congruent to the generating parabola of the paraboloid on which it lies, we see that the distance between its vertex and focus is equal to the distance of the vertex of the corresponding paraboloid from the common focus F. Hence, the locus of the vertices of the family of paraboloids is the limaçon

$$r=a(1+e\cos\theta)\ .$$

Here the pole is at F, and θ is measured from OX in the same sense as before.

ON THE DOUBLE LAYER POTENTIAL[1]

By V. C. POOR, University of Michigan

1. Introduction. The convergence of the double layer potential is of fundamental importance in mathematical physics, in particular in the theory of

[1] Presented to the Am. Math. Soc., Jan. 2, 1926.

magnetism and in the modern proof of "Dirichlet's Principle" through integral equations. A proof of the convergence of the double layer potential is given by Plemelj[1] and a suggested method of proof by Goursat.[2] Students of potential theory should find considerable satisfaction in the following simple proof of so fundamental a theorem.

2. Statement of problem. In the double layer potential

$$\Psi=\int_\sigma \frac{\Phi\cos\theta\, d\sigma}{r^2},$$

where the integral is taken over a finite surface area σ, the density factor Φ is the strength of the shell or layer; r is the distance between two points M and P, the former being the point of integration so that Ψ is a function of the point P; θ is the angle between the normal to the surface σ at M and the vector drawn from M to P. Evidently Ψ is a continuous single valued function of the point P for P outside the surface σ; but if P is a point of the surface the integrand becomes infinite and to the second degree when the point of integration coincides with the point P. Our problem is to show that in this case the integral converges to a unique finite value.

3. Convergence proof. We shall assume that the surface, σ, is regular, *i.e.*, that there is a unique tangent plane and two distinct radii of curvature at every point of the surface. The surface has naturally a positive and negative face depending on the direction of the intensity of magnetization. If we select a point P_0 of the surface and cut the surface by a secant plane perpendicular to the normal at P_0 this plane will slice off a cap which we will call C_1; the remaining portion of the surface $\sigma-C_1$ we will designate as σ_1. Then the potential at a point P external to the double layer breaks up into two integrals so that

$$\Psi=\int_\sigma \frac{\Phi\cos\theta\, d\sigma}{r^2}+\int_C \frac{\Phi\cos\theta\, d\sigma}{r^2}. \tag{1}$$

The cap and that portion of the plane outside the boundary of the cap divides space into two parts. A point P will be considered as outside (on the positive side) or inside (on the negative side) according as it lies in that portion of space adjacent to the positive or negative face of the cap. The first integral in (1) is the potential at P due to the shell σ_1. The value of the second integral may be obtained by circumscribing a unit sphere about P as centre. Then the

[1] J. Plemelj: Ueber linear Randwertaufgaben der Potential Theorie. *Monatshefte für Mathematik und Physik.* (1904) vv. 15–16: 355.

[2] E. Goursat. *Cours d'Analyse Mathêmatique.* III. (1915) p. 251.

Also see M. A. Liapounoff: Sur certaines questions qui se rattachent au problème de Dirichlet. *Jour. de Math.* t.4 (1898)

solid angle $d\Omega$ of the cone whose apex is P and whose base is $d\sigma$, is given by the equation

$$\frac{d\Omega}{1} = \frac{d\sigma \cos\theta}{r^2},$$

the solid angle being also taken as positive or negative according as P lies on the positive or negative side of the cap. If Φ' and Φ'' are the least and greatest values of Φ at points of the cap then for some intermediate value, Φ_1, of Φ the potential at P due to the cap becomes $\Phi_1\Omega_1'$, where Ω_1' is the solid angle at P subtended by the boundary of the cap. The potential at P is thus given by the equation

$$\Psi^{(1)} = \int_{\sigma_1} \frac{\Phi \cos\theta \, d\sigma}{r^2} + \Phi_1\Omega_1' \,. \tag{2}$$

Had we chosen a different cap C_2 cut off by a parallel plane the potential at P, using the index 2, evidently would have been

$$\Psi^{(2)} = \int_{\sigma_2} \frac{\Phi \cos\theta \, d\sigma}{r^2} + \Phi_2\Omega_2' \,. \tag{3}$$

But $\Psi^{(1)}$ and $\Psi^{(2)}$ are equal for every P outside the double layer. So if on some curve drawn on the positive side of the cap to P_0 we select a set of points P_i having P_0 for its limit, then the terms of the sequences $\Psi^{(1)}(P_i)$ and $\Psi^{(2)}(P_i)$ will always be identical; the sequences will therefore have the same limit. If we call this limit Ψ_e then in the former sequence the solid angle Ω_1', approaches Ω_1, the solid angle at P_0 subtended by the boundary of the cap C_1, and similarly Ω_2' will become Ω_2 so that in the limit as P approaches P_0

$$\Psi_e = \int_{\sigma_1} \frac{\Phi \cos\theta \, d\sigma}{r^2} + \Phi_1\Omega_1 \,, \quad \text{or} \quad \Psi_e = \int \frac{\Phi \cos\theta \, d\sigma}{r^2} + \Phi_2\Omega_2 \,. \tag{4}$$

In fact this same argument and the same conclusion holds if the cap C_2 is of zero area, in this case Φ_2 is Φ_0 the value of Φ at P_0 while the solid angle Ω_2 is zero. Thus if we indicate the distance from P_0 to some secant plane by ϵ, we may then say that Ψ is independent of ϵ. We may thus choose form (4) for Ψ_e which is unique, finite and independent of ϵ and let ϵ approach zero. The solid angle Ω_1, evidently approaches 2π as its limit, the mean density factor Φ_1 approaches Φ_0 as its limit, while the area of integration σ_1 approaches the area of the shell as ϵ approaches zero, or in the limit we will have

$$\Psi_e = \int_{\sigma} \frac{\Phi \cos\theta \, d\sigma}{r^2} + 2\pi\Phi_0 \,. \tag{5}$$

Since Ψ_e and $2\pi\Phi_0$ are each finite and unique the integral over the surface σ is finite and unique. This integral is the double layer potential at P_0 and it differs from the value approached by the potential by $2\pi\ \Phi_0$ when the approach is towards the positive face. Had the approach been towards the negative face the solid angle would have been negative and the value Ψ_i approached by the potential as P approaches P_0 will in this case be given by the equation

$$\Psi_i=\int_\sigma \frac{\Phi\cos\theta\, d\sigma}{r^2}-2\pi\Phi_0\ . \tag{6}$$

4. **Summary.** We thus see that the double layer potential has a unique finite value at every point of space. It is discontinuous at points of the double layer, since on approaching a point, P_0, of the shell the potential function approaches its value at the point plus or minus $2\pi\Phi_0$ depending on whether the point is approached from the positive or negative side of the shell.

If we combine equations (5) and (6) we obtain the sum and difference of potential or

$$\Psi_e-\Psi_i=4\pi\Phi_0\ ,\quad \Psi_e+\Psi_i=2\int_\sigma\frac{\Phi\cos\theta\, d\sigma}{r^2}\ .$$

Equations (5) and (6) may be looked upon as integral equations in Φ where Ψ_e and Ψ_i take on prescribed values at the surface.

QUESTIONS AND DISCUSSIONS.

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. NEW ANNUITY FORMULAS FOR PAYMENTS MADE BETWEEN CONVERSION DATES

By H. W. SIBERT, University of Cincinnati

1. The formulas now used for annuities having payments between the conversion dates are based upon an assumption which does not accord with the usage of banks and other financial agencies. In the development of the formulas it has been assumed that each payment made between conversion dates receives compound interest from date of payment and from this assumption is developed a conversion factor, i/j_p, by which $R(a_{\overline{n|}}$ at $i)$ and $R(s_{\overline{n|}}$ at $i)$ must be multiplied in order to give A and S. The common usage of banks and other financial

agencies is to credit simple interest from date of payment to the next conversion date on each payment made between conversion dates and it is upon this assumption that the present paper will develop a conversion factor which might profitably replace i/j_p.

The following notation is that commonly used in annuity formulas, viz:

R, payment or rent per interest period.

p, number of payments per interest period.

i, interest per interest period.

n, number of interest periods.

A, present value of an annuity.

S, accumulated value of an annuity at the end of the period.

($a_{\overline{n|}}$ at i), present value of an annuity when payment of \$1 is made at the end of each interest period.

($s_{\overline{n|}}$ at i), accumulated value of an annuity when a payment of \$1 is made at the end of each interest period.

$$(a_{\overline{n|}}\text{ at } i)=\frac{1-(1+i)^{-n}}{i}\,. \quad (s_{\overline{n|}}\text{ at } i)=\frac{(1+i)^n-1}{i}\,. \quad j_p=p[(1+i)^{1/p}-1]\,.$$

2. Each payment $=R/p$. For ordinary annuities there is no payment made at the beginning of the annuity but there is one made at its very end so the first payment in any interest period will receive simple interest for $(p-1)/p$ interest periods, the second for $(p-2)/p$ periods, the next to last for $1/p$ periods, and the last for 0 periods. Thus the total interest received on the p payments during any interest period will be

$$\frac{R}{p}\left[\frac{p-1}{p}+\frac{p-2}{p}+\cdots+\frac{1}{p}+0\right]i=\frac{R}{p^2}\,i\,[(p-1)+(p-2)+\cdots+1+0]\,.$$

$$=\frac{R}{p^2}\,i\left[\frac{p(p-1)}{2}\right]=Ri\left[\frac{p-1}{2p}\right]\,.$$

The accumulated sum of these p payments at the end of any interest period will be

$$R+Ri\left[\frac{p-1}{2p}\right]=R\left[1+\frac{p-1}{2p}\,i\right]\,.$$

This last value is the total value of the payments as figured at the end of any interest period. Thus the annuity would have the same A and S if one payment only were made at the end of each interest period, provided this single payment equals $R[1+(p-1)i/2p]$.

Therefore for such an annuity

$$A=R\left[1+\frac{p-1}{2p}\,i\right](a_{\overline{n|}}\text{ at } i)\text{ and } S=R\left[1+\frac{p-1}{2p}\,i\right](s_{\overline{n|}}\text{ at } i)\,.$$

The formulas now in use give

$$A=R\frac{i}{j_p}(a_{\overline{n}|}\text{ at }i)\text{ and }S=R\frac{i}{j_p}(s_{\overline{n}|}\text{ at }i),\text{ where }\frac{i}{j_p}=\frac{i}{p[(1+i)^{1/p}-1]},$$

and is known as the conversion factor. The conversion factor corresponding to i/j_p which has been developed above is $[1+(p-1)i/2p]$. It should be noted that when $p=1$, both

$$\left[1+\frac{p-1}{2p}\,i\right]\text{ and }\frac{i}{j_p}\text{ will }=1\ .$$

3. An annuity due is one where there is a payment at the very beginning of the annuity but none at the end. In such an annuity the first payment in any interest period will receive interest for the entire interest period while the last will receive interest for $1/p$ interest periods. Thus the total interest received on the p payments at the end of any interest period is

$$\frac{R}{p}\left[1+\frac{p-1}{p}+\frac{p-2}{p}+\cdots+\frac{1}{p}\right]i$$

$$=\frac{R}{p^2}\,i\,[p+(p-1)+(p-2)+\cdots+1]=\frac{R}{p^2}\,i\left[\frac{p(p+1)}{2}\right]=Ri\left[\frac{p+1}{2p}\right].$$

The accumulated sum of these p payments at the end of any interest period is

$$R+Ri\left[\frac{p+1}{2p}\right]=R\left[1+\frac{p+1}{2p}\,i\right].$$

Thus the formulas for annuities due will be

$$A=R\left[1+\frac{p+1}{2p}\,i\right](a_{\overline{n}|}\text{ at }i)\text{ and }S=R\left[1+\frac{p+1}{2p}\,i\right](s_{\overline{n}|}\text{ at }i)\ .$$

The formulas now in use for annuities due give

$$A=R(1+i)^{1/p}\left[\frac{i}{j_p}\right](a_{\overline{n}|}\text{ at }i)\text{ and }S=R(1+i)^{1/p}\left[\frac{i}{j_p}\right](s_{\overline{n}|}\text{ at }i)\ ,$$

or else they are in forms which can be transformed into the two above formulas. Thus the conversion factor for annuities due as now used is $[(1+i)^{1/p}(i/j_p)]$ while the one developed above is $[1+(p+1)i/2p]$. It should be noted that when $p=1$ each of these will $=1+i$.

4. If i/j_p be expanded in powers of i it becomes

$$1+\frac{p-1}{2p}\,i-\frac{p^2-1}{12p^2}\,i^2+\cdots$$

This series is convergent when $p>1$ and $i<1$, which requirement is fulfilled for annuities having payments between conversion dates. The signs of this series alternate $+$ and $-$ and the terms steadily decrease in numerical value. Therefore i/j_p cannot differ from $[1+(p-1)i/2p]$, which is the corresponding factor in this article, by an amount greater than the third term of the series, which equals $-(p^2-1)i^2/12p^2$.

The factor$[(1+i)^{1/p}(i/j_p)]=i/p+i/j_p$. Using the above expansion for i/j_p, the factor $[(1+i)^{1/p}[i/j_p)]$ will equal $1+[(p+1)i/2p]-[(p^2-1)i^2/12p^2]+\cdots$. Thus the difference between $[(1+i)^{1/p}(i/j_p)]$ and $[1+(p+1)i/2p]$, which is the corresponding factor derived in this article, will be exactly the same as the difference between i/j_p and $[1+(p-1)i/2p]$.

The conversion factors derived in this article do not differ greatly from those now in use but it should be noted that these new conversion factors are based upon common practice in financial institutions. These new factors are much easier to calculate than those now in use. As these new factors are commensurable numbers they are easier to multiply by than are the factors now in use.

II. Functions of two Variables for Which the Double Integral does not Exist

By R. L. Jeffery, Acadia University

It is pointed out by E. W. Hobson[1] that even though a function of two variables is continuous in each variable separately, the double integral in the sense of Riemann, need not exist. This is obvious if the function is unbounded, but there seems to be nowhere in the literature a bounded function that illustrates Hobson's observation. In this note such a function is constructed.

The region of definition will be the unit square. We shall first construct on the square $A\equiv(-1,\ -1)$, $B\equiv(1,\ -1)$, $C\equiv(1,\ 1)$, $D\equiv(-1,\ 1)$, an auxiliary function $Q(x,\ y)$. Draw the line joining the origin to the point $C\equiv(1,\ 1)$. On the part of $ABCD$ which lies in the first quadrant and above or on this line (except the point C),

$$Q(x,y)=\frac{x(1-y)}{1-x}\ .$$

In the first quadrant and below this line,

$$Q(x,y)=\frac{(1-x)y}{1-y}\ ,\quad \text{and} \quad Q(1,1)=0\ .$$

On the parts of $ABCD$ which lie in the second, third, and fourth quadrants, $Q(x,\ y)$ is defined by reflecting the first quadrant in the y axis, in the origin,

[1] *Theory of Functions of a Real Variable*, 2d ed., vol. I, p. 486.

and in the axis of x respectively. It is easily verified that Q is bounded and continuous in each variable separately, and is zero on the bounding lines of the square $ABCD$. Furthermore, Q is discontinuous at each of the points A, B, C, and D, with saltus equal to unity.

We next place interior to the unit square a Harnack[2] set of squares, $S \equiv S_1, S_2, \cdots$, where ΣS_n, the total area of S, is equal to m, $0<m<1$. The corner points of the set of squares S we designate by E. Now, between the points of the square S_1 and the points of the square $ABCD$, set up a one-to-one correspondence by means of a projective transformation which carries A, B, C, and D, respectively into the corresponding corners of the square S_1. At a point of S_1 let $f(x, y)$ have the value of $Q(x, y)$ at the corresponding point of $ABCD$. On each of the remaining squares of the set S define $f(x, y)$ in the same manner. On the part of the unit square exterior to S let $f(x, y)=0$.

A consideration of the structure of the function Q, together with the fact that any line parallel to either axis can meet at most a finite number of the squares of the set S,[2] shows us that f is bounded and continuous in each variable separately. Also, since Q is discontinuous at each of the points A, B, C, and D, with saltus equal to unity, f is discontinuous at each point of the set E with saltus equal to unity. Hence, to prove the non-existence of the double integral of f, it is sufficient to show that the content of the set E is different from zero. This follows from the manner in which E was constructed. The Cantor nondense closed set G complementary to S on the unit square clearly has measure equal to $1-m$. Hence, since the measure and content of a closed set are the same, the content of G is $1-m$. But it is evident that every point of G is a limit point of E. Consequently, since the content of a set of points is the same as the content of its derivative,[3] the content of E is $\geqq 1-m$.

III. "Long" Division

By L. S. Dederick, University of British Columbia.

In this age of calculating machines it may not seem timely to discuss methods of computation with pencil and paper. But the machines are not always available. The repeated divisions called for in Professor Lehmer's note on cube roots and fifth roots in this department of the Monthly for August - September are of course meant for a machine. If, however, similar ones are to be performed on paper, they suggest with unusual force the wastefulness, both in writing and space, of the ordinary method of "long" division. They go straggling across the paper in a way entirely unjustified by the actual work necessary. Yet they are outside the realm usually assigned to "short" division.

[2] See Pierpont: *Theory of Functions of a Real Variable*, vol. II, §354–3.

[3] Hobson: *loc. cit.*, §118.

The following method for dividing by a number of two or more figures condenses the work in a manner somewhat similar to that of synthetic division in algebra, and demands only the very slightest increase in the mental operations involved. It may be described as follows. No partial product is written down. Instead, each figure of the first partial product is subtracted from the proper figure of the dividend as soon as found, and the remainder written immediately below this figure. In this subtraction, if "borrowing" is necessary, the effect is merely to add one to the amount "carried" in finding the next figure to the left in the partial product. The subtraction of the multiple of one figure involves precisely the same mental operation as that performed in "short" division. The addition of one to the figure "carried" is the only increase in the amount of mental arithmetic over that commonly performed.

The only other unusual feature is the manner of reading the remainders or partial dividends. The first must be read with its last figure in the line of the dividend and its preceding figures on the line below. Each subsequent remainder is obtained from the preceding as the first is from the original dividend. It will in general have most or all of its figures on different lines. As the operations on it, however, involve each figure separately, this is no great disadvantage. It constitutes the essential feature of the condensation. There is no difficulty in identifying the figures that belong to a particular remainder, as each is the last figure in its column at the time it is to be used. When the figure in any column is finally reduced to zero, the zero need not be written; but the last preceding figure should be underlined or marked in some other way to show that it is not to be read in the next remainder.

If space is left for the remainders amounting to one line for each figure of the divisor, the quotient may be written on the next line below. Its location on this line may be chosen in various ways, as for example so as to bring its decimal point under that of the dividend, or its first significant figure under that of the dividend. For the operation itself the most convenient placing is probably to put each figure of the quotient in the column of the last figure of the corresponding partial product. For repeated divisions, however, this would have the disadvantage of moving the significant figures continually to the right.

The following example exhibits the division of 35678 by 42.1 after obtaining each of the first three remainders, and also the completed work after obtaining twenty figures in the quotient.

42.1 | 35678	42.1 | 35678.0	42.1 | 35678.00	42.1 | 35678.000000000000000000
199	1994	19943	1994 3652678716372316335
	31	319	319 4583921 3 212
		1	12 311 1
8	84	847	847.45843230403800475

The first remainder or partial dividend is 1998, the second 3140, the third 1930, and so on.

The completed work occupies 5 lines as compared with 39 in the usual arrangement, and has the additional advantage that the quotient is suitably placed for any subsequent operation.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Einstein's Theory of Relativity. By MAX BORN, translated by H. L. BROSE, M.A. (Translation of third German edition, 1922). New York, E. P. Dutton and Co., 1924. xi+293 pages. Price $5.00.

Professor Born has written a thoroughly readable account, for non-specialists, of the Einstein theory, with particular emphasis on the history of physics and the restricted principle of relativity. He remarks in his preface that "the difficulty which persons not conversant with mathematics and physics experience in understanding the theory of relativity seems to me to be due for the most part to the circumstance that they are not familiar with the fundamental conceptions and facts of physics, in particular of mechanics." Accordingly, with a minimum of mathematics, the author proceeds from an apology for his occasional square roots to a clear explanation of those physical theories, now in part superseded or amended, out of which relativity evolved as naturally as have any of the greater physical theories. This presentation is avowedly prepared for that all-but-hypothetical class of educated laymen which includes neither ignoramuses nor those whose secondary and college training has been entirely literary or humanistic in the snobbish sense. To the latter the theory of relativity must remain a mystery among mysteries until such time as they are born again.

Assuming that his readers are neither idiots nor sloths, the author devotes 184 pages (about 3/5 of the book) to a careful exposition of so much as is required for his purpose of geometry and cosmology, the fundamental laws of classical mechanics, the Newtonian world-system, the fundamental laws of optics, and the fundamental laws of electrodynamics. Even for those already acquainted with the theory of relativity, this unified review of the physics and mathematics in which the special principle originated, will be of considerable interest. An admirable feature of the entire discussion is the exceptionally clean, clear-cut drawing of the diagrams. Many of these depict the physical phenomena represented in remarkably vivid fashion. This part of the book could be read with profit by college students and possibly, if it may be hinted without impropriety, by some pure mathematicians, whether they are interested or not in relativity. Often the clarity of the presentation is reminiscent of the style of Felix Klein. An occasional equation appears in an unfamiliar dress where the the translator has wisely refrained from altering formulas to their usual English equivalents, as for instance $K=mb$ for Newton's second law of motion. This however will cause no inconvenience to those who follow the text. The free use of "curl" and "div" in the electrodynamics may temporarily puzzle a freshman,

but the sooner he learns to think vectorially, the better for his physical soul. As elsewhere in this section the graphic explanations of these dark things are a material aid to the uninitiated. In fairness to the author it should be emphasized that there is not a differential or an integral sign in the book. As stated in the preface, higher mathematics (from the layman's standpoint) are taboo.

On the ground thus carefully cleared is built up in chapter VI the imposing and classical (but already somewhat antiquated) edifice of the special principle of relativity. Whether physically adequate or not at this period of scientific history, the special theory as created by Einstein and Minkowski, for aesthetic reasons if for no other, still repays the most careful scrutiny. It was here that theoretical physics first began, in 1905, to rub the sand of centuries out of its eyes. For an understanding of modern mathematical work on spectral series the special theory is indispensable. In certain parts of atomic physics the theory of special relativity still is creative, although for astronomical or cosmological investigations it is effete. The reader approaching this stage of relativity for the first time will find an easy introduction in Professor Born's book.

The last chapter is given over to Einstein's Theory of Relativity—the general theory. It is gratifying to see that this title is reserved for the heart of the subject, and not indiscriminately bestowed upon preparatory attempts which now have lost much of their first interest. Rather apologetically the author introduces the concept of space curvature "which usually excites aversion among non-mathematicians," and points out that no one except "the person of untrained mind" asks that "it be imagined; can invisible light be imagined, or inaudible tones?" Thus are the brickbats which many intensely conservative haters of general relativity hurl at the theory heaved back with compound interest. Relativity no longer is on the defensive.

Elsewhere in the book the author is almost impolite to those "simple minds" who, seeing an imaginary difficulty in the relativistic determination of the length of a rod, "condemn the theory of relativity at one stroke." These difficulties are magnified a thousand-fold for the unsympathetic spectator who essays a birdseye view of the general theory from the illusive heights (or depths) of common so-called sense. The sense requisite for an appreciation of this final chapter is anything but common, and it is doubtful whether any but laboriously trained mathematicians will see what much of it means. For example, the formula on p. 272 for the "generalized Pythagorean theorem" might be grievously misunderstood; it would look less shocking if s, x, y, z, t were clothed with the customary d's. But, faithful to his preface, the author shuns d's as the devil is alleged to shun holy water. Nevertheless, the chapter as a whole is lucid.

The experimental verifications are duly if lightly sketched, and the author remarks, on the basis of a probable increase in refinements of measurements, that "we may expect that the new theory will be brought more and more into

harmony with observation." This temperate prophecy has been partially justified since the book was printed. Surely there is no more beautiful piece of scientific work, from either the abstract or the experimental aspect, than the comparatively recent verification of the gravitational deflection of light carried out at Mt. Wilson for the massive little companion of Sirius. St. John's definitive work on the special shift dates after the printing of the book.

This book as a whole should at least cause the beginner in relativity to suspect that the theory is not yet as ossified as are some of the forlorn Don Quixotes who gallop forth to overthrow it, while lecturing to college classes or popular audiences, with humor. That the physical universe is a *four*-dimensional continuum of time-space may be one of those rib ticklers which cause us to turn purple with mirth and to applaud the speaker vigorously if somewhat foolishly, but if so, sober reflection should convince us that *four* is not inherently more ludicrous than the sacred *three*. To all those who, like the reviewer, have suffered under such sorties, not infrequently from eminent men in high places, the sanest advice probably is to cut the lectures and read Professor Born's book. It may earn a flunk, but it will be more amusing. Einstein is still busy; to apprehend his latest (July, 1925) profound modification will presuppose a reading knowledge of what has gone before since 1916. To turn one's back on work of this calibre is troglodytish.

The translation seems to have been faithfully done, with "safety first" as a watchword. After all, the solemn richness of the Teutonic idiom, "linking all being with the ego" through the rhythmic "beating of the clocks," even though our "railway train encounters an obstacle and becomes shattered," may be a better vehicle than the lighter English for a subject which is almost metaphysical. In what is approximately the language of some parts of the rendering, it so therefore will appear on the one hand that relativization allows itself to be in this way more embeddedly comprehended by the human spirit, and, on the other hand, moreover, it throws a bright ray on the conceptual objectivations. Let us hope that this is indeed so, and that some altruistic mathematical physicist some day will translate Einstein as Clifford translated Riemann. In the meantime those who do not easily read German owe a debt of gratitude to the translator of Professor's Born's "Relativity."

E. T. BELL

Die Mathematische Methode. Logische Erkenntnisstheoretische Untersuchungen im Gebiete der Mathematik, Mechanik, und Physik. By OTTO HOELDER. Berlin, Julius Springer, 1924. IV+563 pages. Price $6.30.

Within the last two decades scientists of renown have published works on the philosophy of mathematics. Most of these carry prominent references to the remarkable *Principles of Mathematics* by Bertrand Russell which no

doubt constituted an important source of stimulation in their production. To this collection has now been added a volume from the pen of the distinguished mathematician, Professor Hoelder.

The object of Professor Hoelder's treatise is to describe and analyze the chains of deduction which are used in mathematical demonstrations and to compare the components into which these chains have been analyzed. The work is divided into three parts. The first two parts have to do with the elucidation of the deductive procedure in mathematics while the third part is concerned with the relation of deduction to experience, in particular, to the empirical foundations of the applied sciences. The first part, containing a sketch of typical mathematical demonstrations as a basis later for logical analysis, has all the system and thoroughness which may be expected from the distinguished author. Likewise in the third part the author has inserted two valuable and lucid chapters on the facts and assumptions of mechanics and physics, including a brief discussion of Einstein's theory of relativity. The logico-philosophical discussion comprises nearly 200 pages and is contained in the second part, "Logical Analysis of Methods" and two appendices, "The Art of Investigation" and "Paradoxes and Antinomies." With reference to idealism and realism Hoelder occupies somewhat of a middle ground, inclining to a pseudo-realism which is, perhaps, a modification of the Kantian idealism. This position, no doubt, causes Hoelder to find much that is good in the philosophies of Natorp and Meinong, and to make extended and acute criticisms of Kant. Hoelder states that the original stimulus to his work was given by the philosophy of Kant.

One of the more important conclusions of Hoelder is that measure is a synthetic concept constructed on the basis of qualitative concepts with the assistance of sequences of operations which involve counting; and in support and illustration of this thesis Hoelder expounds at length an axiomatic theory of the so-called "third element" involving as undefined relations primarily "betweenness" and "precedence." In this theory, Hoelder seems surreptitiously to have introduced the metrical concept by means of his axiom VIII (p. 287) expressing continuity. Again, on p. 291, the statement is made that the same kinds of activity are exercised in the theory of substitutions as in arithmetic; yet the fundamental nature of a substitution is not very clearly expressed by Hoelder. A substitution is quite as fundamental as a number; these concepts, indeed, are bifurcative developments from the same foundations just as the theories of number-fields and substitution groups are coordinated branches of the same general theory. Hoelder refers to the interpretation of counting in terms of correspondence as involving substitutions leading to the concept multiplicity (*Anzahl*). This foundation of multiplicity is worked out in detail for a finite set of elements. The concept of an infinite aggregate is dealt with

only in a fragmentary manner in the context proper and mainly in connection with mathematical continuity. The statement is made that an infinite set of elements can only be given by a "law" but just what a "law" is does not seem to be satisfactorily explained (*cf.* pp. 98, 193, 555). The concept of correspondence leads Hoelder to the concept of representation which is based on it and which in turn brings the author to the discussion of "simplified representation" (sec. 114) and "genetic agreement" (*uebereinstimmende Erzeugung*). The transition from a finite to an infinite set of elements (p. 281) leads Hoelder to introduce the process of superstruction (*Ueberbauung*) of concepts as opposed to substruction (*Unterbauung*). This process is given considerable prominence in Hoelder's book and is intended to be a generalization of the process of complete induction (*cf.* secs. 119-120). When one system of concepts is superstructed upon another, then the superstruction is the form or "representation" (*Darstellung*) and the substruction (p. 318) is the content or "meaning" (*Deutung*). The opposites, "representation" and "meaning" are introduced in section 98. The principle of superstruction of concepts seems to require examination in reference to any elements of novelty it may have when compared with Poincaré's conception of induction as the mathematical process. At least this is so in the case of Hoelder's examples in sections 116, 117, 119, 120.

Among his more general interpretations, Hoelder's estimate of the power and economy of symbolism in mathematics is inadequate. Strangely, he does not seem to mention in his book the *Principia Mathematica* of Russell and Whitehead. The concept of a working hypothesis in mathematics is apparently not recognized, nor is the logical continuation of such a concept developed from the "working hypothesis" in the empirical realms of observation although the latter concept is mentioned (p. 434). With regard to the mathematical antinomies Hoelder is disposed lightly to dismiss them by the use of such words as "inadmissible" (p. 193), "indeterminate" (p. 552), etc. A more intelligible discussion of antinomies would require the development of the theory of the domain of a concept which has been employed so extensively by Bolzano and others and is, in fact, mentioned by Hoelder himself in section 96.

Lack of space forbids more than a passing reference to the excellent chapters on mechanics and physics. Apart from their controversial interest, these chapters are of real value in assisting the teacher and student to a better understanding of these subjects.

The preceding remarks give, of course, only a very inadequate impression of the richness and scholarly character of Hoelder's work. The chief merit of the book, as it appears to the reviewer, is its practical, working quality. Without straining to support any particular philosophical hobby, the book supplies a wealth of valuable material for the ready use of both the teacher

and research student in the mathematico-philosophical field. Hoelder's specific choice of theories from the literature may naturally be controversial. It is a remarkable fact that Hoelder's subject matter could have been presented in an admirably unified manner by departing somewhat from tradition and utilizing the work of Grassmann in the light of modern development. Hoelder's tendency to follow tradition is accentuated in his restraint in citing the works of American mathematicians; so far as the reviewer has been able to observe, only one American mathematician is mentioned. Typographically, the book is very attractive and remarkably free from errors. One such error must be pointed out: "Russell" is persistently spelled "Russel."

A. R. SCHWEITZER.

Essentials of Applied Calculus. By R. G. THOMAS. New York, D. VanNostrand Co., 1924. XVII+408 pages. Price $2.50.

The preface states that this text is an abridgment and re-arrangement of *Applied Calculus*, published in 1919; and is issued to supply the demand for a text book better adapted to the time available in the usual college course. It is suitable for a first course in calculus in colleges and technical schools.

An admirable feature of the work is the definition of a differential in accordance with the ideas of Newton. In the hands of the author the differential does not become a "ghost of a departed quantity" for the puzzled student; but is used in a satisfactory manner to simplify many derivations that are troublesome to the beginner when presented from the standpoint of limits.

The exercises are fairly numerous, well graded, and contain many practical problems. A large number of illustrative examples are solved, some of which have numerical results. The print is clear and the matter is well arranged on the page.

It is doubtful if the text is better adapted to class use than the author's *Applied Calculus*, for an instructor has no difficulty in omitting sections not suitable to his purpose. Then, too, the text is not a revision of the *Applied Calculus*, but is made up of the major part of that work with the addition of some new matter, for instance, exercises 11-20, page 11; Exercises II_2, page 39; illustrative examples and exercises, pages 118-120; and a part of article 159. Some new articles are interpolated without numbers so as not to make it necessary to change the numbering as given in the older work. Other articles, as a whole or in part, have been taken from their original setting and placed where it is difficult to understand them. For instance, on page 85, under expansion of $\cosh x/a$ and $\sinh x/a$, it would seem necessary to refer to articles 146 and 147 of the *Applied Calculus* to understand the use of s and $\tan \varphi$. Objection may be made to the forward references of which there are quite a number. Many of the typographical errors noted in the *Applied Calculus*

persist in this text. On the whole the work has many good features, not the least of which is the help given in representing established facts and laws in mathematical language.

C. I. PALMER.

Mathematics for Technical Students. By E. R. VERITY. London and New York, Longmans, Green and Co., 1924. xi+468 pages. Price $4.00.

The preface states that this book has been written primarily to meet the needs of students in evening classes of technical institutions, though it will be found suitable as a short course for day engineering classes.

The text has many of the "ear marks" of several other English texts on practical mathematics published during the past fifteen years. The problems are numerous and well selected. A large number of examples are solved, thus giving the student insight in the method of analyzing a problem. In fact, the subjects are developed to a large extent by solving typical examples, paying little attention to the general theory.

The field covered is algebra, trigonometry, and calculus both differential and integral; and these are quite as distinct as they are when put between separate covers. It is not a coordinated or a combined course in mathematics, though the trigonometric ratios are defined and used in the chapter that treats of ratio and proportion. The first 200 pages are devoted to algebra including graphs; pages 202-276 deal with trigonometry; and the remainder of the book concerns itself with the ideas of calculus and its applications. At the close of the text are found tables of logarithms, antilogarithms, and natural trigonometric functions, all to four places. These are followed by answers to the exercises of the text.

Most of the exercises appear to be "desk" problems in that they are not applications that have actually occurred. The text should be found suitable for classes where the learning is done mainly through practice gained in working problems; but, for a class of inquiring students, a good teacher would be needed to answer questions that might arise regarding fundamental principles.

C. I. PALMER.

Der Gegenstand der Mathematik im Lichte ihrer Entwicklung. By H. WIELEITNER. Leipzig, B. G. Teubner, 1925. 61 pages. Price 1 Reichs-Mark.

This paper-covered pamphlet of 61 pages constitutes "Band 50" of the "Mathematisch-Physikalische Bibliothek" brought out by Teubner. It is one of many booklets prepared by authors of prominence, and sold at the nominal price of one Reichs-Mark each or 2 Reichs-Mark for double numbers.

The pamphlet under review starts out with general remarks on mathematics and its development, then discusses ancient geometry, algebra, modern

geometry, higher analysis, mathematics and reality. Though mathematics is viewed in the light of its growth, the publication is not a history of the science. There is no adherence to chronological presentation. Greek geometry is approached from the modern view point. The concepts of the infinitely small and the infinitely great, as set forth by Anaxagoras, are considered along with the ideas of Leibniz. The statement of theorems in Euclid is discussed in the light of Berkeley's criticism of the possibility of a general, wholly arbitrary, triangle or figure.

All in all, the treatment is excellent and indicative of full mastery of the results of recent historical research and philosophic thought. One curious feature is brought to mind by the perusal of the book. In interpreting historical questions which are rendered doubtful on account of insufficiency of data, there are apt to be changes of "fashion" at different periods. For example, in telling of the discovery of the algebraic solution of cubic equations, Wieleitner mentions del Ferro and Cardan, but omits the name of Tartaglia. The same subject, based on substantially the same data, was presented in Hankel's history of 1874, and there Tartaglia is made the chief actor. There are a few places where our author failed to refer to important results of recent research. Thus the vital subject of Zeno's arguments on motion is given as sophistry and no hint is afforded of Paul Tannery's masterly exposition which presents Zeno as severely rigorous. On the restoration of the tract on indivisibles, attributed by some to Aristotle, reference is given to the work of O. Apelt (1891) but H. H. Joachim's later and fuller researches are overlooked. The formula which we now write $e^{i\theta} = \cos\theta + i \sin\theta$ is attributed to Euler without mention of the earlier appearance of the theorem in the logarithmic form in the writings of Cotes. The name "Integral" is connected with Johann Bernoulli, though it was first used in 1690 by Jakob (James) Bernoulli.

FLORIAN CAJORI.

Théorie Mathématique de l'Electricité, By TH. DE DONDER. Part I. Introduction to Maxwell's Equations. Paris, Gauthier Villars, 1925. 198 pages, 76 figures in the text.

This mathematical treatise on electricity contains nearly 700 definite formulas, besides many other auxiliary equations. It is presented as a fundamental textbook on electromagnetic theory from the mathematicians' standpoint, especially directed towards the establishment of Maxwell's equations defining electromagnetic fields between bodies at rest.

The book is notable for the rigor and precision of its treatment. Space is treated as strictly Euclidean, and the dielectric constant of free space appears in all the formulas to which it belongs. In less rigorous textbooks, it is only too often overlooked or omitted.

An interesting and important distinction is made between the electric force distribution and the resultant vector force, in the case of an inductively polarised region. In nearly all other cases, the distinction is shown to disappear. Coefficients of capacitance are developed through the electric energy of the associated charges. This seems to be a new and interesting method of approach.

In leading up to Maxwell's equations of the general electromagnetic field, a side-by-side comparison is conducted into the properties of superficial electric currents and of volume currents. This presentation is serviceable, in clearing up the definition of current density. It offers a new gateway to the current vector potential, whose curl at any point is the local magnetic force. A very helpful parallelism is maintained in the treatment of magnetics and electrics. It is well known that such a parallelism exists; but it is presented here in an unusually forceful way.

The first section of the book, with eight chapters, discusses the stationary electric field. The second section, with four chapters, discusses the stationary electromagnetic field. The third section, with two chapters, deals with the variable electromagnetic field.

The book ends with an excellent chapter on electric and magnetic systems of units, including the Heaviside-Lorentz system. It would be made more useful for ready reference, if an index were added to the table of subjects in the final pages.

The work is a valuable compendium of exact mathematical relations in the fundamentals of classical electricity and magnetism, without involving direct use of the electron. It will appeal particularly to the mathematical physicist; but it will also aid the electrical engineer in questions that probe the fundamentals of his science.

A. E. KENNELLY.

A Graphic Table Combining Logarithms and Anti-logarithms. By A. LACROIX and C. L. RAGOT. New York, The Macmillan Company, 1925. 56 pages. Price $1.40.

This book represents an innovation in logarithm tables. These graphical tables are constructed with a numerical scale and a logarithmic scale having a common line as a base. The values on each scale are represented by graduations so that a value on either scale can be read directly in terms of the other scale without interpolation. The principal table of the book consists of forty pages. From this table, it is possible to read to five places logarithms of all five place numbers and to read the numbers to five places corresponding to all five place logarithms without interpolation. One accustomed to reading scales or a slide rule can read to six places by estimating the distance between the point of reading and the nearest graduation. A numerical table having these qualities

would be between four and five times as large. This table is followed by a similar four place table of six pages.

For persons using logarithms of numbers involving five places or less, the reviewer believes that this table will prove decidedly useful. The fact that interpolation is done away with not only saves time and energy but removes a fertile source for mistakes. The cumulative error caused by interpolation which sometimes renders the result inaccurate in the last place is avoided. The tables do not give logarithms of the trigonometric functions in terms of the angles. For this reason, its usefulness to the student and the engineer using trigonometric functions is limited. D. S. MORSE.

Four-figure Mathematical Tables. By G. W. C. KAYE and T. H. LABY. London, Longmans, Green and Company, 1925. 26 pages. Price $0.40.

This small book contains thirteen tables which are all four place tables with the exception of one five place table of logarithms. The tables involving trigonometric functions have entries at $6'$ intervals. From the logarithm tables, it is possible to read without interpolation the logarithms of all three figure numbers. The other tables have corresponding entries. The four figure tables include tables of logarithms; antilogarithms; reciprocals; squares; natural sines, cosines, and tangents; and logarithms of sines, cosines, and tangents. A table for changing from degrees to radians and a table of squares, cubes, square roots, cube roots, and reciprocals of the numbers from 1 to 100 complete the list. The tables are well arranged and easy to read.

D. S. MORSE.

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Annals of Mathematics, second series, volume 27, no. 1, September, 1925: "A contribution to the theory of finite differences" by J. Kaucky, 1–2; "On the asymptotic expressions of certain definite integrals" by J. A. Shohat, 3–11; "The frequency law of a function of variables with given frequency laws" by E. L. Dodd, 12–20; "The four term diophantine arccotangent relation" by A. A. Bennett, 21–24; "On Weyl's treatment of the parallel displacement of a vector around an infinitesimal closed circuit in an affinely connected manifold" by T. Y. Thomas, 25–28; "Concerning the subsets of a plane continuous curve" by H. M. Gehman, 29–46; "On the oscillations of a line near a position of equilibrium" by F. H. Murray, 47–53; "Note on trinomial congruences and the first case of Fermat's last theorem" by H. S. Vandiver, 54–56; "Definite linear dependence" by L. L. Dines, 57–64; "Least distance from a point to a linear $(n-k)$-space, both in a linear n-space" by H. S. Uhler, 65–68.

Journal of the Franklin Institute, volume 200, no. 6, December, 1925: "An analogy between pure mathematics and the operational mathematics of Heaviside by means of the theory of H-functions" by J. J. Smith, 775–814.

The Monist, volume 35, no. 4, October, 1925: "Infinity and the infinitesimal" by W. Parkhurst and W. T. Kingsland, Jr., 633–666.

School Science and Mathematics, volume 25, no. 9, whole no. 218, December, 1925: "Note on the Phillips system of multiplication" by W. E. Pyke, 949–951; "All geometrical constructions may be made with compasses" by Michael Goldberg, 961–965.

Transactions of the American Mathematical Society, volume 27, no. 4, October, 1925: "Cone cubic configurations of a ruled surface" by A. F. Carpenter, 397–415; "Concerning upper semicontinuous collections of continua" by R. L. Moore, 416–428; "On the oscillation of a continuum at a point" by W. A. Wilson, 429–440; "On the conditions of integrability of covariant differential equations" by J. A. Schouten, 441–473; "Complete groups of points on a plane cubic curve of genus one" by M. I. Logsdon, 474–490; "On the existence of the Stieltjes integral" by H. L. Smith, 491–515; "The mutual inductance of two square coils" by T. H. Gronwall, 516–536; "On the development of continuous functions in series of Tchebycheff polynomials" by J. A. Shohat, 537–550; "Analytic functions in three dimensions" by E. R. Hedrick, 551–555; "The Beltrami equations in three dimensions" by E. R. Hedrick and L. Ingold, 556–562; "Fields of parallel vectors in a Riemannian geometry" by L. P. Eisenhart, 563–573; "A symbolic treatment of the geometry of hyperspace" by L. Ingold, 574–599.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 2910 Harris Park Ave., Austin, Texas.

CLUB TOPICS

The Pasturage Problem of Sir Isaac Newton

By B. H. Brown, Dartmouth College.

The Pasturage Problem[1] of Sir Isaac Newton is not unworthy a place beside the older and numerically greater cattle problem of Archimedes discussed in the Monthly[2] by Professor Archibald.

If 12 oxen eat up $3\frac{1}{3}$ acres of grass in 4 weeks, and 21 oxen eat up 10 acres in 9 weeks, how many oxen will eat up 24 acres in 18 weeks, the grass being at first equal on every acre, and growing uniformly? Ans. 36 oxen.

Apparently someone, somewhere, believed that Newton's *Arithmetica Universalis* was a book on arithmetic, and hence any problem in it was fair game for the young arithmetician. In the course of time the problem was incorrectly copied, the $3\frac{1}{3}$ in the first condition became $3\frac{1}{2}$ with the resulting answer of 37 113/175 oxen. Let us add parenthetically that in this form the problem seems to us neither less nor more absurd than the original. At least as early as 1835 the problem had crossed the Atlantic, its source was forgotten, and with either $3\frac{1}{2}$ or $3\frac{1}{3}$ became the *pièce de résistance* of American arithmetics. The later history of this problem forms a very sad commentary on

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[1] Newton, *Arithmetica Universalis*, 1704.
[2] This Monthly, (*1918*, 411–414).

the scientific and educational standards of early nineteenth century American mathematics teachers. The details are set forth in engaging fashion in Cajori's *The Teaching and History of Mathematics in the United States.*[3]

Let us solve this problem for the literal case, stating the problem in the following form:

If a_i oxen eat b_i acres of grass in c_i weeks, $i=1, 2, 3$, the grass being at first equal on every acre and growing uniformly, what relation exists between a_i, b_i, c_i; and what inequalities are necessary to insure a real solution?

Suppose x oxen eat the original stand on 1 acre in 1 week; and y oxen eat 1-week's growth on 1 acre in 1 week. Then

$$\frac{b_i}{c_i}x+b_iy=a_i\,, \qquad i=1,2,3\,;$$

whence

$$\begin{vmatrix} \frac{b_1}{c_1} & b_1 & a_1 \\ \frac{b_2}{c_2} & b_2 & a_2 \\ \frac{b_3}{c_3} & b_3 & a_3 \end{vmatrix}=0\,,$$

and this relation may be written

$$\frac{a_1}{b_1}c_1(c_2-c_3)+\frac{a_2}{b_2}c_2(c_3-c_1)+\frac{a_3}{b_3}c_3(c_1-c_2)=0\,.$$

If this condition obtains

$$x=\frac{c_ic_j\left(\frac{a_j}{b_j}-\frac{a_i}{b_i}\right)}{c_i-c_j}\,, \qquad y=\frac{\frac{a_i}{b_i}c_i-\frac{a_j}{b_j}c_j}{c_i-c_j}\,,$$

and, assuming a_i, b_i, c_i all positive, and $c_1>c_2>c_3$ it is necessary and sufficient that x and y be positive, that is

$$\frac{a_3}{b_3}>\frac{a_2}{b_2}>\frac{a_1}{b_1}\,; \qquad \frac{a_1}{b_1}c_1>\frac{a_2}{b_2}c_2>\frac{a_3}{b_3}c_3\,.$$

SUGGESTIONS: (*a*) Compare this solution with the prize-winning "lucid analytical solution" of Mr. James Robinson,[4] and with the analysis of Newton.

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[3] Washington, 1890, pp. 109, 110.

[4] *Hendricks's Analyst*, vol. III, p. 75; also the *Mathematical Magazine*, edited by Artemas Martin, vol. I, pp. 17 and 43; also *The Mathematical Monthly*, vol. II, pp. 83–85.

(*b*) This problem is often given in the old arithmetics under the heading "False Position." Solve this problem by double false position.

(*c*) The student with a thirst for generalization may try the further introduction of horses, sheep, etc. into these pastures, and he might then let the animals eat in proportion to their size and grow exponentially.

CLUB ACTIVITIES

JUNIOR MATHEMATICS CLUB of the University of Wisconsin, Madison, Wisconsin.

The program for the college year 1924–1925 was the following:

October 2, 1924.	Picnic on the Drive.
November 23.	"Graphs" by Professor A. Dresden.
November 6.	"Ancient Chinese mathematics, the Chinese calendar, and the place of western mathematics in Chinese secondary schools" by F. H. L. Chang.
November 20.	"Unilateral surfaces" by Professor L. Dowling.
December 4.	"Life of Pythagoras" by Miss A. H. Krause. "Life of Newton" by R. B. Schwenger. "Life of Leibnitz" by Miss M. A. Summers. "Mathematical fallacies" by E. R. Heineman.
December 11.	"The half-regular polyhedrons of Archimedes" by Professor E. B. Van Vleck.
December 25.	Christmas program consisting of a piano recital and a sleight-of-hand performance.
January 15, 1925.	"Closure in mathematics" by Professor Ingraham.
February 26.	"Desargues' theorem" by Professor Hart.
March 12.	"Women mathematicians" by Miss H. E. Urschel, Miss D. H. Haskins, Miss G. L. Fries.
March 26.	"Mathematics in secondary schools" by Professor H. L. Miller.
April 30.	Joint meeting with senior mathematics club.
May 14.	"Factoring and the highest common divisor" by Professor E. B. Skinner.
May 26.	Picnic.

(Report by Professor Skinner.)

THE HARVARD MATHEMATICAL CLUB, Harvard University, Cambridge, Massachusetts.

During the year 1924–1925, the following papers were presented at the meetings of the Harvard Mathematical Club:

October 22, 1924.	"Weierstrass' second implicit function theorem" by Professor W. F. Osgood.
November 5.	"Lagrange's multipliers" by Mr. Maurice Marden.
November 19.	"Study's dual vector and line congruences" by Mr. M. M. Slotnick.
December 3.	"Projective geometry" by Professor W. C. Graustein.
December 17.	"Stereographic projection" by Mr. T. L. Smith.
January 14, 1925.	"The twisted quartic" by Mr. M. S. Demos.
February 26.	"A famous problem of construction" by Mr. E. T. Virata.
March 11.	"Quaternion multiplication" by Mr. D. E. Whitford.
March 18.	"Regular transformations of sequences" by Professor L. L. Silverman of Dartmouth College.
April 1.	"Implicit functions in general analysis" by Dr. L. M. Graves.
April 14.	"Differential equations from the group standpoint" by Mr. H. B. Curry.
April 29.	"The Riemann problem for linear differential equations and its generalization" by Mr. J. J. Wolfender.

May 13.	"Mathematics at Harvard a hundred and fifty years ago" by Professor J. L. Coolidge.

The Executive Committee for the year 1924–1925 consisted of

Mr. K. W. Halbert, president; Mr. E. W. Perkins, secretary-treasurer; Professor W. F. Osgood, faculty adviser.

The following were elected members of the Executive Committee for the year 1925–1926:

Mr. H. B. Curray, president; Mr. T. L. Smith, secretary-treasurer; Dr. H. W. Brinkman, faculty adviser.

(Report by F. W. Perkins.)

The Mathematics Club of the University of Maine, Orono, Maine.

The following is the program of the Mathematics Club of the University of Maine for 1924–1925:

November 12, 1924.	"Mathematical philosophy" by Mr. Engstrom.
December 3.	"The development of number systems" by Professor Lucas.
February 17, 1925.	"The birth of the moon" (Darwin's theory) by Professor Willard.
March 3.	"The nine-point circle" by Miss Savage.
March 17.	"The volume of a regular icosahedron" by Mr. Beale.
April 29.	"The history of astronomy" by Miss Mossler.

(Report by Miss Elizabeth Laughlin.

The Mathematics Club of Hunter College, New York City.

The following programs were given at the meetings of the Mathematics Club of Hunter College during the year 1924–1925:

October 2, 1924.	Reception to Freshmen.
October 7.	"Mathematics in colleges" by Dr. Tomlinson Fort.
October 21.	"The development of symbolism in mathematics" by Margaret Cronin '25.
November 3.	"The development of algebraic notation" by Florence Gitlin '25. "Geometric fallacies" by Nannette Klein '26.
November 18.	"The making of books" by Dr. D. E. Smith of Columbia University.
December 2.	"Logarithms to the base e" by Sophie Rotkowitz '25. "Logarithms to the base 10" by Olga Marshaw '25.
December 16.	Debate: Resolved that freshman mathematics is advisable.
January 6, 1925.	"Algebra in the colonial schools" by Professor Simons.
February 24.	Business meeting. Election of Miss Carolyn Eisele as faculty advisor of the club.
March 3.	"Algebra in the colonial schools" continued by Professor Simons.
March 17.	"Archimedes' cattle problem" by Ethel Garfunkel '27.
March 31.	"Hyperspace" by Bertha Odessky '27.
April 14.	"The human significance of mathematics" by Rose Richter '25.
May 5.	"Mathematical fallacies" by Gertrude Kornhauser '26.
May 14.	Musicale and tea.
May 19.	"Mathematics and astronomy" by Harriet Griffin '25.
June 2.	Election of officers.

The officers for the year 1924–1925 were:

Elizabeth Draper, president; Rose Richter, vice-president; Mary Draper, secretary; Florence Gitlin, treasurer; Rosalind Honig, publicity manager.

The officers for the year 1925–1926 are:

Gertrude Kornhauser, president; Mary Draper, vice-president; Bertha Odessky, secretary; Lena Kalan, treasurer; Nannette Klein, publicity manager.

(Report by Miss Bertha Odessky, secretary.)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

(N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.)

3173. Proposed by Samuel Beatty, University of Toronto.

If X is a positive irrational number and Y its reciprocal, prove that the sequences

$$\begin{array}{lll} (1+X)\,, & 2(1+X)\,, & 3(1+X)\,,\cdots \\ (1+Y)\,, & 2(1+Y)\,, & 3(1+Y)\,,\cdots \end{array}$$

contain one and only one number between each pair of consecutive positive integers.

3174. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Construct a triangle given an angle, the difference of the including sides, and the altitude to the third side.

3175. Proposed by Otto Dunkel, Washington University.

Is the following statement correct?

The intersections of two curves whose equations are given in polar coördinates (ρ, θ) are obtained by solving the two equations simultaneously for ρ and θ.

3176. Proposed by A. Pelletier, Montreal, Canada.

Using the straight-edge only, can a perpendicular be drawn from a given point to a given straight line?

3177. Proposed by Frank Irwin, University of California.

Prove the identity

$$\frac{1}{\sin(y+z)}\left[\sin y+\frac{\sin x \sin z}{\sin(x+y+z)}\right]=\frac{1}{\sin(x+z)}\left[\sin x+\frac{\sin y \sin z}{\sin(x+y+z)}\right].$$

3178. Proposed by J. L. Riley, Ouachita College, Arkadelphia, Ark.

Find the smallest integral values of x and y which satisfy the equation

$$x^2-67y^2=1\,.$$

3179. Proposed by Philip Fitch, Denver, Colorado.

A and B mail letters to $m+n$ people, A mailing m letters and B mailing n letters. If no person can receive more than one letter from each of the senders, and if two letters, no letter, or the wrong letter to a person is counted as an error, how many errors are possible?

3180. Proposed by D. M. Yost, California Inst. of Technology.

Evaluate the definite integral

$$\int_0^\infty \frac{dx}{x^x}\,,$$

where x^x may be written $e^{x\log x}$.

SOLUTIONS

2677 [1918, 75]. Proposed by R. K. Morley, Worcester, Mass.

A quarter-mile track is to be constructed, having semi-circular ends and straightaway sides. It is required to have the rectangular part of enclosed field referred to in No. 12, Granville's *Calculus*, page 116, as large as possible. Find length of the straightaways.

Also, required to inscribe the maximum rectangle in a track of length l, with semi-circular ends and straightaway sides, assuming that two sides of the rectangle are parallel to the straightaways. Find the length of the straightaways and the dimensions of the rectangle.

Solution by the Proposer

Part 1. It should be noted that the reference to Granville's *Calculus* does not apply to the present printing of that book as the faulty race-track problem (see this Monthly, *1917*, 431) has now been replaced.

Let x=length of the straightaways; r=radius of the semicircular ends. Then the area to be a maximum is $2xr$. Also $2x+2\pi r=1/4$ mile.

Hence area is

$$f(x)=\frac{2}{\pi}\left(\frac{x}{8}-x^2\right), \text{ and } f'(x)=\frac{2}{\pi}\left(\frac{1}{8}-2x\right),$$

Setting $f'(x)=0$, we obtain $x=1/16$ mile, which is easily shown to give the required maximum.

Part 2. Let x and r have the same meanings as before, let $x+2y$ be the length of the inscribed rectangle, $2z$ its breadth, and θ the angle between the central axis of the figure parallel to the straightaways and the radius r to a corner of the inscribed rectangle. Then the area to be made a maximum is $A=2z(x+2y)$; and also $l=2x+2\pi r$, $y=r\cos\theta$, $z=r\sin\theta$. Thus

$$A=lr\sin\theta-2\pi r^2\sin\theta+2r^2\sin 2\theta$$

is a function of the two independent variables r and θ. The partial derivatives are

$$\frac{\partial A}{\partial r}=l\sin\theta-4\pi r\sin\theta+4r\sin 2\theta, \quad \frac{\partial A}{\partial \theta}=lr\cos\theta-2\pi r^2\cos\theta+4r^2\cos 2\theta,$$

and, if we exclude the trivial results $r=0$ and $\sin\theta=0$, the only value of θ making both zero is given by

$$\cos\theta=\frac{2}{\pi}, \qquad \sin\theta=\frac{\sqrt{\pi^2-4}}{\pi}.$$

From these values we find

$$r=\frac{\pi l}{4(\pi^2-4)}, \quad x=\frac{(\pi^2-8)l}{4(\pi^2-4)}, \quad y=\frac{l}{2(\pi^2-4)}, \quad z=\frac{l}{4\sqrt{\pi^2-4}},$$

$$\frac{\partial^2 A}{\partial r^2}=-\frac{4}{\pi^2}(\pi^2-4)^{3/2}, \quad \frac{\partial^2 A}{\partial r\partial\theta}=-\frac{2l}{\pi}, \quad \frac{\partial^2 A}{\partial\theta^2}=-\frac{(\pi^2+8)l^2}{8(\pi^2-4)^{3/2}}.$$

Thus

$$\frac{\partial^2 A}{\partial r^2}\frac{\partial^2 A}{\partial\theta^2}-\left(\frac{\partial^2 A}{\partial r\partial\theta}\right)^2=\frac{l^2}{2}$$

and, since both $\partial^2A/\partial r^2$ and $\partial^2A/\partial\theta^2$ are negative, these critical values make A a maximum. The maximum area is then

$$\frac{l^2}{8\sqrt{\pi^2-4}}=.0516l^2.$$

It is interesting to compare this area with that of the square inscribed in a circle of circumference l. This area is

$$\frac{l^2}{2\pi^2}=.0507\,l^2$$

NOTE BY OTTO DUNKEL. The proof above shows that there is only one relative maximum within the given region for r and θ. To show that this is the absolute maximum in the region we may proceed as follows. The region is defined by $0 \leqq r \leqq l/2\pi$, $0 \leqq \theta \leqq \pi/2$, and the function A is continuous in this region. It has therefore an absolute maximum within or on the boundary. On the parts of the boundary $\theta=0$ and $r=0$, $A=0$; on the part $\theta=\pi/2$ the maximum of A is $l^2/8\pi$; on the part $r=l/2\pi$ the maximum is $l^2/2\pi^2 > l^2/8\pi$. Hence the maximum on the boundary is $l^2/2\pi^2$. Now the necessary conditions above show that at a point within the region the value of the area is $.0516\, l^2$ which is greater than the maximum on the boundary $.0507\, l^2$, and hence the absolute maximum must lie within the region. Also at this point the first partial derivatives which are here continuous must be zero. But since there is only one point within the region where both first partial derivatives are zero the absolute maximum must be at this point. It has thus been shown that the value of A at this point is greater than at any other point of the region and no use has been made of the second partial derivatives.

Also solved by J. B. REYNOLDS, and F. L. WILMER.

2720 [1918, 302]. Proposed by A. A. Bennett, Lehigh University.

Given three points A, B, C, in a plane, draw from an arbitrary fourth point D the segments AD, BD, CD. Also draw rays AA', BB', CC' making equal (small) angles respectively with segments AD, BD, CD. The triangle determined by the three rays does or does not contain the point D according as the original triangle ABC does or does not contain D.

Prove the theorem, considering also the case in which A, B, C, D are concyclic.

SOLUTION BY RUFUS CRANE, Ohio Wesleyan University.

It is assumed that the point D lies in the plane of ABC, also that the three rays make the angle θ with AD, BD, CD in the same sense.

Let AA' and CC' meet in A', BB' and AA' in B' etc. Then, since $\angle DAA' = \angle DCA'$ or its supplement, the points D, C, A, A' are concyclic; similarly $DABB'$, and $DBCC'$. Let these circles have centers X, Y, Z, respectively. Since AD, BD, CD are perpendicular respectively to XY, YZ, ZX, the rays AA', BB', and CC' make the same angle $(90°-\theta)$ with XY, YZ, ZX, respectively, and in the same sense. Hence $A'B'C'$ and XYZ are directly similar. Now extend DX till it meets the circle DCA at R. The $\angle DAR$ is a right angle, and RAA' is either $90°+\theta$ or $90°-\theta$. In either case RDA', that is, XDA' is $90°-\theta$. Similarly YDB' and ZDC' are $90°-\theta$. Also

$$\frac{DA'}{2DX} = \frac{DB'}{2DY} = \frac{DC'}{2DZ} = \sin\theta.$$

Hence the triangles $A'B'C'$ and XYZ have D as their center of similarity, and their ratio of similarity is $2\sin\theta$. Hence $A'B'C'$ will or will not enclose D according as XYZ does or does not enclose D. Obviously XYZ will or will not enclose D according as D lies within or without the triangle ABC. Hence the theorem.

If D lies on the circumcircle of ABC, the triangle XYZ, and therefore the triangle $A'B'C'$, reduces to a point. In the statement of the problem, the word *small* is redundant. As the angle θ increases from 0° to 180°, the points A', B', C' traverse the circles DCA, etc., making one complete circuit. When $\theta=90°$, XYZ and $A'B'C'$ are homothetic, ratio $1:2$.

2723 [1918, 303]. Proposed by G. Y. Sosnow, Newark, N. J.

The feet of the perpendiculars from the intersection of the diagonals on the sides of a cyclic quadrilateral M are joined to form a second quadrilateral N. Prove that N is a quadrilateral of minimum perimeter inscribed in M.

SOLUTION BY MICHAEL GOLDBERG, Philadelphia, Penn.

Let the given cyclic quadrilateral M be $ABCD$. Locate the points C', D', A' as follows: reflect the figure on the axis BA, giving the new points D_1, C'; then reflect this second figure on BC' giving A_2, D';

reflect the third figure on $C'D'$ giving A', B_2. The angle between AD and $A'D'$ is $A-B+C-D=(A+C)-(B+D)=0$. Therefore, AD and $A'D'$ are parallel.

Draw any line PP' parallel to AA' cutting the lines $AD, AB, BC', C'D$ and $D'A'$ in the points P, R, S' T', and P', respectively. Locate the points S and T on BC and CD respectively corresponding to S' and T'. Then $PRST$ is a minimum quadrilateral inscribable in $ABCD$. Since P was taken at any point along AD, there is an infinity of equal minimal inscribable quadrilaterals. The proof follows from the fact that the perimeter of any other inscribed quadrilateral would develop into a broken line between P and P' and, therefore, would be longer than the straight line.
$\angle ARP=\angle BRS$, $\angle BSR=\angle CST$, $\angle CTS=\angle DTP$, $\angle DPT=\angle APR$, since their corresponding angles are vertical angles in the developed figure. The equality of these angles is a necessary and sufficient condition for a minimum perimeter. It is the path that a ray of light would take in order to return to its source, when the sides AB, BC, CD, and DA are considered as mirrors.

NOTE BY THE EDITORS: The problem, as proposed, is only a special case of the foregoing general theorem. Let O be the intersection of the diagonals AC and BD, and E, F, G, H the feet of the perpendiculars from O to the sides AB, BC, CD, DA. Then $\angle FGO=\angle FCO=\angle ODH=\angle OGH$, since we have three cyclic quadrilaterals. Hence FG and GH make equal angles with CD. A similar proof applies to the sides of the second quadrilateral through F, E, H.

2724 [1918, 303]. **Proposed by Frank Irwin, University of California.**

Show that there is a unique set of values, $x_1, x_2, \ldots, x_n$ that satisfy the equation

$$x_1^2+x_2^2+\cdots+x_n^2-x_1x_2-x_2x_3-\cdots-x_{n-1}x_n-x_n+\frac{n}{2(n+1)}=0\,.$$

SOLUTION BY THE PROPOSER.

The left side of the equation given in the problem may be written in the following form by the simple process of completing the squares

$$\sum_1^{n-1}\frac{i+1}{2i}\left[x_i-\frac{i}{i+1}x_{i+1}\right]^2+\frac{n+1}{2n}\left[x_n-\frac{n}{n+1}\right]^2=0\,.$$

Hence the only real solution is given by setting $x_i=ix_{i+1}/(i+1)$, and $x_n=(n/n+1)$. From these equations follow at once

$$x_i=\frac{i}{n+1}\,,\qquad i=1,2,\cdots,n\,.$$

The problem was suggested by similar problems in Wolstenholme's *Mathematical Problems.* (Is the rising generation of mathematicians acquainted with that fascinating book ?) In fact the left member of our equation differs from that in problem No. 218 in Wolstenholme in the constant term only. A totally different method for attacking such problems may be found in the *Intermédiaire des Mathématiciens*, vol. 24, 1917, p. 67 (my solution of the same problem, by the method used here, on page 136 of the same volume).

2769 [1919, 171]. **Proposed by B. J. Brown, Kansas City, Mo.**

Expand in powers of x as far as x^2 the function $\dfrac{\cosh \lambda x}{\cosh \lambda}-x\dfrac{\sinh \lambda x}{\sinh \lambda}$ in which λ is a positive constant.

Prove that, if $\lambda \tanh \lambda>2$, the function has only one maximum value for $x>0$ and that the value of x for which the maximum occurs is less than 1. (India Civil Service, 1912.)

SOLUTION BY R. H. SCIOBERETI, Berkeley, California.

(In this solution, sinh x will be replaced by sh x, cosh x by ch x, and tanh x by th x.)

The function $f(x, \lambda)=(\text{ch}\,\lambda x/\text{ch}\,\lambda)-(x\,\text{sh}\,\lambda x/\text{sh}\,\lambda)$ may be expanded in a series of the form $\varphi_0+\varphi_1x+\varphi_2x^2+\cdots+\varphi_nx^n+\cdots$, where the φ_k are functions of the parameter λ. Since $f(x, \lambda)$ is an even function both of x and λ, it follows that $\varphi_{2k+1}=0$ and that φ_{2k} will be an even function of λ. Further-

more, the proposed function has derivatives of all orders, and can be developed in a Maclaurin series as follows:

$$f(x,\lambda)=\frac{1}{\text{ch}\lambda}+\frac{x^2}{2!}\frac{\lambda}{\text{sh}\lambda}(\lambda\,\text{th}\,\lambda-2)+\cdot\cdot\cdot+\frac{x^{2p}}{2p!}\frac{\lambda^{2p-1}}{\text{sh}\,\lambda}(\lambda\,\text{th}\,\lambda-2p)+\cdot\cdot\cdot\ ,$$

since

$$f^{(2p)}(x)=\frac{\lambda^{2p-1}}{\text{sh}\,\lambda}\text{ch}\,\lambda x(\lambda\,\text{th}\,\lambda-2p)-x\frac{\lambda^{2p}}{\text{sh}\,\lambda}\text{sh}\,\lambda x\ ;\qquad f^{2p}(0)=\frac{\lambda^{2p-1}}{\text{sh}\,\lambda}(\lambda\,\text{th}\,\lambda-2p)\ ,$$

$$f^{(2p+1)}(x)=\frac{\lambda^{2p}}{\text{sh}\,\lambda}\text{sh}\,\lambda x[\lambda\,\text{th}\,\lambda-(2p+1)]-x\frac{\lambda^{2p+1}}{\text{sh}\,\lambda}\text{ch}\,\lambda x\ ;\qquad f^{2p+1}(0)=0\ .$$

The remainder of the series after the $2p$th term, may be written in the form

$$-\frac{x^{2p+1}}{(2p+1)!}\frac{\lambda^{2p}}{\text{sh}\,\lambda}[\,\text{sh}\,(\lambda\theta x)(2p+1-\lambda\,\text{th}\,\lambda)+\lambda\theta x\,\text{ch}\,(\lambda\theta x)]\ ,\ \text{or}$$

$$R_{2p}=\frac{-x}{\text{sh}\,\lambda}\frac{(\lambda x)^{2p}}{2p!}\left[\text{sh}\,(\lambda\theta x)\left(1-\frac{\lambda\,\text{th}\,\lambda}{2p+1}\right)+\frac{\lambda\theta x}{2p+1}\text{ch}\,(\lambda\theta x)\right]\ ;\qquad 0<\theta<1\ ,$$

which shows that $\lim\limits_{p\to 0} R_{2p}=0$.

NOTE BY OTTO DUNKEL. If the original function be denoted by $f(x)$, then

$$f'(x)=\frac{\lambda\sinh\lambda x}{\sinh\lambda}\varphi(x)\ ,\qquad \varphi(x)=\tanh\lambda-\frac{1}{\lambda}-x\coth\lambda x\ .$$

Then

$$\varphi'(x)=-\frac{1}{2\sinh^2\lambda x}\left[\sinh 2\lambda x-2\lambda x\right]<0\ ,\ \lim_{x\to 0}\varphi(x)=\tanh\lambda-\frac{2}{\lambda}>0\ ,\ \varphi(1)=-\left[\frac{1}{\lambda}+\frac{2}{\sinh 2\lambda}\right]<0\ .$$

Hence $\varphi(x)$ decreases from a positive value at $x=0$ to zero at $x=x_0<1$, and then continues to decrease. It now follows that $f'(x)>0$, $0<x<x_0$ and $f'(x)<0$, $x>x_0$. Hence $f(x)$ has only one maximum for $x>0$ and it occurs at $x=x_0<1$.

Since $2/\lambda<\tanh\lambda<1$, $\lambda>2$, and it is easily shown that $\lambda>2.0653$.

2818 [1920, 135]. Proposed by S. A. Corey, Des Moines, Iowa.

What is the maximum error that could occur in computing the common logarithm of $(1+x)$ by the formula:

$$\log_{10}(1+x)=.144,764,827\frac{x}{1+x}+.579,059,309\frac{x}{2+x}+.072,151,17\frac{x}{1+x}\ ,$$

where $0\leqq x\leqq 6/10$?

SOLUTION BY OTTO DUNKEL, Washington University.

This may be regarded as a problem in maxima and minima and as such the only difficulty in its solution is that encountered in the tedious numerical computations. Consider the error

$$f(x)=\log_{10}(1+x)-x\left[B+\frac{D}{x+1}+\frac{C}{x+2}\right]\ ,$$

where $D=.072613657$, $C=.579059309$, and $B=.07215117$. Then

$$f'(x)=\frac{-x[Bx^3-(M-6B)x^2-(M+3D-9B)x-4(D-B)]}{(x+1)^2(x+2)^2}\ ,$$

$$=\frac{-x(x-a)\varphi(x)}{(x+1)^2(x+2)^2}\ ,$$

where $M=D+B+C/2=.434294482$, and $a=.3454374$, and where $\varphi(x)$ is a quadratic expression which here is always positive. Hence $f'(x)<0$, $x<0$; $f'(x)>0$, $0<x<a$; and $f'(x)<0$, $x>a$. Thus $f(x)$ for

negative values of x decreases to its minimum value 0 at $x=0$; it then increases to its maximum value .000012228 at $x=.3454374$; as x increases further it decreases to zero at $x=.4960^+$ and then continues to decrease. Since $f(.6)=-.000029912$, we have .000029912 for the maximum of the absolute value of the error in the interval $0\leqq x\leqq .6$.

3032 [1923, 276]. Proposed by Otto Dunkel, Washington University.

If $a_1, a_2, \cdots, a_n$ are any real or complex quantities which satisfy the equation

$$x^n - na_1x^{n-1} + {}_nC_2a_2^2x^{n-2} + \cdots + (-1)^i{}_nC_ia_i^iX^{n-i} + \cdots + (-1)^na_n^n = 0$$

where ${}_nC_i = n!/(n-i)!i!$, prove that $a_1 = a_2 = \cdots = a_n$.

Solution by the Proposer.

Set $|a_i| = \rho_i$. It will be shown that all the ρ_i's have the same value ρ. For suppose that they are not all equal, that ρ_k is as great as any other ρ and that there is at least one ρ, say ρ_i, such that $\rho_k > \rho_i$. Since

$$a_k^k = \frac{\sum a_{i_1}a_{i_2}\cdots a_{i_k}}{{}_nC_k},$$

we have

$$\rho_k^k \leqq \frac{\sum \rho_{i_1}\rho_{i_2}\cdots\rho_{i_k}}{{}_nC_k} < \rho_k^k .$$

Here we have a contradiction, and hence the ρ_i's must all have the same value ρ. If $\rho=0$ the theorem is proved.

Suppose that $\rho\neq 0$: it will be shown that all the a_i's have the same value. For $na_1 = \Sigma a_i$, and if there are any two a's not equal then

$$n\rho = \left|\sum a_i\right| < \sum |a_i| = n\rho .$$

There results the contraction $\rho<\rho$. The theorem of the problem is then proved.

Also solved by F. L. Wilmer.

3128 [1925, 204]. Proposed by J. Rosenbaum, Milford, Connecticut.

Given the mid-points of the sides of a polygon, to construct the polygon.

Solution by J. P. Ballantine, Columbia University.

Let $M_1, M_2, \cdots M_n$ be the given mid-points, in any number of dimensions. From any point A_1, draw A_1M_1 and produce it its own length to A_2. and from A_2 draw A_2M_2 and produce it its own length to A_3 etc., thus obtaining a final point A_{n+1}. From another arbitrary point B_1, construct another sequence $B_1B_2\cdots B_{n+1}$, in the same way. The line segments A_1B_1, A_2B_2, etc., are all parallel and of equal lengths, but alternately oppositely sensed. Hence if n is even A_1B_1 and $A_{n+1}B_{n+1}$ are sensed the same way, and if n is odd, the opposite way.

In the case that n is odd, if B_1 is taken as the mid-point of the line-segment A_1A_{n+1}, then B_1 will coincide with B_{n+1}, and the required polygon is constructed. In the case that n is even, if A_1 and A_{n+1} happen to coincide, the polygon is constructed and B_1 will coincide with B_{n+1}. If A_1 and A_{n+1} do not happen to coincide, then the construction is impossible.

Note by the Editors. If n is odd there is only one polygon. For if there were two with vertices A_i and B_i, respectively, then vector $A_{n+1}B_{n+1}$ = vector B_1A_1; but this is impossible since $A_{n+1}B_{n+1}\equiv A_1B_1$.

If n is even and if there exists one polygon with the given mid-points, there are an infinite number of such polygons obtained by taking any point for B_1 in the above construction. For let A_i be the vertices of the first polygon, then vector A_1B_1 = vector A_1B_{n+1}. Hence $B_{n+1}\equiv B_1$ and the B figure closes.

Also solved by J. C. Bennett, Leonard Carlitz, Michael Goldberg, A. Pelletier, and F. L. Wilmer.

3130 [1925, 204]. Proposed by Paul Capron, U. S. Naval Academy.

Show that if d, D are the double polar distances, or diameters, of the inscribed and circumscribed circles of a spherical triangle, and d', D' the corresponding diameters for its polar triangle,

$$d'+D=180^\circ=d+D'\ .$$

Solution by R. H. Sciobereti, Berkeley, California.

Let $a, b, c; A, B, C; r, R$ be the sides, angles, radius of the inscribed circle and radius of the circumscribed circle of a spherical triangle. Denote the corresponding parts of its polar triangle by the same letters accented. For the given triangle we have the well known formula

$$\tan\frac{A}{2}=\frac{\tan r}{\sin(s-a)}\ , \qquad 2s=a+b+c\ ; \tag{1}$$

while from one of the right triangles of its polar triangle we have

$$\cos(S'-A')=\cot R'\tan\frac{a'}{2}\ , \qquad 2S'=A'+B'+C'\ . \tag{2}$$

Setting in (2) $A'=\pi-a$, $a'=\pi-A$, $S'-A'=\pi/2-(s-a)$ we obtain

$$\tan\frac{A}{2}=\frac{\cot R'}{\sin(s-a)}\ . \tag{3}$$

Hence from (1) and (3) $\tan r\tan R'=1$; and, since we suppose that the poles of the circles are chosen so that the radii are less than $\pi/2$, we must have

$$r+R'=\frac{\pi}{2}\ .$$

In the same manner we find that $r'+R=\pi/2$ and hence

$$d+D'=d'+D=\pi\ .$$

Also solved by J. A. Bullard, Theodore Bennett, Leonard Carlitz, and A. Pelletier.

3134. [1925, 261] Proposed by J. Rosenbaum, Milford, Conn.

In a given circle to inscribe a hexagon of assigned angles which shall have the maximum area.

Solution by the Proposer.

The assigned angles A, B, C, etc., determine the arcs FAB, ABC, BCD, etc.

As regards the area of the variable hexagon, we can consider either set of the three alternate arcs as fixed in position. Accordingly, we take the arcs FAB, BCD and DEC as fixed; thus obtaining the fixed triangle FBD, with the variable points A, C, and E lying respectively on the three fixed arcs.

It is our problem to locate these points on these arcs so as to satisfy the requirements as to the angles F, B, and D, and so that the area $ABCDEF$ shall be the maximum.

The triangle FBD being fixed, it is only necessary to consider the sum of the areas of the three triangles FAB, BCD, and DEF. Denoting the angular measure of the arc FA by x, the remaining five arcs are by vitrue of the given angles determined as linear functions, L, of x. This, in turn, determines the six chords, FA, AB, BC, etc., as functions, $\sin L(x)$, and the sum of the areas of the three triangles which is equal to $\frac{1}{2}FA\cdot AB\sin A+\frac{1}{2}BC\cdot CD\sin C+\frac{1}{2}DE\cdot EF\sin E$, is thus expressible as a function, F, of x.

Equating $F'(x)$ to zero, and solving, we obtain

$$\tan x=\frac{\sin^2 A+\sin C\sin(2A-2B+C)-\sin(A+C)\sin(A-2B+C-2D)}{-\sin A\cos A-\sin C\cos(2A-2B+C)+\sin(A+C)\cos(A-2B+C-2D)}\ . \tag{1}$$

Furthermore, since at least one of the arcs FB, BD, DF, is less than or equa to $120°$, the arc FB can be chosen as being less than or equal to $120°$, and since, for a convex polygon, arc FA is less than arc FB, we can write $x<120°$.

Equation (1) together with (2) determines x uniquely. It is seen that the point A can be constructed with ruler and compasses, and then the points C and E can also be located.

We notice that if the value of x as determined by (1) and (2) is less than or equal to zero, or is equal or is equal to or greater than arc FB, then a maximum hexagon does not exist. There will also exist no maximum if either of the points C and E as determined by the point A and the given angles B and D do not fall within the arcs BD and DF respectively. The upper limit, on the other hand, always exists, and can be constructed and completed.

By considering the altitudes from the vertices A, C, and E of the triangles FBA, BDC, and DFE, it is easy to prove that when these vertices rotate so that the sum of the areas of the three triangles is decreasing then the rate of decrease is increasing; from which follows that *there is no minimum.*

The solution is equally applicable to any polygon of an even number of sides. In case of an odd number of sides, the angles determine the polygon if it is to be inscriptible, and there is no problem

Also solved by W. J. PATTERSON.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

At the Kansas City meeting of the American Association for the Advancement of Science, Professor L. H. BAILEY was elected president, Professor E. V. HUNTINGTON, vice-president of Section A (mathematics), Professors M. I. PUPIN and F. R. MOULTON members of the Executive Committee, and Professor H. H. MITCHELL member of the committee of Section A. Professor R. C. ARCHIBALD is Secretary of Section A. The third annual prize of the Association has been awarded to Professor D. C. MILLER of the Case School of Applied Science for his address on the Michelson-Morley ether drift experiment, its history and significance, this being the presidential address of the American Physical Society, presented at a joint session of that society with the Association.

Professor L. E. DICKSON has been elected an honorary member of the Union of Mathematicians and Physicists of Czecho-Slovakia.

Professor R. A. MILLIKAN has been elected correspondent of the Paris Academy of Sciences in the section of physics.

Hampden Sidney College has conferred an honorary doctorate on Professor J. E. WILLIAMS of the department of Mathematics of the Virginia Polytechnic Institute.

Dr. H. M. GEHMAN who took his Ph.D. degree at the University of Pennsylvania in February, 1925, is at the University of Texas as a National Research Fellow in mathematics.

At the University of Tennessee, Associate Professor J. D. BOND has been promoted to a full professorship in mathematics.

At the University of Chicago, Assistant Professor E. P. LANE has been promoted to an associate professorship. Dr. MAYME I. LOGSDON, who is studying in Rome, will return to resume her duties in October, 1926.

At Harvard University, Professor G. D. BIRKHOFF has been given the additional title of Cabot Fellow, and will be relieved of part of his teaching so that he may devote more time to original work.

Dr. ECHO D. PEPPER is studying in Oxford under Professor Hardy (and not in Paris as previously announced in these notes). She holds an international fellowship of the General Education Board.

At Pennsylvania State College, the following promotions have been made: Associate Professor T. E. GRAVATT has been promoted to a full professorship; Assistant Professor C. C. WAGNER to an associate professorship; Mr. JABIR SHUBLI to an assistant professorship. Professor J. H. TUDOR and Assistant Professor L. S. JOHNSTON are on leave of absence this year. Professor Johnston is studying at the University of Chicago.

At the new Texas Technological College, Adjunct Professor J. N. MICHE of the University of Texas has been appointed professor and head of the department of mathematics. Mr. D. A. FLANDERS of the University of Pennsylvania has been appointed professor. Assistant Professor W. M. WHYBURN of Texas Agricultural and Mechanical College, and Dr. L. D. AMES formerly associate professor at the University of Missouri have been appointed associate professors. Miss E. T. STAFFORD, instructor at the University of Texas has been appointed adjunct professor.

At Pomona College the departments of astronomy and mathematics have been separated and Professor F. P. BRACKETT, who was head of the department of mathematics, has been made professor of astronomy and head of the department. Professor W. P. RUSSELL, who was associate professor of mathematics, has been made professor of mathematics and head of the department.

Assistant Professor W. F. SHENTON of the department of mathematics at the United States Naval Academy, Annapolis, has resigned to accept the chair of mathematics at the American University, Washington, D. C.

Assistant Professor C. O. WILLIAMSON, of the College of Wooster, has been granted leave of absence for the second semester of the current year, in order to carry on graduate work at the University of Chicago.

At Shorter College, Professor RUBY HIGHTOWER has resumed her work as head of the department of mathematics after a year's leave of absence spent

in study at the University of Missouri. Miss MABEL THOMPSON has been appointed instructor in mathematics.

At Middlebury College, Miss ELLEN WILEY and Mr. BERT HAZELTINE have been promoted to assistant professorships of mathematics.

Mr. M. F. JORDAN has been appointed associate professor of astronomy at the University of Maine.

The following appointments to instructorships of mathematics are announced:

University of Maryland, Mr. R. W. RICHESON.

The State College of Washington, Mr. CONSTANTINE COLOGERIS and Miss ALICE ANN GRANT.

THE SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The tenth summer meeting of the Mathematical Association of America will be held by invitation at Ohio State University, Columbus, Ohio, on Tuesday and Wednesday, September 7–8, 1926, in conjunction with and followed by that of the American Mathematical Society. Two sessions of the Association will be held on Tuesday, a joint session of the Association with the Society on Wednesday morning, and sessions of the Society on Wednesday afternoon and on Thursday morning and afternoon. A public reception will be given on Tuesday evening and the joint dinner of the two organizations will be held on Wednesday evening.

All the meetings will be held in the new Faculty Club building of the University, unusual facilities being furnished for the social and program sides of the meeting. The dining room and recreation rooms will be available for this occasion. Lodging and a substantial breakfast can be obtained at Oxley and Mack residence halls at a cost of $1.50 per day. Tenting space and free parking for automobiles will be provided directly in the rear of the residence halls, where a police officer will be in charge day and night. It is also hoped to arrange for tennis courts near the Stadium, without charge to members and guests. A landing field for aeroplanes is also available to our progressive members. Columbus is centrally located, being easily accessible by railroad and automobile.

An attractive program is being arranged by the Program Committee of the Association. The details of the program and blanks for registration for rooms and meals will be sent to the Association members in the early summer.

W. D. CAIRNS, *Secretary-Treasurer.*

The Carus Mathematical Monographs

The CARUS MONOGRAPHS have already begun to fulfill their mission as intended by the generous donor, MRS. MARY HEGELER CARUS and her son, DR. EDWARD H. CARUS.

The first edition of Monograph Number One, "Calculus of Variations," by Professor Bliss, will soon be exhausted.

Nearly 1000 members of the Association have taken advantage of the distribution at cost. Those who neglected to do so at the start may still have the privilege by applying to the Secretary. This includes institutional as well as individual members.

It would be a great tribute to the donor and an honor to the Association if every member would subscribe for the complete series. Every dollar thus paid by members is a contribution to the permanent Publication Fund of the Association.

It is believed that the Association is rendering a great service to mathematics by this enterprise, and a one hundred per cent support from the membership would constitute an appropriate vote of confidence in the undertaking.

Monograph Number Two, "Analytic Functions of a Complex Variable," by Professor Curtiss, has already been delivered to those who have sent in their subscriptions to the Secretary. All distribution to the general public will be made through

THE OPEN COURT PUBLISHING COMPANY
122 South Michigan Avenue, Chicago, Illinois

DESIRED COPIES OF THE MONTHLY

THROUGH an oversight the Association has not reserved a sufficient number of copies of the MONTHLY for October, 1920, and for August, September, and October, 1921, also May, September, and October, 1924. Cash, or credit toward future dues, will be given by the Secretary for such copies at the rate of forty-five cents per copy, up to a limited number of copies.

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary.

W. D. Cairns, Oberlin, Ohio

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.

Eleventh Annual Meeting, Philadelphia, Pa., December, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7-8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., May 8.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Fort Collins, Colo., April 16-17.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE TENTH ANNNAL MEETING OF THE ASSOCIATION

The tenth annual meeting of the Mathematical Association of America was held at the Junior College, Kansas City, Mo. on Wednesday and Thursday, Dec. 30–31, 1925, in conjunction with the Western meeting of the American Mathematical Society and in affiliation with the American Association for the Advancement of Science. One hundred and sixty were present at the various sessions, including the following 122 members of the Association:

ELBERT ALLEN, University of Missouri.
FLORENCE E. ALLEN, University of Wisconsin.
N. ALTSHILLER-COURT, University of Oklahoma.
R. C. ARCHIBALD, Brown University.
G. N. ARMSTRONG, Ohio Wesleyan University.
C. H. ASHTON, University of Kansas.
C. S. ATCHISON, Washington and Jefferson College.

WEALTHY BABCOCK, University of Kansas.
TALMON BELL, Sterling College.
FLORENCE BLACK, University of Kansas.
W. C. BRENKE, University of Nebraska.

W. D. CAIRNS, Oberlin College.
FLORIAN CAJORI, University of California.
THEODOSIA T. CALLAWAY, Stephens Junior College.
A. D. CAMPBELL, University of Arkansas.
R. D. CARMICHAEL, University of Illinois.
C. C. CARTER, Bluffs, Ill.
E. W. CHITTENDEN, University of Iowa.
J. L. COOLIDGE, Harvard University.
BYRON COSBY, State College, Kirksville, Mo.
A. R. CRATHORNE, University of Illinois.
L. H. CUTTING, Westport High School, Kansas City.

MARIAN E. DANIELLS, Iowa State College.
L. S. DEDERICK, University of British Columbia.
I. M. DELONG, University of Colorado.
W. W. DENTON, University of Michigan.
E. L. DODD, University of Texas.
LUCY T. DOUGHERTY, Junior College, Kansas City, Kan.
ARNOLD DRESDEN, University of Wisconsin.
OTTO DUNKEL, Washington University.

W. E. EDINGTON, Purdue University.
C. A. EPPERSON, Kansas City Day School for Boys.
H. S. EVERETT, Bucknell College.

EDNA M. FELTGES, Orlando, Fla.
J. C. FIELDS, University of Toronto.
B. F. FINKEL, Drury College.
R. R. FLEET, William Jewell College.
T. M. FOCKE, Case School of Applied Science.

M. G. GABA, University of Nebraska.
W. H. GARRETT, Baker University.
F. J. GERST, St. Louis University.
G. D. GORE, South Dakota State College.
G. W. GORRELL, University of Denver.
H. C. GOSSARD, University of Wyoming.
CORNELIUS GOUWENS, Iowa State College.

W. A. HARSHBARGER, Washburn College.
J. O. HASSLER, University of Oklahoma.
OLIVE C. HAZLETT, University of Illinois.
E. R. HEDRICK, University of Calif., Southern Branch.
KATHERINE E. HELWIG, State Teachers College, Maryville, Mo.
T. H. HILDEBRANDT, University of Michigan.
A. J. HOARE, Fairmount College.
INA E. HOLROYD, Kansas State Agric. College.
GOLDIE P. HORTON, University of Texas.
J. M. HOWIE, Nebr. Wesleyan University.
JEWELL C. HUGHES, University of Arkansas.
E. V. HUNTINGTON, Harvard University.
EMMA HYDE, Kansas State Agric. College.

BYRON INGOLD, Culver-Stockton College.
LOUIS INGOLD, University of Missouri.

DUNHAM JACKSON, University of Minnesota.
G. H. JAMISON, State Teachers College, Kirksville, Mo.
W. C. JANES, Kansas State Agric. College.
H. E. JORDAN, University of Kansas.

J. R. KLINE, University of Pennsylvania.
L. C. KNIGHT, College of Wooster.

KURT LAVES, University of Chicago.
C. F. LEWIS, Kansas State Agric. College.
FLORENCE P. LEWIS, Goucher College.
R. G. LUBBEN, University of Texas.
W. A. LUBY, Kansas City Junior College.
A. C. LUNN, University of Chicago.
W. H. LYONS, Kansas State Agric. College.

J. V. MCKELVEY, Iowa State College.
S. L. MACDONALD, Colorado State Agric. College.
WILLIAM MARSHALL, Purdue University.
ANNA MARM, Bethany College.
C. A. MESSICK, University of Iowa.
G. R. MIRICK, Lincoln School, New York.
U. G. MITCHELL, University of Kansas.
R. L. MOORE, University of Texas.
THIRZA A. MOSSMAN, Kansas State Agric. College.
F. R. MOULTON, University of Chicago.
PAUL MUEHLMAN, Loyola University.
F. D. MURNAGHAN, Johns Hopkins University.
ARRIA MURTO, Carthage (Mo.) High School.

I. F. NEFF, Drake University.

JESSE OSBORN, Harris Teachers College.

H. P. PETTIT, Illinois Wesleyan University.
A. W. PHILIPS, Kansas State Agric. College.
E. C. PHILLIPS, Georgetown College Observatory.
T. A. PIERCE, University of Nebraska.
A. D. PIERSON, Kansas City Junior College.

C. A. REAGAN, University of Kansas.
P. R. RIDER, Washington University.
H. L. RIETZ, University of Iowa.
W. J. RISLEY, Colorado School of Mines.
W. H. ROEVER, Washington University.
J. B. ROSENBACH, Carnegie Inst. of Technology.
H. E. RUSSELL, University of Denver.

FAITH SAUNDERS, Alderson (West Virginia) Junior College.
G. T. SELLEW, Knox College.
J. I. SHANNON, St. Louis University.
J. A. G. SHIRK, Pittsburg (Kans.) State Teachers College.
L. S. SHIVELY, Mt. Morris College.
C. H. SISAM, Colorado College.
E. R. SMITH, Iowa State College.
G. W. SMITH, University of Kansas.
I. W. SMITH, North Dakota State College.
ELIZABETH T. STAFFORD, Texas Tech. College.
R. C. STALEY, Kansas State Agric. College.
EDITH STEININGER, Coffeyville (Kans.) Junior College.
E. B. STOUFFER, University of Kansas.
C. J. STOWELL, McKendree College.
W. T. STRATTON, Kansas State Agric. College.

J. S. TURNER, Iowa State College.

R. A. WELLS, Park College.
J. J. WHEELER, University of Kansas.
A. E. WHITE, Kansas State Agric. College.
S. J. WIFVAT, Drake University.
FRANCES M. WRIGHT, University of Oklahoma.

JESSICA M. YOUNG, Washington University.

The attendance from the Middle West as shown in the foregoing list indicates that many utilized the opportunity afforded by the meetings at Kansas City. Some colleges and universities were represented by nearly the complete staff of teachers of mathematics and many expressions were heard as to the pleasure in making the acquaintance of other mathematicians and the profit gained from listening to reports of research work and of other topics relating to collegiate and graduate teaching. It was withal a disappointment to the officers of the Association that there was not a larger attendance from this central region; only about one-fourth of the members of the Association living within four hundred miles of Kansas City were present at the meetings. The inclement weather and other cogent causes operated to lower the attendance but the same shortage in attendance has been evident at other meetings of the Association and the Society, and it is the feeling of your officers that programs so attractively brought together by able and busily occupied program committeemen should have fuller support, since the inspirational element in

our organizations is perhaps the greatest justification for their existence. Let us encourage the members of our teaching staffs to make an extra effort to attend our meetings and in this, as in other ways, to share actively in promoting the helpfulness of these gatherings. In particular, let us plan definitely to take part in the September meetings at Ohio State University.

The sessions of the American Association for the Advancement of Science began with the formal opening on Monday evening in the Auditorium of the Junior College, Dr. J. McKeen Cattell, editor of *Science*, giving the retiring presidential address on "Some psychological experiments." This address appeared in *Science* for January 1 and 8. The Kansas City meetings were noteworthy for the large number of lectures which were offered after the pattern of the British Association lectures, popular presentations of recent advances in science given by outstanding men in various fields. On Tuesday afternoon Professor D. C. Miller of the Case School of Applied Science delivered an address as president of the American Physical Society on "The Michelson-Morley ether-drift experiment, its history and significance"; he gave the finished results of his experiments on the repetition and refinement of the Michelson-Morley work, which he has been conducting on Mount Wilson for the past two years, and reported his conclusion, on the basis of extensive computations, that the solar system is moving toward the constellation of the Dragon with a velocity of 200 kilometers per second, the earth dragging the ether along with it to the extent of 95 per cent with a residual "slip" of 5 per cent. This address was regarded as a notable contribution to the advancement of science by the American Association committee on award and as such was awarded the American Association Prize of one thousand dollars.

On Tuesday evening Professor R. A. Millikan, director of the Norman Bridge Physical Laboratory, lectured on "The stripped atom." On Wednesday afternoon the American Association paid the American Mathematical Society the compliment of making the third annual Josiah Willard Gibbs lecture a general session of the American Association. Before an audience of 600 or more the lecture was delivered by Professor James Pierpont on "The history of man's effort to solve the problem of space, and the effect of relativity on our views"; this address will be published in the *Bulletin*. At the same hour in another auditorium four lectures were given under the auspices of the American Association's special committee, on the place of the sciences in education; these papers were by Dr. O. W. Caldwell, of the Lincoln School, Columbia University, chairman of the special committee, Dr. B. E. Livingston, Johns Hopkins University; Dr. E. E. Slosson, of Science Service, and Professor M. I. Pupin, Columbia University, president of the Association. Among the other general sessions and addresses may be mentioned a session on the relations of engineering to the fundamental sciences, a session under the auspices of

the committee of one hundred on scientific research, and a lecture on Thursday evening by Professor F. R. MOULTON, University of Chicago, on "The origin and evolution of worlds." Full accounts of the American Association meetings and numerous papers on the programs will appear in early issues of *Science*.

Professor L. H. BAILEY of Cornell University was elected president of the American Association for 1926. On nomination by the section committee for Section A, Professor E. V. HUNTINGTON was chosen vice-president and Professor H. H. MITCHELL member of the section committee, Professor R. C. ARCHIBALD continuing his term as secretary.

The joint dinner of the mathematicians with 127 persons present was held at the City Club on Wednesday evening. Professor F. R. Moulton introduced the speakers in a happy fashion. Professor Hedrick told of the transfer of the printing of the *Bulletin* and *Transactions* to this country, of the subsidizing of these journals by a grant obtained through the National Academy of Sciences, and of the promise of excellent work which is shown at the outset of this new venture. Professor Pierpont gave delightful reminiscences of Josiah Willard Gibbs, under whom he had studied at Yale. Dr. Jewell Hughes recalled graduate days at the University of Chicago and various courses under Professor Moulton. Professor Coolidge as the representative of the Mathematical Association spoke of the pleasant experiences at Kansas City, of the recognition by the American Association of mathematics as one of the sciences, adding with true eloquence a tribute to mathematics as an art, to which we are drawn by its beauty, by its cultural and ideal spirit and by the symbolic character of mathematics which it shares with the other arts. Professor Carmichael discussed the possibility of mathematical meetings at Nashville in Dec. 1927 in affiliation with the American Association; he expressed his judgment that there is promise enough in such a meeting to make it profitable and pleasant to all concerned. Various speakers followed with expressions of agreement, of the feasibility of meeting at Nashville which is not so far distant as are other of our meeting places, and of the desirability of holding mathematical meetings as far south as possible for the benefit and inspiration of teachers in the South. Those present at the dinner voted unanimously in favor of holding the meetings at Nashville. At the conclusion of this dinner the thanks of Section A, the Mathematical Association, and the Mathematical Society were voted especially to Professors Luby and Stouffer for the excellent preparations and the delightful appointments for the dinner. By formal vote at one of the sessions, the cordial greetings of the mathematicians at Kansas City were sent to their colleagues assembled at Hunter College.

In a joint meeting of Section A with Section L (Historical and Philological Sciences) Professor W. H. ROEVER of Washington University read an interesting paper on "William Chauvenet and his mathematical contribution to astronomy."

The American Mathematical Society held its twenty-fourth Western meeting together with that of the Southwestern Section on Tuesday and Wednesday. Forty-four papers were read at the two sessions on Tuesday. The third Gibbs lecture has already been noted above.

The meetings of the Mathematical Association consisted of a joint session on Wednesday morning and two separate sessions on Thursday. The program was prepared under the following Committee: L. W. DOWLING, chairman; W. C. BRENKE, OLIVE C. HAZLETT, G. H. LIGHT. This association joined with the other two organizations on Wednesday morning, but without a formal representative, in order that there might be adequate time for the two important addresses of that session. Professor H. J. Ettlinger, who was to have given a paper on "Mathematical clubs for colleges and universities," was prevented from attending the meetings through sickness in the family, and the committee decided to postpone this paper to a later meeting at which it might be presented by the author. Abstracts of some of the papers are given, numbered in accordance with the numbers of the papers.

JOINT SESSION OF THE ASSOCIATION WITH THE AMERICAN MATHEMATICAL SOCIETY AND SECTION A OF THE AMERICAN ASSOCIATION

(1) "The Heine-Borel theorem and allied problems" by Professor T. H. HILDEBRANDT, University of Michigan, Vice-president of the American Mathematical Society.

(2) "The algebraic numbers and division" by Professor J. C. FIELDS, University of Toronto, retiring vice-president of Section A.

1. Professor Hildebrandt's address will appear in the *Bulletin* of the American Mathematical Society.

2. Professor Fields' paper will appear in *Science.*

FIRST SESSION OF THE ASSOCIATION.

(3) "Robert Adrain and the beginnings of American mathematics,"—retiring presidential address by Professor J. L. COOLIDGE, Harvard University.

(4) "The definition of function and its effect on elementary and advanced instruction" by Professor E. R. HEDRICK, University of California, Southern Branch.

3. President Coolidge's paper appeared in full in the February issue.

4. Professor Hedrick reviewed the elementary definition of function in its classical sense, and certain closely associated concepts. The modern extensions of these concepts were stated and the effect of the concept and its associated ideas on both elementary and advanced instruction was emphasized, with a considerable number of specific instances. It was shown that the content of courses in mathematics, and the method of instruction, would be greatly affected by realization, on the part of teachers, of the fundamental connections which exist with the function concept.

SECOND SESSION OF THE ASSOCIATION

(5) "Determinants and their principal minors" by Professor E. B. STOUFFER, University of Kansas.

(6) "The course in statistics in the department of mathematics," by Professor A. R. CRATHORNE, University of Illinois.

(7) "Some applications of mathematics to architecture" by Professor E. C. PHILLIPS, Georgetown University.

(8) "A new method of determining a series solution of linear differential equations with constant or variable coefficients" by Mr. W. O. PENNELL, Chief engineer, Southwestern Bell Telephone Company, St. Louis.

5. Professor Stouffer indicated briefly the work which has been done in discovering relations among the principal minors of determinants. He pointed out that a certain theorem by McMahon published in 1893 is incorrect. This theorem was apparently fundamental in the study of the relations among the principal minors, and as a consequence of its acceptance by later writers, only results of little importance have been obtained. In the present paper McMahon's error is pointed out, and complete systems of principal minors for the general determinant are derived and also complete systems of relations among the principal minors are obtained. The whole idea is generalized to apply to certain combinations of any number of determinants. Several applications of the results to other fields of mathematics are indicated.

6. Professor Crathorne's paper appears in full in this number of the MONTHLY.

7. After a brief historical account of the origin and development of Gothic architecture and a description of the three main styles of decorative tracery used especially in the windows of Gothic buildings, namely geometrical tracery, flowing tracery and flamboyant tracery, Professor Phillips gave a simple mathematical expression for the general groundwork of all such tracery which resembles the outlines of a flat flower with pointed petals; from this simple expression which represents very well the earliest forms of flowing tracery there were obtained, by suitable modifications of the arbitrary constants of the equations, the mathematical expressions to represent the somewhat more complex forms which architects introduced as the art of Gothic decoration advanced in perfection. A method was then given for changing the shape of the petal-like elements of the decorative scheme without altering the mutual relations of these elements among themselves; the varying artistic effect thus produced in the design was studied and an attempt was made to find out which of these forms was most pleasing. The last portion of the paper treated briefly the mathematical equations representing the latest, and probably the least perfect, type of tracery, the one which was used in flamboyant gothic.

Mathematically the entire paper may be summed up in the brief statement that all forms of Gothic window tracery are fairly well represented by the equations:

$$\theta=(c/p)(t-\tfrac{1}{2}\sin 2t)+b\rho\,,$$
$$\rho=1+k\cos t-m\sin^2 t\cos t\,,$$

where θ and ρ are polar coördinates, t is the parametric variable, and c, p, b, k and m are arbitrary constants.

8. The method described by Mr. Pennell is an operational method for the solution of any linear differential equation. Heretofore operational methods have been developed for the solution of linear equations with constant coefficients. The method outlined in this paper is applicable to any linear equation with constant or with variable coefficients. The method is one of operational division. The differential equation is first rewritten substituting for the differential coefficients d^n/dx^n the operational symbol p^n. The equation is then solved by algebraic methods for the unknown quantity which is expressed in terms of p and the independent variable. This is called the operational solution. The series solution is then obtained from the operational solution by a method of operational long division. This method is the key to the entire process and reduces the series solution of any linear equation to what is substantially an algebraic process. For many equations it is believed that this gives a much more simple and direct solution than any hitherto known.

The theory was developed by showing that there is an algebra which obeys the commutative, distributive and associative laws and with one exception parallels in its operation the results obtained by integration and differentiation so that the operational division by which the solution of the differential equations is obtained is merely long division following the rules of the operational algebra.

Meeting of the Board of Trustees of the Association.

Nine members of the Board were present at the various sessions.

The following twenty-nine persons, on applications duly certified, were elected to individual membership:

Violet M. Andrews, Junior Student, Oberlin Coll., Oberlin, Ohio.

J. W. Blincoe, A.M. (Virginia). Prof., Math. and Physics, Central Wesleyan Coll., Warrenton, Mo.

M. Sue Burney, A.M. (Chicago). Instr., Jr. Coll., St. Joseph, Mo.

W. H. Buxton, A.M. (Oregon). Prof., Whitworth Coll., Spokane, Wash.

Joseph Canning, A.B. (Intermountain Union Coll.). Railroad Mach., Helena, Mont.

D. S. Dearman, M.S. (Millsaps), A.M. (Vanderbilt). Head of Dept., Ky. Wesleyan Coll., Winchester Ky.

Mrs. Flora H. Eaton, A.M. (North Carolina). Registrar and Head of Dept. of Math., Mars Hill Coll.. Mars Hill, N. C.

Frances Gilmore. Breaux Bridge, La.

G. D. Gore, M.S. (Chicago). Instr., S. Dak. State Coll., Brookings, S. Dak.

Margaret E. Harris, A.M. (Teachers Coll., Columbia). Chair of Math., Grenada Coll., Grenada, Miss.

A. H. Jekel, A.B. (Michigan). Actuary's Dept., Mo. State Life Ins. Co., St. Louis, Mo.

Vladimir Karapetoff, C.E., M.M.E. (Russia). Prof., Elec. Eng., Cornell Univ., Ithaca, N. Y.

Mary C. Martin. Teacher, High School, Mount Morris, Ill.

T. A. Martin, A.M. (Ohio Wesleyan; Yale). Head of Dept., Berea Coll., Berea, Ky.

T. W. MOORE, B.S. (Washington and Jefferson). Asst., Yale Univ., New Haven, Conn.
SISTER M. PRUDENTIA MORIN, A.B. (Catholic Univ. of Amer.). Teacher, Coll. of St. Scholastica, Duluth, Minn.
Miss ARRIA MURTO, B.S. (Missouri). Teacher, High School, Carthage, Mo.
E. G. OLDS, A.M. (Pittsburgh). Asst. Prof., Carnegie Inst. of Tech., Pittsburgh, Pa.
D. D. PEELE, A.M. (Chicago). Dean and Prof. of Math., Columbia Coll., Columbia, S. C.
J. M. RICE, E.E. (Lehigh). Instr., Penna. State Coll., State College, Pa.
SUSAN V. RICHMOND, A.B. (Randolph-Macon W. Coll.). Teacher, Western High School, Washington, D. C.
KATHERINE J. RYNO, A.M. (Tennessee). Asso., Lander Coll., Greenwood, S. C.
R. C. STALEY, Instr., State Agric. Coll., Manhattan, Kans.
P. L. STEVENSON, A.M. (Colorado). Prof., Westminster Coll., Salt Lake City, Utah.
EVELYN R. THOMPSON, A.B. (Mt. Holyoke). Teacher, Western High School, Washington, D. C.
C. H. VEHSE, B.S. (Brown). Asst., Brown Univ., Providence, R. I.
P. L. WELTON, A.M. (Michigan). Instr., East High School, Rochester, N. Y.
W. M. WIBLE, A.M. (Indiana). Prof., Intermountain Union Coll., Helena, Mont.
J. B. WINSLOW, A.B. (Toledo Univ.). Instr., Physics and Math., Toledo Univ., Toledo, Ohio.

The following were appointed associate editors of the MONTHLY for the year 1926:

N. H. ANNING	H. J. ETTLINGER	H. W. KUHN
H. E. BUCHANAN	H. S. EVERETT	C. N. MILLS
W. B. CARVER	B. F. FINKEL	F. D. MURNAGHAN
OTTO DUNKEL	TOMLINSON FORT	D. E. SMITH

The Trustees voted to re-appoint the Secretary and Dean T. M. FOCKE of Case School of Applied Science as the representatives of the Association on the Council of the American Association for 1926 and to appoint Professor DUNHAM JACKSON as the representative of the Association on the National Research Council for a three year term, to succeed Professor H. L. RIETZ whose term expires June 30, 1926. They also voted to hold the annual meeting at Nashville in December 1927 in affiliation with the A.A.A.S.

The Trustees acted on a proposition under which regional associations of teachers of mathematics may be affiliated with this association, members of such associations to be admitted to the Mathematical Association without the payment of the customary initiation fee. Fuller details will be announced when mutual arrangements are more completely made.

In recognition of the services of Mr. Curtis C. Carter, of Bluffs, Illinois, in interesting a friend in the work of the Association, leading to the making of her will in favor of the Association, a bequest of large import for the future, the Trustees voted to make Mr. Carter a life member.

The Trustees directed the sending of a letter to Professor David Eugene Smith voicing the sentiments of the Association on the occasion of his retirement from teaching. The letter follows, together with a reply from Professor Smith:

DEAR PROFESSOR SMITH:

The Mathematical Association of America desires to convey to you its congratulations on your completion of twenty-five years of service in Teachers College, and its deep sense of the significance of your service to the Association and to the teaching of mathematics throughout the country.

You have thrown light on the history of our science, not as a dead record of things gone by, but as the heritage of experience which must be our present guide; you have surveyed the life of our time, not as a spectacle apart, but as a progress in which we must all find our proper place; and you have striven with unfailing devotion and with conspicuous success to share with all men in fullest measure the wisdom that you have achieved.

The Association is proud to claim as one of its leaders a man who has personally contributed so much toward the accomplishment of the purposes to which it is dedicated.

For the Trustees of the Association:
DUNHAM JACKSON, President,
W. D. CAIRNS, Secretary-Treasurer.

MY DEAR COLLEAGUES:

I cannot tell you how deeply I appreciate the message which I have received from you through the hands of Professor Jackson, as President, and Professor Cairns, as Secretary-Treasurer. I cannot fully comprehend the fact that today is the last one of my direct association with this college and university. My closing lecture is given this afternoon. The disagreeable sensation of closing my active connection with our students is rendered less acute by such letters as come to me daily from my friends throughout the country.

I wish to say that one of the delightful experiences in my life has been my connection with the Mathematical Association of America, and with the AMERICAN MATHEMATICAL MONTHLY. I hope at some future time that this appreciation will be more fully understood and in a more tangible way than at present.

Yours very sincerely,
DAVID EUGENE SMITH.

THE CHAUVENET PRIZE.

The committee on the award of the first Chauvenet Prize for excellence in mathematical exposition, Professors W. C. GRAUSTEIN, ANNA PELL WHEELER, and E. B. VAN VLECK, chairman, recommended that the award be made to Professor G. A. BLISS of the University of Chicago for his paper on "Algebraic functions and their divisors," published in the *Annals of Mathematics*, volume 26, Numbers 1 and 2, September and December, 1924. The Trustees voted to approve this choice and to thank the members of the committee for their arduous but very valuable efforts. The award was announced at the business meeting and the prize of one hundred dollars, furnished by a member of the Association, was presented to Professor Bliss following the meetings. His comment on the significance of this prize award is worthy of being shared with the membership of the Association:

DEAR PROFESSOR CAIRNS:

I received recently your letter describing the newly established Chauvenet Prize for excellence in mathematical exposition, and announcing the decision of the Trustees. This, and the substantial check which preceded it, have given me much pleasure, as you will well understand, but I should like to add that the award of the prize for the particular paper you mention has had for me an especially encouraging interest. For many years past the so-called "arithmetic method" in the theory of algebraic functions has seemed to me accessible in the literature with a difficulty quite out of proportion to its completeness and beauty, and I have often wished for a presentation of it which would lead with directness to the fundamental results of the algebraic function theory. When the simplifications in my paper finally occurred to me, after a good deal of study at various times, I thought it would be worth while to set them down in print where they might perhaps aid some future inquirer into this unusually interesting mathematical domain. You will readily understand from these remarks, I think, why the discovery of

this paper by your Committee, and their expression of approval of it, have gratified me especially. I congratulate the Association on the inauguration of the Chauvenet Prize, and hope that it may give to many others in the future the same pleasant impetus which it gives to me.

Yours very sincerely,

G. A. BLISS.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The Secretary-Treasurer announced the names of those elected to membership. He reported also the death of the following members:

G. P. ALDRICH, Instructor in mathematics, University of Iowa (October 8, 1925).

W. W. R. BALL, Trinity College, Cambridge, England (April 4, 1925).

A. G. HALL, Registrar, University of Michigan (January 10, 1925).

W. A. HAMILTON, Professor of mathematics, Antioch College. (June 25, 1925).

W. E. HEAL, U. S. Coast and Geodetic Survey, (October 9, 1925).

P. A. LAMBERT, Professor of mathematics, Lehigh University (February 15, 1925).

MARCIA LATHAM, Instructor in mathematics, Hunter College (May 9, 1925).

L. E. LUNN, Superintendent of schools, Heron Lake, Minnesota (June 18, 1924).

MANSFIELD MERRIMAN, Consulting engineer, New York (June 7, 1925).

M. T. PEED, Professor of mathematics, Emory University (August 28, 1925).

C. E. STROMQUIST, Professor of mathematics, University of Wyoming (April 3, 1925).

G. W. SUBLETTE, Consulting and construction engineer, Minneapolis (November 6, 1925).

W. P. YANCEY, Professor of mathematics and physics, St. Ambrose College (March 13, 1925).

The election of officers for the year 1926 was conducted by mail and in person at this meeting, Professor Crathorne and Mr. Pierson acting as tellers, the result of the ballot being as follows:

For President: A. A. Bennett, 225 votes; Dunham Jackson, 342 votes.

For Vice-Presidents: E. L. Dodd, 212 votes; W. B. Ford, 341 votes; Clara E. Smith, 266 votes; J. W. Young, 298 votes.

For additional members of the Board of Trustees, to serve until January 1929: Florian Cajori, 445 votes; A. B. Chace, 170 votes; J. L. Coolidge 320 votes; B. F. Finkel, 241 votes; C. F. Gummer 189 votes; U. G. Mitchell, 170 votes; E. H. Moore, 366 votes; Oswald Veblen 343 votes.

The following were accordingly declared elected:

President: DUNHAM JACKSON, University of Minnesota.

Vice-Presidents: W. B. FORD, University of Michigan; J. W. YOUNG, Dartmouth College.

Additional members of the Board of Trustees: FLORIAN CAJORI, University of California; J. L. COOLIDGE, Harvard University; E. H. MOORE, University of Chicago; OSWALD VEBLEN, Princeton University.

FINANCIAL REPORT OF THE SECRETARY-TREASURER AS OF DATE, DEC. 17, 1925.

RECEIPTS.

Balance Dec. 16, 1924		$ 6,847.35
1924 indiv. dues	390.60	
1924 instit. dues	28.00	
1925 indiv. dues	6,036.89	
1925 instit. dues	648.40	
1925 subscriptions	616.11	
Initiation fees	332.00	
Advertising	875.00	
Life membership fees	106.14	
Sale copies MONTHLY	34.87	
Sales copies Register	8.00	
Sale reprints	1.43	
Sales first Monograph	1,166.50	
Gift for Chauvenet Prize	100.00	
For *Annals* subscriptions	4.90	
Interest Oberlin Savings Bank	110.27	
Interest Peoples Bkg. Co.	89.80	
Interest Treasury Note	10.94	
Interest Liberty Loan	10.63	
Interest Hardy Fund	120.00	
Total 1925 receipts		10,690.48
Total assets to the end of 1925 business		$17,537.83
Total expenditures		11,806.20
Balance to the end of 1925 business		$ 5,731.63
Received on 1926 business		$ 1,568.15
Book balance Dec. 17, 1925		$ 7,299.78

EXPENDITURES.

Publishers' bills (Oct. '24–Oct. '25)		$ 6,885.50
President's office		43.82
Manager's office		19.96
Editor-in-Chief's office		559.02
Committee on Membership		114.25
Joint Committee on Membership		176.09
Printing 1924 Register		550.00
Part expense 1925 Register		101.75
Printing Peirce reprints		98.85
Secretary-Treasurer's office:		
Postage	$231.45	
Bond	5.00	
Safety deposit	4.00	
Office supplies	29.36	
Express, tel., etc.	43.00	
Clerical work	740.50	
Printing	284.10	
Library expense	53.92	
Refund on subscriptions	7.65	
Washington meetings	88.05	
Ithaca meeting	50.00	
Paid copies of MONTHLY	28.35	
		$1,565.38
Annals subvention		300.00
Paid *Annals* subscriptions		7.90
Paid to sections from initiation fees		135.93
Paid to B. F. Finkel int. Hardy Fund		120.00
Carus Monograph honorarium		300.00
Delegate to Vanderbilt Univ. inauguration		25.00
Transfer Monograph income to certif. of deposit		798.75
Paid Open Court Pub. Co. for Monograph sales		4.00
Total expenditures		$11,806.20
Cash on hand		$ 43.25
Checking account		1,028.37
Oberlin Savings Bank account		2,609.02
Peoples Banking Company acct.		1,519.14
Liberty Bond		500.00
U. S. Treasury Note		500.00
Treas. Savings Certs.		1,000.00
Certificate of dep. Chauv. Fd.		100.00
Bank balance Dec. 17, 1925		$ 7,299.78

Of the funds on hand, $311.39 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date January 1, 1926; $1,000 is held, as the beginning of a permanent endowment fund, in the form of U. S. Treasury Savings Certificates, the surrender value on Jan. 1, 1926 being $1,072.50; $167.75 belongs to the Carus Monograph Fund; and a certificate of deposit for $100 is held temporarily as the first Chauvenet Prize Fund.

Aside from the above-mentioned funds on hand, the Carus Monograph Fund, increased by sales and interest, amounts at present to $2,120.17.

When the accounts were closed December 17, 1925, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1925 the following items:

BILLS RECEIVABLE.

1925 individual dues	$150.00
1925 institutional dues	42.00
Advertising	100.00
Interest Liberty Loan	10.00
Interest Treasury Note	11.00
	$313.00

BILLS PAYABLE

Publishers' bills (Nov.–Dec. 1925)	$1,400.00
President's office	50.00
Manager's office	30.00
Editor-in-Chief's office	90.00
Other editors' postage	20.00
Committee on Membership	50.00
Secretary-Treasurer's office	140.00
Initiation fees due to sections	500.00
Subvention *Annals*	100.00
Printing 1925 Register	550.00
Printing annual ballots, programs, etc.	150.00
Life Membership Fund	311.00
Carus Monograph Fund	167.75
Chauvenet Prize Fund	100.00
	$3,658.75

If to the balance on 1925 business shown in this report, $5,731.63, there be added the bills receivable, $313.00, and there be subtracted the estimated bills payable, $3,658.75, there results an estimated final balance on 1925 business of approximately $2,380 including the endowment fund of $1,000; the corresponding figure one year ago on 1924 business was $2,500. It is hoped that this somewhat lower figure for this year may be bettered by receiving unpaid dues in arrearage; in spite of the importunity of the treasurer, too many of the Association members have not attended to the payment of their dues or arranged with the treasurer concerning these payments.

W. D. CAIRNS, *Secretary-Treasurer.*

A PROPOSED AMENDMENT TO THE BY-LAWS OF THE MATHEMATICAL ASSOCIATION

At the Kansas City meeting of the trustees of the Mathematical Association of America, it was voted by a preponderating vote to recommend to the Association that the By-Laws be amended so as to provide for a two-year term for president, the president not to be eligible for re-election. Article III, Section 3 would then read: "The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for re-election. The Vice-Presidents shall be elected by the Association's members annually etc." As there was a diversity of views on this question, it was voted to present this for the consideration of the members of the Association with a concise statement of the arguments for each side, the amendment to be voted on at the summer meeting in September, 1926. The following two statements have been prepared by two officers of the Association.

Affirmative. "The time has passed when the presidency should be used as a means of honoring members of the Association. The essential thing is that the president should serve the Association, not that the Association should honor the president. Experience has shown that a one year term is too short for any president really to serve as he should. Several of our presidents have expressed the view that they would have liked to do more but the period was too short. The experience of the American Mathematical Society is the same, certain recent presidents having rendered the Society notable service: for example Professor Veblen in raising the endowment and Professor Bliss in enlarging the membership; neither of these services could have been performed by a one year president.

This same result could not be reached by electing the president of the Association for one year, with the possibility of re-election. Since nominations are made in September, it would be quite impossible for the bulk of the membership to appreciate at that time whether or not the president elected at the beginning of the current year was, or was not, particularly capable and interested in his duties. A president would not care to undertake a two year task when he was entirely uncertain as to whether or not he would enjoy a two year term. It is not reasonable to expect any president to undertake any work for the Association unless he can feel reasonably sure that he will have time to complete it before he goes out of office. In general, a man elected for two years will realize his opportunity and take his duties more seriously than one elected for a short term with no particular certainty of re-election."

Negative. "The By-Laws of the Association provide that the president and vice-presidents "shall be eligible for re-election but not for more than two consecutive terms." In the early years of the Association it seemed wise to the

respective presidents to decline in advance the nomination for a second term for the following reasons: (1) in order that, from the very outset, there might be no appearance of undue centralization of power; (2) in order that, as rapidly as possible, there might be a geographic distribution of responsibility and corresponding development of interest in the work of the Association all over the country; (3) in order that the presidential honor might be conferred upon as many as possible of the large number of highly worthy candidates. It seems perfectly evident to me that these objects have been attained to a higher degree by the administrations of the Association's ten presidents than would have been the case if there had been only half as many presidents and each had served two years. Even the interest which has been maintained through these really democratic elections each year should not be lost sight of in this connection.

However, there is weight to the contention that a president who has an aggressive program may find a single year too short a time in which to work out his plans, and it may well be that the precedent followed in the first decade has become too rigid and is in need of modification. This can easily be done by simply giving wide publicity to the fact that the by-laws now provide for re-election and that the growing complexity of the Association's activities may sometimes justify a second presidential term. But this question should be determined in democratic fashion on the merits of the case at any given primary election.

The above plan preserves a desirable flexibility, whereas the proposal to amend the by-laws so as to make the presidential term two years would establish a rigidity from which there would be no escape. For example, the time will soon come when the growth of the Association through its sections (now sixteen in number) will warrant the selection of a president from the far south or the far west, but it would probably be unwise for a long period yet to have the chief executive officer for two consecutive years so far from the center of Association activities. The same considerations apply to our Canadian membership. We would do honor to ourselves as well as to our neighbors on the north if we should sometime select a president from among them, but neither would they nor we think it wise to take such action for a two-year period. The success of the Association has been due in no small measure to the flexibility and democratic adaptibility of its machinery and many of us would regret exceedingly to see these characteristics restricted by the proposed amendment."

This notice of the proposed amendment with two points of view is submitted to the members of the Association, in accordance with Article VIII, Section 1, of the By-Laws.

W. D. CAIRNS, *Secretary-Treasurer.*

THE MAY MEETING OF THE MINNESOTA SECTION.

The regular spring meeting of the Minnesota Section was held at St. John's University, Collegeville, Minnesota, on Saturday, May 16, 1925. Professor W. H. Kirchner, University of Minnesota, the Chairman of the Section, presided.

The attendance was 45 at the dinner, while 73 attended the reading of the papers, and included the following 21 members of the Association: W. O. Beal; A. Bogard; Sister Brigetta; W. H. Bussey; Elizabeth Carlson; H. H. Dalaker; Gladys Gibbens; W. L. Hart; D. Jackson; W. H. Kirchner; W. H. McEwen; E. L. Mickelson; M. A. Norgaard; G. C. Priester; H. L. Rietz; R. R. Shumway; H. L. Smith; F. J. Taylor; Ella Thorp; A. L. Underhill; G. Winkelmann.

The following officers were elected for the coming year: Chairman, M. A. NORGAARD, St. Olaf College, Northfield; Secretary, A. L. UNDERHILL, University of Minnesota.

A motion was passed, expressing the appreciation of the Section for the hospitality of St. John's University.

The 1926 meeting will be held in May at St. Olaf College, Northfield.

The following six papers were read:

(1) "A survey of recent progress on random sampling," by Professor H. L. RIETZ, University of Iowa, National President of the Association, 1923-24.

(2) "The dynamics of correlation," by Professor DUNHAM JACKSON, University of Minnesota.

(3) "A survey of mathematics in the Benedictine Order," by Rev. GILBERT WINKELMANN, O. S. B., St. John's University.

(4) "A paradox in the realm of college grades and honor points," by Professor W. H. BUSSEY, University of Minnesota.

(5) "An experiment with prognostic test in college freshmen mathematics," by Professor M. A. NORGAARD, St. Olaf College.

(6) "Anti-polars in perspective," by Professor W. H. KIRCHNER, University of Minnesota.

Abstracts of papers, numbered as in the above list of titles, follow:

1. The plan of this paper was to give first a general view of those parts of the theory of random sampling which constitute an appropriate setting for remarks on recent progress along this line. The paper first dealt with the Bernoulli theory as the simplest sampling theory and next considered Bernoulli, Poisson and Lexis distributions. With this setting the nature of the extension of the theory by Coolidge published in 1921 was discussed. The paper next dealt with the Tchebycheff inequality and discussed the generalization of this inequality published by Pearson in 1919 and the further generalizations by Camp and Midell published in 1922. The paper next dealt with the inversion of

the Bernoulli-Laplace theorem and with the recent contributions to this subject. The final section of the paper was devoted to the recent contributions of the Pearson school of statisticians to the problem of determining the distribution of averages obtained from small samples.

2. This paper was an elementary account of the principal axes of a statistical distribution, and of the geometric interpretation of coefficients of correlation as based on the formulas pertaining to ellipsoids of inertia in dynamics. The treatment was general, being in no way restricted to normal distributions.

3. The monastery of Monte Cassino was founded by St. Benedict in 529. Professor Winkelmann pointed out that in fourteen centuries over three hundred Benedictines are recognized as having made contributions to the advance of mathematics. Eminent among these were: the venerable Bede for his work on the calendar and on finger calculation; Adelard of Bath, one of the greatest of the translators from Arabic into Latin; Alcuin, because his school and those of his followers led directly to the founding of the University of Paris; and Gerbert of Aurillac (Pope Sylvester II) the most learned man of his time, and, because of his treatises on arithmetic and geometry, perhaps the greatest mathematician of the Order. Reference was made to the Benedictine University of Salzburg, to a long line of skillful teachers, and to mathematicians who, like Galileo and Fourier, came under Benedictine influences although less intimately connected with the Order.

4. In many colleges, requirements for graduation are expressed quantitatively in terms of credits and qualitatively in terms of grades and honor points. In the marking system referred to in Mr. Bussey's paper there are four so called passing grades, A, B, C, D, and a grade F to represent failure; and honor points are given to students according to the following schedule:

With every credit of grade A the number of honor points given is 3,
With every credit of grade B the number of honor points given is 2,
With every credit of grade C the number of honor points given is 1,
With every credit of grade D the number of honor points given is 0,
With every credit of grade F the number of honor points given is -1.

The paper was an attempt to answer questions similar to the following: "If every freshman in a college takes three courses, and if every teacher of those freshmen gives out grades A, B, C, D, F approximately according to a specified grade distribution schedule (for example, 5 per cent of grade A, 15 per cent of grade B, 45 per cent of grade C, 20 per cent of grade D, and 15 per cent of grade F), how many of the freshmen may reasonably be expected to have an average grade of C, that is an average of one honor point per credit, for a year's work."

5. In colleges where sectionizing of freshman classes has been attempted, the grouping has generally been done at the end of the first semester or at the end

of the first six weeks. This delay seems very undesirable; and especially is it detrimental to the bright freshman, who through lack of proper competition may develop an improper attitude to study. For three years Professor Nordgaard has tried out the reliability of a prognostic test, covering the fields of arithmetic, algebra and geometry and given at registration time. He has worked out the correlation between scores thus obtained and the first semester's work in freshman mathematics. Generally the Pearson correlation coefficient falls between .40 and .50, which is somewhat higher than that shown between semester results and tests on current work given at the end of five weeks. Two years the students were sectionized according to the results of the prognostic tests, and one year they were left unsectionized, in order to examine the problem from a different angle.

A. L. UNDERHILL, *Secretary.*

THE COURSE IN STATISTICS IN THE MATHEMATICS DEPARTMENT[1]

BY A. R. CRATHORNE, University of Illinois

In these days of crowded classrooms and diversified curricula the duplication or overlapping of courses has become an important factor in collegiate economics. Many an administrative officer in reviewing the courses in the catalog has paused to consider the various courses containing some form of the word statistics—economics statistics, educational statistics, mathematical statistics, vital statistics, theory of statistics, statistical biology, statistical mechanics, or just "statistics"—and has exhibited undue curiosity as to the amount of overlapping and the possibility of combining or in some way reducing the number of courses. In some institutions this has led into a discussion of the advisability of the formation of a department of statistics.

A great deal of the blame for this repetition of the word statistics in the catalogs can be placed upon the word "statistics" itself. It is not what in mathematics we call "well defined." To many people the first syllable is stressed, the relation with the word "state" is uppermost so that a sociological or economic meaning is given to the word. To others the word has a broader meaning in that it is thought to be concerned with the study of methods for finding the salient characteristics of quantitative data without making any statement about the subject matter of the data. In most dictionaries one finds two or more definitions which usually include these two. It is not my purpose to discuss the precise definition of statistics, or to differentiate between statistics, statistical methods, or theory of statistics, but simply to discuss some features of the courses in statistics given in the mathematics department.

[1] Read Dec. 31, 1925, at the Kansas City meeting of the Mathematical Association of America.

I know of no better introduction to this part of the paper than to quote the carefully worded statement by the committee on the mathematical analysis of statistics of the National Research Council in the preface to its *Handbook of Mathematical Statistics*;—"The study of a problem by statistical methods usually involves three stages: (1) the collection of material or data; (2) the mathematical analysis of the data thus collected; (3) the interpretation of the results, for the particular purpose in view. As to stage (1), the best methods of collecting data depend almost entirely on the nature of the particular field of inquiry. . . . . The same is true in regard to stage (3): the problems connected with the interpretation of statistical results are necessarily very different in different fields of inquiry. . . . . The problems of stage (2), on the other hand, are in a sense common to all fields of statistical inquiry. Whatever the content of the data may be, the form of the mathematical analysis is essentially the same."

The courses in statistics given in the mathematics department should be confined in most part to the second stage although there are times when the mathematical statistician is consulted as to methods of collection of data and more often as to interpretation of results. College courses emphasizing these stages should be given by the departments of education, economics, or sociology and should be considered as courses in education, economics and sociology. Biologists and anthropologists have long recognized the principle implied here and have invented the words biometry and anthropometry. We might ask the economists to introduce the word econometry, but it would be very difficult to argue them out of their rather well grounded historical claim to some share in the word statistics.

The courses in mathematical statistics vary almost as much as courses in mathematics. There are courses for freshmen, sophomores, juniors and seniors and purely graduate courses, but in general they can be grouped under two heads—elementary courses and advanced courses, using the prerequisite of calculus to mark the rather vague boundary line between the two. It is not my purpose to discuss in any detail the content of these courses. In a paper[1] before this association some three years ago, Professor Rietz considered this phase of the statistics course quite fully. In general the content of the elementary course can be broadly sketched by the mention of a few topics: averages, measures of dispersion, frequency distributions, normal and binomial distributions, interpolation, sampling, correlation. By adding to this list the names of Bayes, Lexis, Tchebycheff, Pearson, Edgeworth, Yule, Charlier, Sheppard, Bortkiewitsch, R. A. Fisher, and the topics associated with these names, we sketch in broad outline the advanced course. This does not mean that the

[1] On the Subject Matter of a Course in Mathematical Statistics, by H. L. Rietz, (*1923*, 155-166).

two courses overlap to any extent. For example, the topic "frequency distributions" should occur in any course in statistics, elementary, intermediate or advanced, but in the advanced course it takes on a meaning that would be entirely lost to a student of the other courses.

As for a standard course, even the elementary course has not become standardized like the courses in calculus for example. Very few of those who teach the subject give the same course two years in succession. The students who take the first college course in mathematical statistics are usually those who do not profit so much from lectures as they do from a text book. The subject of text books in statistics is a timely one. In the last year or two there has been almost a deluge of texts on statistics, and the theory of probability. Many of these are excellent but the best are written for students in other departments—economics, education, psychology, sociology. Teachers of statistics from the mathematical side seem to be too individualistic to unite to any extent on a text book. Whenever two of them get together, the conversation usually turns to the book that was used last year, the book now in use and the book that will be tried out next year. The great American text in mathematical statistics still remains to be written; at least it is not yet published. Sometimes the same text is used for a course in the mathematics department and a course in some other department, usually economics. One might look for overlapping here, but with experienced instructors two very distinct courses would be given—one in mathematics, the other in economics. One of the good English books for a junior-senior course in mathematical statistics is really a text book in psychology.

As for the students who take the first course in statistics, they are of several classes. Some few elect it as a general culture course. Many students in the mathematical group elect it to fill out the requirement for a major. Then there are the students in other groups who use the subject as a sort of technical elective. There is one large group of students which seems to be peculiar to the subject. They are older men, often graduate students in some other field with little or no mathematics, who have discovered that they have come to a point where they are very much handicapped by their lack of mathematical training. It is hard to tell such men to review their high school mathematics and then take two years of college mathematics. It is equally hard to resist the request of their department or college for a combined course in mathematics and statistics suitable for this type of student. The mathematics instructor, right down in his heart, takes a somewhat Machiavellian satisfaction out of the plight of these students, but he has enough missionary spirit to compromise and the result is a course with prerequisites: "college algebra and junior standing, or upon consultation with the instructor." There is another and worthier kind of satisfaction for the mathematics instructor in

connection with this statistics course. It comes from those students in the course who happen to have had a course in calculus. The way in which they stand out among the others even when no calculus is used in the course is very gratifying. This seems to be true in other courses in statistics besides those given in the department of mathematics. At the University of Illinois we give each year a course requiring college algebra and junior standing, but year by year the number of students who have had calculus has increased until it has reached a point where there should be two kinds of first courses in statistics—one for those who have had calculus and one for those who have not had it.

Now and then we have the serious and earnest student who takes the elementary course in statistics on top of the minimum amount of mathematics. He often develops into a man who acquires a skill in finding averages, measures of dispersion, coefficients of correlation and other constants of statistics. Substitution in a formula is to him the height of mathematics and the conditions under which the formula was derived are of no particular moment. He calculates columns of probable errors without a thought as to the kind of distribution with which he is working. And correlation coefficients, how he can calculate and interpret them! Here, for example, is a correlation of 0.7 between a time series of prices of pig iron and one of prices of pork. That this correlation has a tremendous and deep lying significance is easily shown. Just look at the probable error, which is .04463!

From many points of view, the most interesting student in the statistics courses is the student who is preparing to become a statistician. Too often he decided on this profession very late in his college course and thinks that a single course in statistics will be sufficient preparation. If he selects the course in the mathematics department, he is usually a student in the mathematics group and has seldom had much economics. Such a man is very one-sided in his preparation to do statistical work. He has an exaggerated idea of the usefulness of mathematics in all departments of statistics and comes late to an appreciation of the fact that knowledge of the subject matter of the statistical problem is very necessary. Then he is often disappointed when he starts work and finds that at first he doesn't seem to have much use for his advanced mathematics. There is a whisper going about the campus nowadays that statistics is a good thing to go into and sometimes it is necessary to dampen the enthusiasm of an undergraduate who expects immediately upon graduation to integrate, interpolate, and correlate his way to a five-figure salary. Statistics is a good thing for the right person to look forward to, but, like any other worthwhile profession, it needs careful preparation and a hard climb afterwards. The prospective statistician should keep in mind that there are three phases of the general problem of statistics and that the course in the mathematics de-

partment is mainly concerned with only one. If he could arrange his college course from the beginning, he should elect mathematics and accounting in his freshman year, mathematics and economics in his sophomore year. In his junior year he should take some mathematical statistics and mathematics of finance and courses in economics. In his senior year he should take a course in economic statistics and somewhere in his course, even if he has no particular liking for natural science, he might take some courses in biology to get the point of view of the biological statisticians or biometricians as they call themselves. This work of course takes up only a small part of his time. It is assumed that he is taking a general college course with opportunity to elect the courses mentioned. This sketchy suggestion as to the arrangement of a college course to suit the needs of a future business statistician does not have in mind the research worker who will need a more intense and pointed preparation in the graduate school.

In the advanced course in mathematical statistics there is not much question of overlapping. It is a course in pure and applied mathematics. The students are usually graduate students and are either from the mathematics department or are taking mathematical statistics as a minor. Now and then a student appears who is looking forward to a research position in statistics. I have indicated roughly a few of the characteristics of this course but I doubt if any of the men who give it ever follow the same outline two years in succession. The theory of statistics as we now know it is rather young and undeveloped. The growing pains are very evident. We notice their effect in the symbolism and language of statistics which have been adopted in considerable part direct from the theory of probabilities after hard usage in the theory of errors.

In his review of Keynes' *Theory of Probability*, Sheppard, the well-known English statistician, makes some pertinent remarks bearing on the relation between the theory of probability and statistics.

"The theory of rational belief is one thing: the mathematical theory of probability—which is practically dead—except as a mathematical exercise . . . . its place having been taken by the frequency calculus—is another: and the science of statistics—an inductive science, based or to be based on observation and experiment, and using the frequency calculus as its deduction machine—is another. This science is being built up slowly and Mr. Keynes does not give much help by putting up fragments of the old theory of probability and knocking them down again. . . . . It is quite true that statisticians do use such phrases as "probable error" which seem to connote probability. But probability, if the word is used in the ordinary sense, is one thing, frequency is another. You cannot infer frequency from probability, nor probability from frequency, unless in either case, probability means frequency, which is practically what it does mean to the statistician."

A great deal of what Sheppard has said may be summed up by saying that for many years in its youth the theory of statistics was dressed by its parents in old clothes belonging to a kind uncle called the theory of probability. The nephew endured it patiently for many years, but of late having earned a little

money of his own, he is busy making inquiries as to the best tailors and haberdashers and hopes soon to surprise his uncle who has retired to his country home on a pension. To put it more concisely, we can say that one of the important problems of the present and future is the untangling of the theories of probability and statistics.

A fundamental problem in statistics which is badly in need of this untangling treatment is as follows: An event has happened p times and failed q times out of $p+q$ trials. We know nothing of the underlying frequency of the event. We wish to know what will happen in the next m trials. This problem leads us to the subject of inverse probabilities and the not-to-be-suppressed controversy over Bayes' theorem. I cannot do better on this point than quote R. A. Fisher:[1]

"Several reasons have contributed to the prolonged neglect into which the study of statistics in its theoretical aspects has fallen. In spite of the immense amount of fruitful labor which has been expended in the practical applications, the basic principles of this organ of science are still in a state of obscurity and it cannot be denied that during the recent rapid development of practical methods, fundamental problems have been ignored and fundamental paradoxes left unsolved. This anomalous state of statistical science is strikingly exemplified by a recent paper in which one of the most eminent of modern statisticians presents what purports to be a general proof of Bayes' postulate, a proof which in the opinion of a second statistician of equal eminence "seems to rest upon a very peculiar—not to say hardly supposable—relation." . . . . It has happened that in statistics a purely verbal confusion has hindered the distinct formulation of statistical problems: for it is customary to apply the same name, mean, standard deviation, correlation coefficient, etc., both to the true value which we should like to know, but can only estimate, and to the particular value, at which we happen to arrive by our methods of estimation. It is this last confusion in the writer's opinion, more than any other, which has led to the survival to the present day of the fundamental paradox of inverse probability, which like an impenetrable jungle arrests progress towards precision of statistical concepts. The criticisms of Boole, Venn and Chrystal have done something towards banishing the method at least from the elementary text-books of Algebra; but though we may wholly agree with Chrystal that inverse probability is a mistake (perhaps the only mistake to which the mathematical world has so deeply committed itself), there yet remains the feeling that such a mistake would not have captivated the minds of Laplace and Poisson if there had been nothing in it but error."

Fisher is constructive in his criticism, for the quotations are taken from the beginning of his memoir in which he reviews with care Bayes' theorem and the recent criticism, and proposes his method of maximum likelihood.

Another illustration showing how the science of statistics is still in the process of being slowly built up, is given by the theory of skew frequency curves and surfaces. Every student of mathematical statistics knows of the attempts to represent frequency distributions analytically and of the progress during the last thirty years which has produced the two systems of curves usually associated with the names of Pearson and Charlier. The corresponding problem of representing skew frequency surfaces[2] analytically has had a slower growth. It would seem natural that Pearson after writing his first paper on univariate

[1] On the Mathematical Foundations of Theoretical Statistics, by R. A. Fisher, *Phil. Trans.*, vol. 222 A, p. 309.

[2] Skew Frequency Surfaces, by Karl Pearson, *Biometrika*, vol. 15, p. 223.

skew variation should turn to the question of bivariate variation. In the case of normal distributions one can multiply together two normal functions in different variables and by rotation of the axes arrive at the normal surface and with it our present day theory of correlation. In Pearson's first attempts to solve the problem of bivariate skew variation the results were in a way negative. By analysing skew correlation data he surmised from the study of special cases that we could not proceed by multiplying together two skew functions and following with a rotation of axes. Later by using a double hypergeometric series he tried to determine a surface which was related to the double series in somewhat the same way that his skew curves were related to the single hypergeometric series. From the resulting differential equations he inferred that the regression lines were not straight lines but were cubic curves intersecting at a point which was not the mean of the surface. These differential equations were of the form

$$\frac{1}{z}\frac{dz}{dx} = \frac{\text{cubic in } x \text{ and } y}{\text{quartic in } x \text{ and } y}, \quad \frac{1}{z}\frac{dz}{dy} = \frac{\text{cubic in } x \text{ and } y}{\text{same quartic in } x \text{ and } y}.$$

During the next twenty years various men worked on the problem with Pearson, —Filon, Isserlis, Rhodes[1]—with results that were special and no very satisfactory advances toward the solution of the general problem were made. Gradually Pearson drifted away from his attack on the skew correlation problem by means of his differential equations to one in which he assumed the forms of the regression and scedastic curves, and from this starting point, under reasonable hypotheses, he hoped to arrive at the equation of the corresponding frequency surface. This method of attack was suggested to Professor Narumi, a Japanese mathematician, and the result was the interesting series of memoirs in *Biometrika*, as yet uncompleted, in which Narumi[2] has made great advances in the problem of bivariate skew frequency distributions.

A recent advance of far reaching importance in practical statistics is the publication of tables of the incomplete gamma function[3] after some twenty years of preparation. These tables give the probability of the occurrence of a deviation from the mean that is greater than any given deviation. They do for certain skew frequency curves what the ordinary normal probability table does for the Gaussian curve. The tables do not cover all Pearson's skew variation curves, but they cover a very large class of some hundreds of curves of which the normal curve is a single member. Tables of incomplete beta functions are desirable. The gamma function tables are two-fold while the

[1] On a certain skew correlation surface, by E. C. Rhodes, *Biometrika*, vol. 14, p. 355.

[2] On the general forms of bivariate frequency distributions which are mathematically possible when regression and variation are subjected to limiting conditions, by Seimatsu Narumi, *Biometrika*, vol. 15, p. 77, and p. 209.

[3] Tables of the incomplete gamma function, edited by Karl Pearson, London, 1922.

beta function table would have to be three-fold so the cost of computation and printing will be well nigh prohibitive. Right here is a very practical problem for present and future workers in the mathematical side of statistics. We all know how the beta function can be expressed in terms of gamma functions, but what about the incomplete beta and gamma functions? Can the incomplete beta function be expressed exactly or approximately in terms of the incomplete gamma function?[1]

Progress along purely mathematical lines is gratifying in the highest degree to workers in theoretical statistics who feel that they have a worthwhile field of their own and rather resent being considered as hewers of wood and drawers of water for others. Nevertheless they all take added pleasure in those advances in their science which prove useful in a practical sense, though one might add that any new advance in the theory of statistics will be sure sooner or later to have its application in some other field. This is well illustrated in the work now being done in the statistical department of the Bell Telephone laboratories. For an example, we roughly sketch some of the applications of statistics to the carbon microphone[2] which is the essential part of the telephone transmitter. Something over one and a half million transmitters are manufactured every year by the Bell system. Variations in the efficiency of the carbon microphone have many causes. There are all the variations connected with the carbon particles, the variations due to the process of assembling the parts, and to differences in the individual parts, variations due to temperature, humidity, personality, and last but not least the variations in the human ear which measures the efficiency. The physicist usually works in a laboratory under controlled conditions, but the telephone engineer must work under commercial conditions. It is not possible to inspect every transmitter. Instead, random samples are used. Certain variations will be found. Can these be explained by the laws of chance? If not, what is the explanation? Is there a general trend upward or downward in any particular characteristic? If there is, what is the cause? Then comes the important economical question: "What should be the size of the sample to be inspected?" To answer these questions satisfactorily is the work of the statistics department. In most problems the first step is the determination of the law of distribution about some mean value of the variable being investigated, and from this law to find the most probable value and the frequency of values occurring between two given limits and then to answer the question, "Are the data such that the variation would follow from a random system of causes?" Here it is necessary to call in the aid of the whole subject of frequency curves skew and non-skew, the theory of random sampling

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[1] The Fundamental Problem of Practical Statistics, by Karl Pearson, *Biometrika*, vol. 13, p. 16.

[2] Some Applications of Statistical Methods, by W. A. Shewhart, *Bell System Technical Journal*, vol. 3, No. 1.

for large and small samples. In fact one of the men in the Bell Telephone laboratories wrote me that he did not recall many theorems in mathematical statistics which had not been used at some time or other in their study of the carbon microphone.

In a paper in the current number of the *Journal of the American Statistical Association* Dr. Shewhart[1] has an interesting article on the use of statistics in maintaining the quality of a manufactured product. When the production of a single kind of article runs into the millions it is a great advantage to be able to discover the existence of causes of non-random variation in a product as early as possible. Among other things in this paper, the author shows that the Pearson goodness-of-fit test is a very useful tool in this problem.

The size of the sample is a question of great economic importance. At the present time a theory of small samples is in process of development which promises to be of the greatest value.[2] Then the problem of making the most efficient use of data is an important economic problem. The cost of analysis of data is small as compared with the collection of the data. Fisher[3] gives a simple illustration of economy in the use of data. In estimating the standard deviation of a normally distributed population from an ungrouped sample, we may use one of two methods—the mean error method or the root-mean-square error method. The use of the latter method with a sample of 100 is equivalent to the use of the former with a sample of 114. This is not a negligible economy when the total number of observations runs into millions.

These references to recent advances in the purely theoretical side on the one hand and to the extremely practical on the other are but what might be the beginning of a very long list. They are simply those things that have especially interested me during the past year or so, and are touched upon here to give emphasis to the fact that the course in statistics in the mathematics department should be considered as a course in mathematics in its best sense —not a mathematics of drudgery and long computations,—a course which leads to a frontier where either the pure mathematician or the practical statistician may find a fertile field. The teacher of even the most elementary course in statistics in the mathematics department is most interested in the students who show aptitude on the real mathematical side. He does his best work when teaching mathematics. There is in the mind of some students and some other people the idea that a course in statistics is a course in statistics. Some people expect the instructor in the mathematics of statistics to teach economics,

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[1] The Application of Statistics as an Aid in Maintaining Quality of a Manufactured Product, by W. A. Shewhart, *Journ. Amer. Stat. Assn.*, vol. 20, p. 546.

[2] The Probable Error of a Mean, by Student, *Biometrika*, vol. 6, p. 1. On the "Probable Error of a Coefficient of Correlation Deduced from a Small Sample," by R. A. Fisher, *Metron*, vol. 1, part 4, p. 82.

[3] *Phil. Trans. Roy. Soc.*, vol. 222, series A, p. 316.

biology or what not. Some others accuse him of teaching economics, biology or what not and teaching it very badly.

The serious part of this whole matter in my mind is not a question of overlapping of courses but a question of coördination among departments. In a physical and geographical sense there can be considerable coöperation. It is not a mere coincidence that the increase of interest in statistics has come at the same time as a decrease of drudgery due to improvements in statistical apparatus. A statistical laboratory is a necessity in the teaching of any course in statistics. Expensive computing machines and readily accessible library equipment may be collected at a place convenient to all departments where it can be used in common. There will be real economy here. Perhaps the best way in which the mathematics department can coöperate is to teach mathematics in our courses in statistics and at the same time to remember and point out that we are giving a narrow and one sided view of the broad field of statistics which should be supplemented by other courses. If the mathematician thinks it too presumptuous to suggest that a course in mathematical statistics be made a prerequisite for other courses in statistics, he may ease his conscience by reminding himself that if the mathematics instructor does not stress the mathematical side no one else will.

THE STURM AND FOURIER-BUDAN THEOREMS AND MIXED DIFFERENTIAL-DIFFERENCE EQUATIONS

By TOMLINSON FORT, Hunter College

I recently tried with indifferent success to teach the Sturm and Fourier-Budan theorems to undergraduates from the proofs given by Dickson in his *First Course in the Theory of Equations*. I then substituted the proofs that are given here. It is to be noticed that they are primarily graphical and with the aid of the black board are easily explained. I have now used them in two different classes with success. In fact the students seem a little amused at the idea that previous classes had found the theorems hard. These proofs are given to teachers of the theory of equations for what they are worth. It is also to be noticed that in section 4 wide generalizations of the Fourier-Budan theorem are given. These are simple but so far as I know are new and may be found of mathematical interest. To one who has read the previous sections proofs would be very easy and so I have not given them in detail.

1. Sturm's Theorem as given by Dickson[1] is:

"Let $f(x)=0$ be an equation with real coefficients and without multiple roots. Modify the usual process of seeking the greatest common divisor of $f(x)$ and

[1] *First Course in Theory of Equations* p. 76.

where the coefficients are as formerly. We note particularly that $p_i(x)>0$. y_0 is chosen at pleasure so long as it is integrable and retains a fixed sign as x varies from a to b. Equations (5) can be solved for the functions y_i successively, choosing each time any particular solution. We have a set exactly like that obtained from (3) only in reverse order. In other words we have a function y_i such that the change in the number of nodes of y_i on the interval 0 to n when x increases from a to b is the number of roots of $y_n(x)$ on the interval a to b or exceeds that number by an even integer, neither a nor b being a root of $y_n(x)$.

We next write

$$p_{i+1}(x)y_{i+1}'+q_{i+1}(x)y_{i+1}'+r_{i+1}y_{i+1}=y_i'+P_i(x)y_i\ , \tag{6}$$

where $p>0$. If

$$p_{i+1}''+2P_ip_{i+1}'+(P_i'+P_i^2)p_{i+1}-q_{i+1}'-P_iq_{i+1}+r_{i+1}=0 \tag{7}$$

by actual substitution it is seen that (6) is satisfied by a function which in turn satisfies an equation of the type,

$$R_{i+1}(x)y_{i+1}'+S_{i+1}(x)y_{i+1}=y_i\ , \tag{8}$$

where $R_{i+1}\equiv p_{i+1}$ and $S_{i+1}=q_{i+1}-P_ip_{i+1}-p'_{i+1}$. We solve (8) and have a sequence of functions which are a solution of (6) and at the same time a Budan sequence.

We can proceed to equations of higher differential order with a condition on the coefficients similar but naturally somewhat more complicated than the above.

A CROSS–DIVISION PROCESS AND ITS APPLICATION TO THE EXTRACTION OF ROOTS

BY DERRICK HENRY LEHMER, University of California

The computer is sometimes confronted with the problem of dividing by a number that is too large to fit in his calculating machine. In this paper we shall present an effective method of overcoming this difficulty, which was used by the writer in calculating the value of the Napierian base to 707 places of decimals, and also give an adaption of the division process for extracting square roots.

There are methods which serve to find the decimal value of a fraction whose denominator is less than twice the key-board capacity of the calculating machine. For a ten-place calculator these methods are of practical use up to perhaps 17-digit denominators. The division method given herewith will be found most useful for denominators beyond this range.

The method is virtually the reverse of a cross-multiplication method given by D. N. Lehmer (*1923*, 67) which may be briefly described as follows. Suppose we are given two large numbers A and B, whose product is required. We break up A and B into equal periods of convenient length. Thus:

$$A=\alpha_1\ ,\ \alpha_2\ ,\ \alpha_3\ ,\ \alpha_4\ ,\ \alpha_5\cdot\ \cdot\ \cdot$$
$$B=\beta_1\ ,\ \beta_2\ ,\ \beta_3\ ,\ \beta_4\ ,\ \beta_5\cdot\ \cdot\ \cdot,$$

where the α's and the β's represent the numbers making up the several periods of A and B.

Now let

$$\sigma_1=\alpha_1\beta_1 \qquad \sigma_2=\alpha_1\beta_2+\alpha_2\beta_1 \qquad \sigma_3=\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$\sigma_n=\alpha_1\beta_n+\alpha_2\beta_{n-1}+\ \cdot\ \cdot\ \cdot\ +\alpha_{n-1}\beta_2+\alpha_n\beta_1\,.$$

The multiplication then assumes this form:

$$\begin{array}{l}
A=\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\cdot\ \cdot\ \cdot\\
\underline{B=\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6\cdot\ \cdot\ \cdot\qquad}\\
\qquad\sigma_1\qquad\sigma_3\qquad\sigma_5\qquad\cdot\ \cdot\ \cdot\\
\underline{\qquad\qquad\sigma_2\qquad\sigma_4\qquad\sigma_6\cdot\ \cdot\ \cdot\qquad}
\end{array}$$
$$C=AB=\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6\ \cdot\ \cdot\ \cdot$$

These σ's are of course two-period numbers and are written on two lines so as to allow for overlapping . The γ's are formed by adding the periods that appear in the two lines directly above.

For division the process is exactly the reverse; we are given the fraction C/A and we wish to find the quotient B. In other words we are given the α's and the γ's from which we are to deduce the β's.

The general plan of attack is as follows. Since the first period of σ_1 is approximately equal to γ_1 we can determine β_1 from the relation

$$\sigma_1=\alpha_1\beta_1\,.$$

Knowing β_1, the exact value of σ_1 is found by multiplication. Now, since we know the last period of σ_1, we obtain the first period of σ_2 by subtracting from γ_2. Using this value as an approximation to σ_2, we determine β_2 from the relation

$$\sigma_2=\alpha_1\beta_2+\alpha_2\beta_1.$$

Having determined β_2, this same equation enables us to find the exact value of σ_2, the last period of which is used to find, from γ_3, the first period of σ_3 from which β_3 is to be determined from the relation

$$\sigma_3=\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1$$

and so on.

For practical use, however, this process must be modified to allow for disturbances arising from two sources: first the uncertainty as to the last digit in any β which is determined from only an approximate value of the corresponding σ, and second, the effect on the first period of any σ caused by the carrying over of a unit or more from the σ immediately following. The modified process is best explained by the use of an example. The reasons for the various steps are easily derived from the cross-multiplication method given above and are not of enough importance to warrant a detailed explanation. It may be remarked that there are several modifications of the above process than can be used successfully. The method given herewith has been found to give the best results.

In this method we make use of a new quantity, σ', which is the result of omitting the first term of the corresponding σ. Thus

$$\sigma_1'=0 \qquad \sigma_2'=\alpha_2\beta_1 \qquad \sigma_3'=\alpha_2\beta_2+\alpha_3\beta_1$$
$$\cdots\cdots$$
$$\sigma_n'=\alpha_2\beta_{n-1}+\alpha_3\beta_{n-2}+\cdots+\alpha_n\beta_1\ .$$

Consequently we have $\sigma_n=\sigma_n'+\alpha_1\beta_n$.

The number of digits in each period is chosen as nine for two reasons. First the successive values of the σ''s tend to increase and soon one is reached that is too large to be written in two periods. A unit must therefore be carried over into the preceding period. Now in a ten-place calculator with only twenty places in the carriage, this carrying over of a unit could not be effected if the number of digits in a period were ten. Using nine digits instead, there remain two dials which serve to hold the amount carried over even when the numbers to be divided are as large as 3,000 digits. The second reason has to do with the checking of the values of the successive σ''s in which, as experience shows, errors are most likely to occur. The process of checking is similar to the familiar method of "casting out the nines" used in ordinary multiplication. Instead of using nine for our modulus however we use 1001 which can be cast out rapidly from a nine or eighteen digit number by splitting it up into periods of three digits each and subtracting the sum of the even periods from that of the odd. An added advantage in using 1001 is that an error will not only be detected but frequently located at once.

As an example, let it be required to find the cube root of 83 by dividing

7512137677415067474067653515153

by

1722149465980481177969345011132

which are solutions corresponding to x and y of the Pellian cubic

$$x^3+Dy^3+D^2z^3-3Dxyz=1$$

in which $D=83$. We put down the outline of our work as follows:

	α_1	α_2	α_3	α_4	α_5
$A=$	172214946	598048117	796934501	113200000	000000000
$B=$	β_1	β_2	β_3	β_4	β_5
$C=$	075121376	774150674	740676535	151530000	000000000
	γ_1	γ_2	γ_3	γ_4	γ_5

We first set the number γ_1, γ_2 in the carriage, that is the number

075121376 774150674 .

This we divide by α_1 and we obtain the following quotient and remainder:

$$q_1=436207067 \qquad r_1=285927292 .$$

We next multiply α_2 by q_1 and obtain

$$\alpha_2 q_1=260872815 \quad 041442839.$$

We observe that if q_1 were β_1 this two-period number would be σ_2'. Now it can be shown that σ_2' lies between r_1 $\alpha_2 q_1=$and $r_1-\alpha_1$. Our r_1 has been properly chosen so as to fulfill this condition. Hence q_1 is β_1 and the above two-period number is σ_2'. We may now put down β_1 and also σ_1 which is $\alpha_1\beta_1$ and which goes directly beneath β_1. The work now appears as follows:

	α_1	α_2	α_3	α_4	α_5
$A=$	172214946	598048117	796934501	113200000	000000000
$B=$	436207067				
	β_1	β_2	β_3	β_4	β_5
	075121376	488223382			
$C=$	075121376	774150674	740676535	151530000	000000000
	γ_1	γ_2	γ_3	γ_4	γ_5

We now set the number r_1, γ_3 in the carriage, that is the number

285927292 740676535

and subtract from it the above value of σ_2'. The remainder when divided by α_1 gives the following quotient and remainder:

$$q_2=145483755, \qquad r_2=688031466 .$$

We next calculate $\beta_1\alpha_3+q_2\alpha_2$. By not clearing the carriage after the first multiplication, the second product is added on automatically. In making this second multiplication we take care to set α_2 in the key-board, not q_2. This then gives us

$$\beta_1\alpha_3+q_2\alpha_2=434634747 \quad 004157902.$$

We observe that if q_2 were β_2 this two-period number would be σ_3'. Unfortunately our value of r_2 is too large to fulfill the condition that the above number, which is an approximation to σ_3', shall lie between r_2 and $r_2-\alpha_1$. To remedy this difficulty we observe that if q_2 were changed

from 145483755 to 145483756

then r_2' would be decreased by the amount α_1 and hence its first two digits would be 51. With this value of r_2 the above condition would be satisfied. Making this slight change in q_2 compels us to change the above two-period number

from 434634747 004157902

to $\sigma_3'=434634747 \quad 602206019$

by a turn of the crank in the positive direction.

Having determined in this way that

$$\beta_2=145483756$$

we are now in a position to obtain σ_2. This is done by adding the product $\alpha_1\beta_2$ to the above value of σ_2. Putting down β_2 and σ_2, the work appears as follows:

α_1	α_2	α_3	α_4	α_5
$A=172214946$	598048117	796934501	113200000	000000000
$B=436207067$	145483756			
β_1	β_2	β_3	β_4	β_5
075121376	488223382			
	285927292	224860015		
$C=075121376$	774150674	740676535	151530000	000000000
γ_1	γ_2	γ_3	γ_4	γ_5

If we add the numbers on the two lines directly above γ_2 we get 774150674 which is γ_2 itself. This then is a check on the work. It should be understood that this is not a check on the σ''s. These numbers must be checked independently as stated above by casting out 1001.

We wish now to determine β_3. The number r_2, γ_4 or

688031466 151530000

is set in the carriage and the value of σ'_3 is subtracted from it. The remainder thus obtained is divided by α_1 and the quotient we get contains a unit in the tenth digit which agrees with the supposition that q_2 as we first obtained it was too small by one in the last place. Disregarding this superfluous 1 we try:

$$q_3=471397947 \qquad r_3=562208119.$$

We now calculate $\beta_1\alpha_4+\beta_2\alpha_3+q_3\alpha_2$ and obtain

447238319 036881555.

It is seen that we have made a fortunate choice of our r_3 for the above number lies between r_3 and $r_3-\alpha_1$. We therefore have $q_3=\beta_3$ and the above two-period number is σ'_4. Knowing β_3 we may now complete σ_3 by adding the product $\alpha_1\beta_3$ to σ'_3. The work now appears as follows:

α_1	α_2	α_3	α_4	α_5
$A=172214946$	598048117	796934501	113200000	000000000
$B=436207067$	145483756	471397947		
β_1	β_2	β_3	β_4	β_5
075121376	488223382	515816519	589321881	
	285927292	224860015		
$C=075121376$	774150674	740676535	151530000	000000000
γ_1	γ_2	γ_3	γ_4	γ_5

If we add the numbers on the two lines directly above γ_3 we get 740676534 which warns us that there will be a unit carried over from the fourth period.

By carrying out the above process the division of C by A might be extended indefinitely; but if we wish to terminate it with β_5 the last two β's can always be obtained in one step. Proceeding as above, we set the number r_3, γ_5 in the carriage and subtract σ'_4. Dividing by α_1 we try:

$$q_4=667594782 \qquad r_4=631106673\,.$$

We now calculate $\beta_1\alpha_5+\beta_2\alpha_4+\beta_3\alpha_3+q_4\alpha_2$ the first term of which is zero. This gives

791395851 138194941.

We now see that r_4 is too small and hence q_4 should be changed from

$$667594782 \text{ to } 667594781=\beta_4$$

if the condition that the above two-period number will lie between r_4 and $r_4-\alpha_1$ is to be fulfilled. By turning the crank one revolution in the negative direction we get

$$\sigma'_5=791395850 \qquad 540146824.$$

We next find that $r_4+\alpha_1-\sigma'_5$ when divided by α_1 gives

$$q_5=069249318 \qquad r_5=899946348.$$

From this we may assume that $q_5=\beta_5$ with doubt only as to the last digit. Knowing β_4 and β_5 we may complete σ_4 and σ_5. Inserting all these numbers and retaining only the first period of σ_5 the work in its final form appears as follows:

	α_1	α_2	α_3	α_4	α_5
$A=$	172214946	598048117	796924501	113200000	000000000
$B=$	436207067	145483756	471397947	667594781	069249318
	β_1	β_2	β_3	β_4	β_5
	075121376	488223382	515816519	589321881	803321619
		285927292	224860015	562208118	196678381
$C=$	075121376	774150674	740676535	151530000	000000000
	γ_1	γ_2	γ_3	γ_4	γ_5

Consequently we have

$$\sqrt[3]{83}=4.36207\ 06714\ 54837\ 56471\ 39794\ 76675\ 94781\ 06924\ 9318.$$

The mistakes in selecting the proper values for q_2 and q_4 might easily have been avoided by forming mentally a rough estimate of σ_3' and σ_5'. For larger divisions however this would not be practical nor indeed necessary since the above procedure solves the difficulty much more readily.

The above process may be adapted for the extraction of the square root of a number. This method depends also upon the proposition that if $n+1$ digits of the square root of a number are known and we wish the root correct to $2n+1$ digits, the remaining n figures may be found by dividing the remainder at that stage of the work by twice the root already found. The general method works equally well whether the number whose square root is to be extracted is an integer or a an irrational number and is a little quicker to perform than division. As in the division process, the method must be modified to allow for carrying over etc. and an explanation is best afforded by the use of an example. The outline of the work may be represented as follows:

$$\begin{array}{l}
\sqrt{N}=\beta_1,\ \ \beta_2,\ \ \beta_3,\ \ \beta_4,\ \ \beta_5\ \cdots \\
\hline
\qquad\quad \sigma_1 \qquad\quad \sigma_3 \qquad\quad \sigma_5\ \cdots \\
\qquad\qquad\quad \sigma_2 \qquad\quad \sigma_4 \qquad\quad \cdots \\
\hline
N=\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5\ \cdots
\end{array}$$

In this case however the σ's are defined as follows:

$$\sigma_1=\beta_1^2 \qquad\qquad \sigma_4=2(\beta_1\beta_4+\beta_2\beta_3)$$

$$\sigma_2=2\beta_1\beta_2 \qquad\qquad \sigma_5=2(\beta_1\beta_5+\beta_2\beta_4)+\beta_3^2$$

$$\sigma_3=2\beta_1\beta_3+\beta_2^2 \qquad\qquad \sigma_6=2(\beta_1\beta_6+\beta_2\beta_5+\beta_3\beta_4)$$

$$\cdots\cdots\cdots\cdots$$

These values may be obtained from the σ's used in division by changing the α's to β's and simplifying. The σ''s are defined as follows:

$$\sigma_1'=\sigma_2'=0 \qquad\qquad \sigma_5'=2\beta_2\beta_4+\beta_3^2$$

$$\sigma_3'=\beta_2^2 \qquad\qquad \sigma_6'=2(\beta_2\beta_5+\beta_3\beta_4)$$

$$\sigma_4'=2\beta_2\beta_3 \qquad\qquad \sigma_7'=2(\beta_2\beta_6+\beta_3\beta_5)+\beta_4^2$$

$$\cdots\cdots\cdots\cdots$$

Here we have

$$\sigma_n = \sigma_n' + 2\beta_1\beta_n \qquad n > 1.$$

As an example let it be required to find the square root of π. We put down the outline of our work as follows:

	β_1	β_2	β_3	β_4	β_5
$\sqrt{\pi}=$					
$\pi=$	031415926	535897932	384626433	832795028	841971694
	γ_1	γ_2	γ_3	γ_4	γ

In order to get β_1 we must first find the square root of γ_1 or

031415926 5.

By inspection we note that the square root lies between 17 and 18. Subtracting 17^2 from γ_1 and dividing the remainder by 17 we find the first digit in half the quotient to be 7. We therefore subtract 177^2 from γ_1 and by dividing the remainder by 177 and taking half the quotient we see that the next approximation to the root is 17724. One more application of this process gives $\beta_1 = 177245385$ and also $2\beta_1 = 345490770$, which last is written on the side for future use.
and also

We now set in the carriage the number γ_1, γ_2 from which we subtract β_1^2. This gives the remainder:

$$r_1 = 032099707\ .$$

The number r_1, γ_3 or

032099707 384626433

is now divided by $2\beta_1$. This gives the following quotient and remainder:

$$q_2 = 090551602 \quad r_2 = 266912893.$$

Squaring q_2 we obtain

$$q_2^2 = 008199592 \quad 624766404.$$

We observe that if q_2 were β_2 this last number would be σ_3'. Now it can be shown that σ_3' lies between r_2 and $r_2 - 2\beta_1$. It is seen that we have chosen r_2 correctly and that therefore $q_2 = \beta_2$. We may now put down β_1, β_2, σ_1, and σ_2 for $\sigma_1 = \beta_1^2$ and $\sigma_2 = 2\beta_1\beta_2$. The work then appears as follows:

	β_1	β_2	β_3	β_4	β_5
$\sqrt{\pi}=$	177245385	090551602			
	031415926	503798225			
		032099707	117713540		
$\pi=$	031415926	535897932	384626433	832795028	841971694
	γ_1	γ_2	γ_3	γ_4	γ_5

Adding the two periods common to σ_1 and σ_2 we get

535897932

which is γ_2 which serves as a check. We also double β_2 obtaining $2\beta_2 = 181103204$ which is set down for future use.

We wish now to determine β_3. The number r_2, γ_4 is set in the carriage and σ_2' subtracted from it. The remainder thus obtained is divided by $2\beta_1$ producing the following quotient and remainder:

$$q_3 = 729816748$$
$$r_3 = 250612664\ .$$

We next calculate $2\beta_2 q_3$ taking care to set $2\beta_2$ in the key-board. This gives:

$$2\beta_2 q_3 = 132172151 \quad 395660592\ .$$

The condition that this number which is an approximation at least to σ_4' shall lie between r_3 and $r_3-2\beta_1$ is satisfied by the r_3 chosen above. We therefore conclude that $q_3=\beta_3$ and that the above two-period number is σ_4'. We are now in a position to calculate σ_3 by adding $2\beta_1\beta_3$ to the value obtained for σ_3'. This being done, the work appears as follows:

	β_1	β_2	β_3	β_4	β_5
$\sqrt{\pi}=$	177245385	090551602	729816748		
	031415926	503798225	266912893	582182364	
		032099707	117713540		
$\pi=$	031415926	535897932	384626433	832795028	814971694
	γ_1	γ_2	γ_3	γ_4	γ_5

Again adding the two periods directly above γ_3 we get

384626433

which is γ_3 itself.

If we wish only 45 significant figures in the square root of π the last two β's can be obtained in one step as follows.

As usual we set the number r_3, γ_5 in the carriage and subtract σ_4'. Dividing the remainder by $2\beta_1$ we try:

$$q_4=334114518 \quad r_4=692312242.$$

We next calculate $\beta_4+2\beta_2 q_4$ remembering to set $2\beta_2$ in the key-board. This gives:

593141695 374011176.

It is seen that we have made a fortunate choice of q_4 since this last number lies between r_4 and $r_4-2\beta_1$. We therefore have:

$$\beta_4=334114518$$
$$\sigma_5'=593141695 \quad 374011176.$$

We next find that $r_4-\sigma_5'$ when divided by $2\beta_2$ gives:

$$q_5=279754946 \quad r_5=407140404 .$$

From this we assume that $q_5=\beta_5$ with doubt only as to the last figure. Knowing β_4 and β_5 we may find σ_4 by adding $2\beta_1\beta_4$ to σ_4' and find σ_5 by adding $2\beta_1\beta_5$ to σ_5'. Inserting these results in the work we finally have:

	β_1	β_2	β_3	β_4	β_5
$\sqrt{\pi}=$	177245385	090551602	729816748	334114518	279754946
	031415926	503798225	266912893	582182364	692312242
		032099707	117713540	250612664	149659452
$\pi=$	031415926	535897932	384626433	832795028	841971694
	γ_1	γ_2	γ_3	γ_4	γ_5

Whence

$$\sqrt{\pi}=1.77245\ 38509\ 05516\ 02729\ 81674\ 83341\ 14518\ 27975\ 4946.$$

The above general process will serve to extract the square root of a number to any degree of accuracy. Simpler methods can be readily derived in which it is only necessary to write down the root, period after period. In fact when the number whose square root is to be extracted is an integer $<10^{10}$, sixty digits in the root can be found in less than five minutes time. These short methods are not satisfactory for extending indefinitely the square root of any number. Moreover we are obliged to dispense with the valuable checks to be had in the more general method. Similar statements may be made with regard to the

division process. We therefore have not given a detailed account of these short methods trusting that the reader can work them out for himself.

COMMUTATIVE ALGEBRAIC INVERSIONS

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1. Introduction. In arithmetic numerous interesting theorems concerning functions of divisors depend ultimately upon the inversion of Dedekind (simultaneously stated by Liouville), which is as follows. Let $p(x)$, $q(x)$ be single valued for integer values of x, and write $\mu(n)=0, 1, -1$ according as the integer $n>0$ is divisible by an integer square >1, or the number of prime factors in n is even or odd respectively; $\mu(1)=1$. Then

$$p(n)=\sum q(d) \text{ implies } q(n)=\sum \mu(d)p(n/d) \ , \tag{1}$$

the summation extending to all divisors $d>0$ of n. Simple proofs of (1) are given in most books on arithmetic; a summary of allied results will be found in Dickson's *History of the Theory of Numbers*, vol. 1, chap. 19.

It was observed by H. F. Baker[1] that (1) is included in a much wider theorem concerning symmetric functions.

Baker's inversion was in turn generalized by L. Gegenbauer.[2] These extensions rapidly become extremely complicated, at least in algebraic expression, and further generalization by ordinary algebra is typographically impracticable. All, however, not only become intelligible at a glance when the appropriate algorithm is introduced, but also admit of indefinite extension in similar or in essentially new directions with equal simplicity. By means of the elementary processes of this algorithm the theorems become self-evident.

2. Notation. Each of $f_i, g_i, \cdots, h_i$ $(i=1, 2, \cdots)$ denotes a symmetric function of precisely i variables chosen from the sets

$$a_1, a_2\ , \ \cdots\ ; \quad b_1, b_2\ , \ \cdots\ ; \ \cdots\ ; c_1, c_2\ , \ \cdots\ .$$

It is essential to remember that by this notation the number of variables in any $f_i, g_i, \cdots, h_i$ is equal to the suffix i. Thus we may have $f_1(a_1), f_1(a_2), \cdots$, $f_2(a_1, a_2), \cdots$ but not $f_3(a_1, a_2)$. Further, the symmetric functions in a given context may be any whatever. For example, our theorems hold for $f_2(a_1, a_2)=a_1^2+a_2^2$, $f_3(a_1, a_2, a_3)=a_1^2+a_2^2+a_3^2$, or for any other symmetric functions of a_1, a_2 and a_1, a_2, a_3 respectively, the point being that different suffixes do not neces-

[1] *Proceedings of the London Mathematical Society*, vol. 21 (1889) pp. 30–32. *Cf.* Dickson, *loc. cit.* pp. 443–4, where the misprints of the original article are rectified. Baker's Theorem is not essentially different from our first example in §5.

[2] *Sitzungsberichte der kaiserlichen Akademie der Wissenschaften, Wien*, vol. 102 (1893) part II, pp. 951–978. Gegenbauer's most general result is included in our second example of §5.

sarily imply that the corresponding symmetric functions are of different *kinds* (in the example both *may* be sums of squares). When convenient the variables will be omitted, thus $f_3(a_1, a_2, a_3)=f_3$.

3. Composition. We now define a process which has the abstract properties of algebraic multiplication. Raise the suffix of each function considered, thus $f_i \equiv f^i$, $g_i \equiv g^i$, etc., so that, for example, $f_2(a_1, a_2) \equiv f^2(a_1, a_2)$ symbolically. Composition is defined by the equations, which obviously are consistent,

$$f^r(a_1, \cdots a_r)f^1(a_{r+1}) = f^{r+1}(a_1, \cdots, a_r, a_{r+1}), \tag{2}$$

$$f^r(a_1, \cdots, a_r)g^1(a_{r+1}) = f_r(a_1, \cdots, a_r)g_1(a_{r+1}), \tag{3}$$

$$\begin{aligned}\{F^r(a_1, \cdots, a_r)+G^s(b_1, \cdots, b_s)\}H^t(c_1, \cdots, c_t)\\ = F^r(a_1, \cdots, a_r)H^t(c_1, \cdots, c_t)+G^s(b_1, \cdots, b_s)\\ \cdot H^t(c_1, \cdots, c_t),\end{aligned} \tag{4}$$

in which F, G, H may be any of $f, g, \cdots, h$. If $f, g, \cdots$ occur with coefficients, as αf^r, βf^s, the $\alpha, \beta, \cdots$ being numbers of any field, the laws of composition are the same as in ordinary algebra, thus

$$\alpha f^1(a_1)\beta f^2(a_2, a_3) = \alpha\beta f_3(a_1, a_2, a_3).$$

Since $f_r, g_r, \cdots, h_r$ are symmetric functions, it follows that composition is commutative and associative. By (4) it is also distributive. Hence composition and algebraic multiplication are abstractly identical.

It follows immediately from the definitions that each of

$$F^r+G^r=H^r, \qquad G^r=H^r-F^r, \qquad F^r=H^r-G^r$$

implies both of the others. This simple remark, as we shall see, is at the root of the inversions described in §1.

To illustrate composition we have

$$\begin{aligned}\{f^1(a_1)+g^1(a_1)\}\{f^1(a_2)+g^1(a_2)\}\\ = f_2(a_1, a_2)+f_1(a_1)g_1(a_2)+f_1(a_2)g_1(a_1)+g_2(a_1, a_2).\end{aligned}$$

Clearly not every sum of symmetric functions can be decomposed in this manner into symbolic factors; it is only those that can that yield inversion theorems of the kinds mentioned.

4. Decomposition. This process can be considered as an inverse of composition. Let

$$p(x, y, \cdots, z) \equiv \sum_{a, b, \cdots, c} a_{a, b, \cdots, c}x^a y^b \cdots z^c$$

be a polynomial (not necessarily homogeneous) in $x, y, \cdots, z$. Replace $x, y, \cdots, z$ respectively by the $F, G, \cdots, H$ of §3 and denote the result by S_i,

where j is the total number of variables occurring as arguments in $F, G, \cdots, H$. We have then

$$S_j = \sum_{a,b,\cdots c,} a_{a,b\,\cdots\,,c} F_a G_b \cdot \cdot \cdot H_c \tag{5}$$

as the *definition* of S_j, or, symbolically,

$$S_j = p(F, G, \cdots, H) . \tag{6}$$

In the right of (5) a particular symbol of symmetric functions, say F, appears with various suffixes, say $a', a'', \cdots$, so that F accounts for symmetric functions having (*cf.* §2) precisely a', or a'', or, $\cdots$ variables. Let n denote an arbitrary constant integer. Then *any* determination of F_n, having n variables, from (6) in terms of functions $S, G, \cdots, H$ (all of those in (6) except F), is called *a decomposition* of (6) or of $p(F, G, \cdots, H)$ with respect to F. Similarly with respect to any one of $G, \cdots, H$.

Note that S, as defined by (5) or (6) is not necessarily a symmetric function in *all* of its j variables; it is symmetric *separately* in the members of each set of variables occurring as arguments of functions F, or functions $G, \cdots$ respectively. In the illustration of §5 we discuss examplès of both cases, S symmetric, S not symmetric, in all of its variables.

In what precedes we have restricted p to be a polynomial. This restriction is inessential; removing it, and making the few obvious changes requisite, we can extend the definition of decomposition to any function p which has a multiple power series expansion. For brevity it may here be assumed that the original series for p and those for each of its decompositions are absolutely convergent in the same region. It is both feasible and profitable, however, to discuss the decomposition of infinite series independently of their convergence, and to devise an interpretation of the results which shall be self consistent. We merely indicate this last possibility; the means for carrying it out are fully discussed in a former paper.[1]

Any set of decompositions of $p(F, G, \cdots, H)$ with respect to $F, G, \cdots, H$ respectively, will be called an *inversion* of $p(F, G, \cdots, H)$ or of (6).

It is not assumed for p general either that decompositions exist or that, if existing, they are unique. If $p(x, y, \cdots, z)$ can be resolved into a product of linear factors with coefficients in any domain of rationality, inversion is always possible and unique in the same domain, as appears from the definitions and the examples next given.

5. Inversions. (I) It will suffice to discuss three of the simplest. Let the variables $a, b, \cdots, c$ be s in number, and write

$$1 + f^1(x) = g^1(x) \qquad (x = a, b, \cdots, c) . \tag{7}$$

[1] *Transactions of the American Mathematical Society*, vol. 25 (1923) pp. 135–154.

Then, by composition, we obtain

$$\{1+f^1(a)\}\{1+f^1(b)\}\cdots\{1+f^1(c)\}=g_s(a,b,\cdots,c)\qquad(s=1,2,\cdots)\,. \tag{8}$$

The structure of the symmetric functions on the right is obvious from the left, if we imagine the symbolic product first distributed before all exponents are degraded to suffixes. To decompose (8) with respect to f we have from (7),

$$f^1(x)=g^1(x)-1\qquad(x=a,b,\cdots,c)\,, \tag{9}$$

and hence by composition,

$$f_s(a,b,\cdots,c)=(-1)^s\{1-g^1(a)\}\{1-g^1(b)\}\cdots\{1-g^1(c)\}\qquad(s=1,2,\cdots) \tag{10}$$

as an inversion of (8).

To see the meaning of the theorem expressed by (8), (10), take the case of 2 variables. Then, by (8),

$$g_2(a,b)=1+f_1(a)+f_1(b)+f_2(a,b)\,,$$
$$g_1(a)=1+f_1(a)\,,\qquad g_1(b)=1+f_1(b\,,$$

and (10) asserts that for these g's,

$$f_2(a,b)=1-g_1(a)-g_1(b)+g_2(a,b)\,,$$

which is correct. For 3 variables the g's given by (8) are

$$g_3(a,b,c)=1+f_1(a)+f_1(b)+f_1(c)+f_2(a,b)+f_2(a,c)+f_2(b,c)+f_3(a,b,c)\,,$$
$$g_2(a,b)=1+f_1(a)+f_1(b)+f_2(a,b)\,,$$
$$g_2(a,c)=1+f_1(a)+f_1(c)+f_2(a,c)\,,$$
$$g_2(b,c)=1+f_1(b)+f_1(c)+f_2(b,c)\,,$$
$$g_1(a)=1+f_1(a)\,,\qquad g_1(b)=1+f_1(b)\,,\qquad g_1(c)=1+f_1(c)\,,$$

and (10) is

$$f_3(a,b,c)=-1+g_1(a)+g_1(b)+g_1(c)-g_2(a,b)-g_2(a,c)-g_2(b,c)+g_3(a,b,c)\,,$$

which is verified by inspection for the foregoing g's.

(II) As a second example we shall take precisely r classes of symmetric functions f_s, g_s, $\cdots$, h_s ($s=1, 2, \cdots$), the symbols f, g, $\cdots$, h being r in number, and decompose

$$\begin{aligned}\{f^1(a)+g^1(a)+\cdots+h^1(a)\}\{f^1(b)+g^1(b)+\cdots+h^1(b)\}\cdots\{f^1(c)+g^1(c)+\cdots+h^1(c)\}\\ =k_s(a,b,\cdots,c)\qquad(s=1,2,\cdots)\end{aligned} \tag{11}$$

with respect to f. As before, the structure of the functions k_s, which evidently by the left of (11) are symmetric in the s variables a, b, $\cdots$, c, is evident for any value of s if we think of the distributed form of the left. In particular, when $s=1$, we have $f^1(x)+g^1(x)+\cdots+h^1(x)=k^1(x)$ $(x=a, b, \cdots, c)$, and therefore as before, for $s=1, 2, \cdots$, we obtain

$$\begin{aligned}f_s(a,b,\cdots c)=(-1)^s\{g^1(a)+\cdots+h^1(a)-k^1(a)\}\{g^1(b)+\cdots+h^1(b)-k^1(b)\}\cdots\{g^1(c)+\\ \cdots+h^1(c)-k^1(c)\}\end{aligned} \tag{12}$$

as a decomposition of (11) with respect to f. Similarly for g, $\cdots$, h. Taken together these r decompositions furnish an inversion of (11).

To verify a formula such as (12) after reduction to non-symbolic form, we eliminate from the decomposition, here (12), that class of functions, here the k_s ($s=1, 2, \cdots$), which was defined by the original composition, here (11).

(III) Consider r symbols f, g, $\cdots$, h of symmetric functions, and s symbols a, b, $\cdots$, c of variables, so that each of the sets

$$\sigma_i\equiv a_i,b_i,\cdots,c_i\qquad(i=1,\cdots,r)$$

contains precisely s variables. Clearly, if the s functions

$$f^1(x_1)+g^1(x_2)+\cdots+h^1(x_r)\qquad(x=a,\,b,\cdots,\,c)$$

be compounded, the result, F_{rs}, is not symmetric in all the variables, but is symmetric separately in the variables of each of the sets σ_i $(i=1, \cdots, r)$. Write

$$F_{rs}(a_1,b_1, \cdots, c_1; a_2,b_2, \cdots, c_2; \cdots; a_r,b_r, \cdots, c_r) = \prod_{x=a,\, b,\, \cdots,\, c} \{ f^1(x_1)+g^1(x_2)+ \cdots +h^1(x_r)\} , \tag{13}$$

so that

$$F^r(x_1; x_2; \cdots; x_r)=f^1(x_1)+g^1(x_2)+ \cdots +h^1(x_r) \qquad (x=a,b, \cdots, c)$$

and hence

$$f_1(x_1)=F^r(x_1; x_2; \cdots; x_r)-g^1(x_2)- \cdots -h^1(x_r) .$$

The decomposition of (13) with respect to f is therefore

$$f_s(a_1,b_1, \cdots, c_1)= \prod_{x=a,\, b,\, \cdots,\, c} \{F^r(x_1; x_2; \cdots; x_r)-g^1(x_2)- \cdots -h^1(x_r)\} , \tag{14}$$

the right of which obviously is symmetric in a_1, b_1, $\cdots$, c_1, as it should be by inspection of the left. Similarly for the decompositions of (13) with respect to $g, \cdots, h$, which can be written down from (14) by suitable changes of suffixes, etc. Thus for g the left of (14) becomes g_s $(a_2, b_2, \cdots, c_2)$.

6. Glancing back we see that the imposition of symmetry upon the functions inverted is necessary in order to preserve commutativity in composition. It is clear that this analogue of multiplication when applied to non-symmetric algebraic functions no longer is commutative. The composition of a symmetric function is isomorphic in its abstract properties with the resolution of an integer into its prime factors. All of the processes are reversible.

QUESTIONS AND DISCUSSIONS

EDITED BY TOMLINSON FORT, Hunter College, Park Ave. and 68th St., New York, N. Y., and H. E. BUCHANAN, Tulane University, New Orleans, La.

The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

REPLIES TO QUESTIONS

55 [*1925*, 506, 510]. Is it possible by ruler and compasses to construct an angle equal to one radian?

REPLY BY A. J. KEMPNER, University of Colorado.

Of course not. If an angle of one radian could be constructed by ruler and compasses, sin 1, cos 1, cos $1+i\sin 1=e^i$ could each be constructed, and e^i would be an algebraic number. But e^x is transcendental for all algebraic exponents $x\neq 0$, by a classical theorem.

NOTE: The proposer of the question informs us that he received replies from "the ends of the earth." EDITOR.

DISCUSSIONS

I. A Cubic Equation of Newton's

By Norman Anning, University of Michigan

1. History. In chapter 13 of his *Arithmetica Universalis* Newton showed how algebra could be applied to the solution of problems in geometry. He used for illustration the problem[1] of finding the diameter $AD(=x)$ of a semicircle when three chords $AB\ (=a)$, $BC\ (=b)$, $CD\ (=c)$ are given. In half a dozen interesting ways he arrived at the equation:

$$f(x)\equiv x^3-x(a^2+b^2+c^2)-2abc=0\ . \tag{1}$$

Having derived the equation he did not concern himself with its solution or discussion. He said[2] in conclusion

"Wherefore if the Method you take from your first Thoughts, for solving a Problem, be but ill accommodated to computation, you must again consider the Relations of the Lines, until you shall have hit on a Way as fit and elegant as possible. For those Ways that offer themselves at first Sight, may often create sufficient Trouble if they are made use of."

2. Nature of the roots.[3] If we neglect a positive numerical factor, the discriminant of (1) is $\quad (a^2+b^2+c^2)^3-27a^2b^2c^2\ .$ (2)

When a, b, c, are real and not all equal, the three numbers a^2, b^2, c^2, are positive and not all equal. For such numbers

$$A.M.>G.M.,\qquad (a^2+b^2+c^2)/3>(a^2b^2c^2)^{1/3},\qquad (a^2+b^2+c^2)^3>27a^2b^2c^2\ .$$

In this case the discriminant (2) is positive and the cubic has three real and distinct roots.

Of the three roots one will be positive when a, b, c, are all positive and two will be positive when any one of a, b, c, is negative. For, in the former case, $f(\infty)$ is positive while $f(0)$ is negative and therefore $f(x)=0$ has an odd number of positive roots. But, by Descartes' theorem, it cannot have more than one. Consequently there is just one. The second case may be proved similarly or by observing that substituting $(-c)$ for c in (1) transforms $f(x)$ into $f(-x)$.

When $a=b$, the roots of (1) are constructible because then

$$x^3-(2a^2+c^2)x-2a^2c=(x+c)(x^2-cx-2a^2)=0\ .$$

When $a=b=c$, the discriminant is zero, and x has the values: $-a, -a,\ 2a$.

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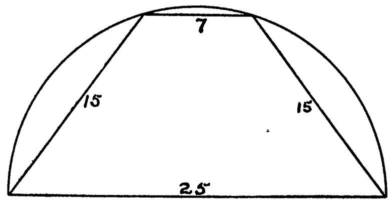
[1] See S. Horsley, *Isaaci Newtoni Opera quae exstant omnia*, London, 1779, vol. I, pp. 83–92. For translation into English see Ralphson and Cunn, *Universal Arithmetick*, London, 1769, pp. 202–216.

[2] Quoted from Ralphson and Cunn, p. 216.

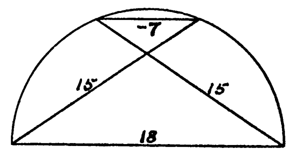
[3] Some of the results in this section were proved in *School Science and Mathematics*, 1911, problems 250, 254.

3. Numerical example.

When $a=b=15$, $c=7$,
$x^3-499x-3150=0$,
$x=25,-18,-7$.



When $a=b=15$, $c=-7$,
$x^3-499x+3150=0$,
$x=-25,18,7$.



This example suggests that two of the roots of (1) are capable of geometric interpretation. This will always be true provided (1) has two roots which are, in absolute value, not less than the greatest of the absolute values of a, b, c.

4. Solution in integers. If we write (1) in the form

$$\left(\frac{a}{x}\right)^2+\left(\frac{b}{x}\right)^2+\left(\frac{c}{x}\right)^2+2\frac{abc}{xxx}=1 ,$$

and compare it with the identity $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1$, which is true whenever $A+B+C=180°$, we see that the problem[1] of finding integers a, b, c, x, to satisfy (1) reduces to that of finding a triangle whose angles have rational cosines. The sides of such a triangle must all be rational multiples of the same quantity and by suitably changing the scale we can, without altering the angles, make the measures of the sides all integers. So if we choose any three positive integers l, m, n, which are the sides of a possible triangle, we may write

$$a=l(m^2+n^2-l^2) , \quad b=m(n^2+l^2-m^2) , \quad c=n(l^2+m^2-n^2) , \quad x=2lmn .$$

For instance, putting $l=4$, $m=5$, $n=6$, yields $a=12$, $b=9$, $c=2$, $x=16$.

II. Another Poristic System of Triangles[2]

By Rufus Crane, Ohio Wesleyan University

A triangle is, in general, determined by any two of its four in- and escribed circles. Again, it is determined by the circumcircle, the nine-point circle, and any one of the in- or escribed circles. In this latter case, if only two of the circles be fixed, the triangle has one degree of freedom and may assume a series of positions. These constitute a poristic system. Gallatly, in his *Modern Geometry of the Triangle*, 1913, discussed the poristic system with the incircle and the

[1] A solution of this problem by P. Bachmann was published in *Archiv der Mathematik und Physik*, series III, vol. 24 (1915), pp. 89–90.

[2] Read before the American Mathematical Society, Chicago meeting, April 10, 1925.

circumcircle fixed. Professor J. H. Weaver, in the MONTHLY, September, 1924, discussed the case where the circumcircle and the nine-point circle were fixed. The present note discusses the case of the nine-point circle and the incircle.

The radii of these two circles may have any values whatever, provided only that the incircle is not greater than the nine-point circle. If the incircle coincides with the nine-point circle, the triangle becomes equilateral and the system degenerates.

In the notation here used the circumcircle has center O, radius R; the nine-point circle has center O', radius $\frac{1}{2}R$; the incircle has center I, radius r, and touches BC at X, CA at Y, AB at Z; the excircle opposite A has center I_1, and touches BC at X_1, CA at Y_1, AB at Z_1; the excircle opposite B has center I_2, etc. The point of tangency of the incircle and nine-point circle is F, the Feuerbach point.

Since the distance OI is constant, in terms of the invariants R and r, the locus of O is a circle with center at I and this distance as radius. This circle will lie within, coincide with, or lie without the incircle according as

$$R \lesseqgtr r(1+\sqrt{2}) .$$

In the first case the triangle is acute angled in all of its positions; in the second case it becomes right angled when O falls at F; in the third case it is acute angled as long as O lies inside the nine-point circle, but is obtuse angled when O falls outside of, and right angled when it falls on that circle.

Obviously, the center of gravity G, the orthocenter H, and the point J (the circumcenter of $I_1I_2I_3$), traverse circles homothetic to that of O. Obviously, also, the Nagel point N traverses the same circle as H.

Aside from these, it may readily be shown that the Gergonne point M traverses a circle. Gallatly has shown that the point σ (the center of similitude of the triangles XYZ and $I_1I_2I_3$) and also the point H_i (the orthocenter of XYZ) are points lying on the line OI at distances from I which are expressible in terms of R and r. He also shows that σ, M, G, are collinear, as likewise H_i, M, N. Also, I, G, N, are collinear and IG is one third of IN. Hence, the lines NH_i and $G\sigma$ intersect the lines IO' and IO at distances which are invariant: hence, their point of intersection M has invariant coordinates referred to these two lines as oblique axes.

Some of the other points of the triangle give us loci of higher degree. The equations of these may be written out, with greater or less ease, by determining the distances of the point in question from some of the points O', I, F, or some other critical point, in terms of R, r, and the angles A, B, C, then taking one of these points as a pole and eliminating A, B, and C. The details of these derivations are routine, but the following results may be stated.

The locus of the vertices A, B, C, is a bicircular quartic, very similar to the nodal form of the circular limaçon. Taking the pole at the node, which lies on $O'I$ at a distance r from F, with $O'I$ as axis, its equation becomes

$$\rho=(R+2r)\cos\theta\pm\sqrt{(R-2r)^2\cos^2\theta+2r(R-2r)}\ .$$

This curve has one focus at I, and another external to the curve, at a distance from I equal to $8Rr/(R-2r)$. We may note that while the vertices are traversing this locus once, the points mentioned above traverse their loci three times.

The excenters I_1, I_2, I_3, also generate a bicircular quartic, in fact, a limaçon, with a focus at O', with a node on IO' distant $2R$ from I. Taking this node as origin, with IO' as axis, its equation is

$$\rho=4R\cos\theta\pm 2\sqrt{R^2-2Rr}\ .$$

The nine points of tangency of the excircles fall in two groups; the three that lie on the sides AB, BC, CA, and which may be called the internal points of tangency, form one group, while the six that lie externally (on the sides extended) form another group. The first group, X_1, Y_2, Z_3, generate a curve of sixth degree passing twice through the Feuerbach point and having a tacnode at that point. The curve has also a crunode lying on the line FIO'. Depending on the relation between R and r mentioned above, the shape of this curve varies. If R is less than $r(1+\sqrt{2})$, the curve is convex throughout: if greater, the outer loop has a sinus at the tacnode. The equation of this curve (and likewise of the following) is too cumbersome to be of value except as showing the degree of the curve.

The six external points of tangency generate a locus consisting of two bicircular curves which are located obliquely to the line FIO', but are symmetrical to each other with respect to this line. This locus proves to be of degree sixteen.

III. A New Method of Determining Bernoulli's Numbers[1]

By C. A. Messick, University of Iowa

1. The methods of determining Bernoulli's numbers seem to group themselves into two general classes, one depending on differentiation,[2] and the other on multiplication of infinite series.[3]

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[1] Presented to the American Mathematical Society at the meeting in Kansas City, December 29, 1925.

[2] Price, *Treatise on Infinitesimal Calculus*, p. 145.
Godefroy, *Théorie Elémentaire des Series*, p. 116.

[3] Bromwich, *Introduction to the Theory of Infinite Series*, p. 233.
Smail, *Elements of the Theory of Infinite Processes*, p. 238.

In the first method, successive differentiation of the function $F(x)=\frac{x}{e^x-1}$ produces the equations:

$$e^x[F'(x)+F(x)]=1+F'(x) \ ,$$

$$e^x\left[F^{(n)}(x)+nF^{(n-1)}(x)+\frac{n(n-1)}{2!}F^{(n-2)}(x)+\cdots+nF'(x)+F(x)\right]=F^{(n)}(x) \ ,$$

where $n=2, 3, 4, \cdots\cdots$. The nth Bernoulli number is defined by $B_n=(-1)^{n-1}[F^{(2n)}(0)]$, all the derivatives of odd order after the first being zero when $x=0$. The Maclaurin series is

$$F(x)=\frac{x}{e^x-1}=1-\frac{x}{2}+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+B_3\frac{x^6}{6!}-\cdots .$$

In the second method, it is assumed[1] that the function has the expansion $x/(e^x-1)=1+A_1x+A_2x^2+A_3x^3+\cdots$. It is found that all the odd numbered A's after the first are zeros, and the even numbered A's are determined by the equations

$$\frac{A_{2n}}{1}+\frac{A_{2n-2}}{3!}+\cdots+\frac{A_2}{(2n-1)!}-\frac{2n-1}{2(2n+2)!}=0 \ ,$$

$$\frac{A_{2n}}{2!}+\frac{A_{2n-2}}{4!}+\cdots+\frac{A_2}{(2n)!}-\frac{2n}{2(2n+2)!}=0 \ ,$$

where $n=1, 2, 3, \cdots$. By comparing with the Maclaurin series, it is seen that we can define the nth Bernoulli number as $B_n=(-1)^{n-1}(2n)!A_{2n}$.

By the methods of the first class a single equation is derived, and by the methods of the second class usually two equations are derived, each containing an arbitrary positive integer n. To find the rth Bernoulli number, set $n=r$ in the equation, substitute the numerical values of the first $r-1$ numbers and perform the indicated operations.

By the new method (which is of the second kind) given below, four equations are obtained which seem different from any given before. From one equation the Bernoulli numbers of odd index can be computed, and from another, the numbers of even index. The advantage in using these equations is that there are only half as many terms as in the equations now in use, so that the arithmetic is much more simple. The other two equations are sums of multiples of an odd or even number, respectively, of the Bernoulli numbers, and can be used to check the numerical accuracy of the results obtained from the first pair.

[1] This is justified by a theorem in Godefroy, p. 96, "If a function, here (e^x-1/x), can be developed into a power series in which the first term is not zero, its reciprocal can be developed into a power series in which the first term is not zero."

An expansion for sin x is also obtained in terms of Bernoulli's numbers, which is believed to be new. This is $F(x)\cdot\varphi(x)$, given below.

2. It is known that if $|x|<1.2$,

$$F(x)=\frac{x}{e^x-1}=1-\frac{x}{2}+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+B_3\frac{x^6}{6!}-\cdots+(-1)^{n-1}B_n\frac{x^{2n}}{(2n)!}\cdots \tag{1}$$

where B_n represents the nth Bernoulli number.

Now let $\varphi(x)$ be $\dfrac{e^{(1+i)x}-e^{(1-i)x}-e^{ix}+e^{-ix}}{2ix}$.

Expanding this function into a power series in x, we have

$$\varphi(x)=x+\frac{3x^2}{3!}-\frac{5x^4}{5!}-\frac{8x^5}{6!}-\frac{7x^6}{7!}+\frac{15x^8}{9!}+\frac{32x^9}{10!}+\frac{33x^{10}}{11!}-\frac{65x^{12}}{13!}-\frac{128x^{13}}{14!}$$
$$\cdots+(-1)^k\left[\frac{[2^{2k}-(-1)^k]x^{4k}}{(4k+1)!}+\frac{[2^{2k+1}]x^{4k+1}}{(4k+2)!}+\frac{[2^{2k+1}-(-1)^{k+1}]x^{4k+2}}{(4k+3)!}\right]\cdots$$

where $k=0, 1, 2, 3, \cdots$. Since this power series is convergent in the interval of convergence of (1), we may multiply it by $F(x)$, obtaining a power series whose interval of convergence is at least as great as that of (1), that is, $|x|<1.2$.

$$F(x)\cdot\varphi(x)=x-x^3\left(\frac{1}{4}-\frac{B_1}{2}\right)+x^4\left(\frac{B_1}{4}-\frac{1}{4!}\right)+x^5\left(\frac{1}{2\cdot 4!}-\frac{8}{6!}-\frac{B_2}{4!}\right)$$
$$-x^6\left(\frac{7}{7!}-\frac{4}{6!}+\frac{B_1}{2\cdot 4!}+\frac{B_2}{2\cdot 4!}\right)-x^7\left(\frac{8B_1}{2\cdot 6!}-\frac{7}{2\cdot 7!}-\frac{B_3}{6!}\right)$$
$$+x^8\left(\frac{15}{9!}-\frac{7B_1}{2\cdot 7!}+\frac{B_2}{4!^2}+\frac{B_3}{2\cdot 6!}\right)+x^9\left(\frac{32}{10!}-\frac{15}{2\cdot 9!}+\frac{8B_2}{4!\cdot 6!}-\frac{B_4}{8!}\right)$$
$$-\cdots\pm C_nx^n\cdots$$

where C_n is as given below.

$$\text{But } F(x)\cdot\varphi(x)=\frac{e^{ix}-e^{-ix}}{2i}=\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

which is convergent for all values of x. We may now equate the coefficients of like powers of x in the two series and compute the numerical values of B_1, B_2, $\cdots$ as far as we wish. This gives

$$B_1=\frac{1}{6},\qquad B_2=\frac{1}{30},\qquad B_3=\frac{1}{42},\qquad B_4=\frac{1}{30},\ \cdots.$$

The coefficient C_n of the nth power of x in the product $F(x) \cdot \varphi(x)$ takes four forms according to whether n is of the form $4k$, $4k+1$, $4k+2$, or $4k+3$, ($k=1, 2, 3, \cdots$). The equation when n is odd is $C_n = 1/n!$, and when n is even, $C_n = 0$. When n is odd

$$C_n = -\frac{B_{\frac{n-1}{2}}}{(n-1)!} + \frac{B_{\frac{n-5}{2}} 2^3}{(n-5)!6!} - \frac{B_{\frac{n-9}{2}} 2^5}{(n-9)!10!} + \cdots$$

ending, when $n = 4k+1$ and B has only even subscripts, as follows:

$$\cdots + (-1)^k \left[\frac{B_2 2^{2k-1}}{4!(4k-2)!} - \frac{1}{2} \frac{2^{2k}-(-1)^k}{(4k+1)} + \frac{2^{2k+1}}{(4k+2)!} \right],$$

ending, when $n = 4k+3$ and B has only odd subscripts, as follows:

$$\cdots + (-1)^{k+1} \left[\frac{B_1 2^{2k+1}}{2!(4k+2)!} - \frac{1}{2} \frac{2^{2k+1}-(-1)^{k+1}}{(4k+3)!} \right].$$

When n is even

$$C_n = \frac{B_{\frac{n-2}{2}}(2^1+1)}{(n-2)!3!} + \frac{B_{\frac{n-4}{2}}(2^2+1)}{(n-4)!5!} - \frac{B_{\frac{n-6}{2}}(2^3-1)}{(n-6)!7!} - \frac{B_{\frac{n-8}{2}}(2^4-1)}{(n-8)!9!} + \cdots$$

ending when $n = 4k$ as follows:

$$\cdots + (-1)^k \left[\frac{B_2[2^{2k-2}-(-1)^{k-1}]}{4!(4k-3)!} - \frac{B_1[2^{2k-1}-(-1)^k]}{2!(4k-1)!} + \frac{2^{2k}-(-1)^k}{(4k+1)!} \right]$$

ending when $n = 4k+2$ as follows:

$$\cdots + (-1)^{k+1} \left[\frac{B_2[2^{2k-1}-(-1)^k]}{4!(4k-1)!} + \frac{B_1[2^{2k}-(-1)^k]}{2!(4k+1)!} - \frac{2^{2k}}{(4k+2)!} + \frac{2^{2k+1}-(-1)^{k+1}}{(4k+2)!} \right].$$

A Crystallographic Illustration of Quadratic Involution

By A. C. Lunn, University of Chicago

Let (**a**, **b**, **c**) be a triple of vectors restricted to be of equal magnitude and inclined at equal angles, so that in the Gibbs notation $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = q$, $\mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} = p$, where q, p are arbitrary except that for real vectors q is positive and p numerically less than q. Then it is apparent from the geometry, and readily provable from the definitions $v\mathbf{a}' = \mathbf{b} \times \mathbf{c}$, $v\mathbf{b}' = \mathbf{c} \times \mathbf{a}$, $v\mathbf{c}' = \mathbf{a} \times \mathbf{b}$, where $v = [\mathbf{abc}]$, that the "reciprocal triple" satisfies similar conditions, only

with a new pair of constants q', p'. One way to show this is to notice that the general resolution theorem, $\mathbf{r}=\mathbf{r}\cdot\mathbf{aa}'+\mathbf{r}\cdot\mathbf{bb}'+\mathbf{r}\cdot\mathbf{cc}'$, gives here

$$\mathbf{a}=q\mathbf{a}'+p(\mathbf{b}'+\mathbf{c}'),\quad \mathbf{b}=q\mathbf{b}'+p(\mathbf{c}'+\mathbf{a}'),\quad \mathbf{c}=q\mathbf{c}'+p(\mathbf{a}'+\mathbf{b}'),$$

whose solution proves to be of similar form, with primed and unprimed interchanged, if q', p' be defined by

$$p'=-p/r\ ,\qquad q'=(q+p)/r\qquad \text{with } r=(q-p)(q+2p)\ .$$

Since the relation between the two triples of vectors is mutual this relation must be an involution, and in fact the interchange of primed and unprimed does give the solution for p', q' in terms of p, q.

To put the result in a different form let x, x' be the cosines of the respective included angles, then $x=p/q$, $x'=p'/q'$, so that

$$qq'=\frac{1+x}{(1-x)(1+2x)}\ ,\qquad x'=-\frac{x}{1+x}\ .$$

Here the second equation must be an involution, and it is by change of sign of x seen to be one of the familiar cross-ratio transformations of period two; and the first expression, being symmetric in q, q', must be an invariant of that transformation.

Two cases occur in the theory of lattices belonging to the cubic system. One is the case of the common orthogonal triple ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) of the simple cube lattice, where the reciprocal triple is the same; here $p=p'=0$, $qq'=1$. The other case can be represented with its first triple $(\mathbf{j}+\mathbf{k},\ \mathbf{k}+\mathbf{i},\ \mathbf{i}+\mathbf{j})/2$, which generates a face-centered cube lattice, with reciprocal triple $(\mathbf{j}+\mathbf{k}-\mathbf{i},\ \mathbf{k}+\mathbf{i}-\mathbf{j},\ \mathbf{i}+\mathbf{j}-\mathbf{k})$, which gives the dual body-centered lattice; here $q=1/2$, $p=1/4$, $q'=3$, $p'=-1$; the cosines are $1/2$, $-1/3$, and the invariant $3/2$.

Various curious extensions of these relations are obtainable by considering n-tuples of vectors in higher spaces and symmetrizing their magnitudes and orientations in various ways.

On the Duals of Metric Theorems

By Louis Weisner, University of Rochester

In dualising theorems involving angular magnitudes we frequently invoke the

'Theorem 1. *The angle between two lines not passing through the center of a circle is equal to the angle subtended at the center of the circle by the poles of the lines with respect to the circle.*

A few theorems[1] involving distances may be dualised with the aid of the

[1] See, for example, Winger, *Projective Geometry*, p. 148, ex. 21–23.

THEOREM 2. *The product of the distances of a point and its polar line from the center of a circle is constant, equal to the square of the radius of the circle.*

The limitations of this theorem being obvious, we proceed to prove a theorem which enables us to dualise such metric theorems as the Pythagorean theorem.

THEOREM 3. *If P_1, P_2 are the feet of the perpendiculars from the center O of a circle, upon the polar lines of two points A_1, A_2 (different from 0) with respect to the circle, then $\overline{A_1A_2}=\overline{P_1P_2}a^2/\overline{OP_1}\cdot\overline{OP_2}$, where a is the radius of the circle.*

The theorem follows from the fact that P_1, P_2 are the inverses of A_1, A_2 with respect to the circle, and from the well-known formula for the distance between the inverses of two points.

The following illustrations exhibit the applications of Theorem 3. The usual machinery of reciprocation is freely used.[1]

1. The dual of the Pythagorean theorem is: If P_1, P_2, P_3 are the feet of the perpendiculars on the sides opposite the vertices A_1, A_2, A_3 of a triangle, from a point O on the circle whose diameter is A_2A_3, then

$$\left(\frac{\overline{P_2P_3}}{\overline{OP_2}\cdot\overline{OP_3}}\right)^2=\left(\frac{\overline{P_1P_2}}{\overline{OP_1}\cdot\overline{OP_2}}\right)^2+\left(\frac{\overline{P_1P_3}}{\overline{OP_1}\cdot\overline{OP_3}}\right)^2.$$

2. The reciprocal of a pair of triangles whose sides are respectively parallel is a pair of triangles perspective from the center of reciprocation. From the properties of similar triangles we may derive properties of perspective triangles. For example, since the homologous sides of similar triangles are proportional, we have the following theorem: Let the triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ be perspective from a point O, the points O, A_i, A_i' $(i=1, 2, 3)$ being collinear. If P_i and P_i' are the feet of the perpendiculars from O on the sides opposite A_i and A_i' respectively, then $(\overline{P_1P_2}/\overline{P_1P_3})\cdot(\overline{OP_3}/\overline{OP_2})=$ the same function of the accented letters, with $O'=O$. This function is therefore an invariant as regards all triangles perspective from O with triangle $A_1A_2A_3$.

3. From the theorem "the tangents to a circle from an external point are equal," we deduce that, if P_1, P_2, P_3 respectively are the feet of the perpendiculars from the focus O of a conic, on two arbitrary tangents and the chord joining the point of contact, then

$$\frac{\overline{P_1P_3}}{\overline{OP_1}}=\frac{\overline{P_2P_3}}{\overline{OP_2}}.$$

[1] An excellent exposition of the subject of reciprocation, a knowledge of which is essential for the comprehension of the illustrations given here, will be found in Winger, *Projective Geometry*, p. 138ff.

4. From Ptolemy's theorem on quadrilaterals, we deduce that if P_1, P_2, P_3, P_4, in cyclic order, are the feet of the perpendiculars from the focus O of a conic, on four arbitrary tangents, then

$$\frac{\overline{P_1P_3}}{\overline{OP_1}\cdot\overline{OP_3}}\cdot\frac{\overline{P_2P_4}}{\overline{OP_2}\cdot\overline{OP_4}}=\frac{\overline{P_2P_3}}{\overline{OP_2}\cdot\overline{OP_3}}\cdot\frac{\overline{P_1P_4}}{\overline{OP_1}\cdot\overline{OP_4}}+\frac{\overline{P_1P_2}}{\overline{OP_1}\cdot\overline{OP_2}}\cdot\frac{\overline{P_3P_4}}{\overline{OP_3}\cdot\overline{OP_4}}.$$

Clearing of fractions we have an equation which proves that P_1, P_2, P_3, P_4 are either collinear or concyclic. Hence, the pedal of a conic with respect to a focus is a line or a circle. (It is a line if the conic is a parabola.)

VI. A Note on the Hindu-Arabic Numerals

By Bibhutibhusan Datta, University College of Science and Technology, Calcutta

In Professor Smith's *History of Mathematics* (vol. II, p. 64) and in the work on the *Hindu-Arabic Numerals* written by him and Professor Karpinski, due credit is given to the evidence of the Hindu origin of these forms. In the latter work, however, it appears (p. 41) that the authors entertain the belief that, although the place value was invented in the sixth century A.D., "not until a considerably later period did it become well known."

It is certain, however, that the decimal system of numeration was familiar to the Vedic Hindus in almost the same form as at present. There are references in Indian literature which prove conclusively that the idea of place value was well known in India in the fifth century A.D., if not a century or two earlier. Fortunately, too, the references are not merely epigraphical, nor are they of doubtful value. One such reference is found in Śaṅkarāchārya's commentary on the Brahma-Sūtra, and also in Vyāsa's commentary on the Yoga-Sūtra of Patañjali. Commenting on Chapter II, Pāda 2, Sūtra 17, Śaṅkarāchārya writes:

"As though the line[1] is one and the same, but being placed in different positions, it indicates the ideas denoted by the different words 'one,' 'ten,' 'hundred,' 'thousand,' etc., so · · · ".

The translation is literal, the text is simple and explicit on the subject of place value, and it cannot, considering the context, be interpreted otherwise.

The passage in the commentary on the Yoga-Sūtra is almost identical and has been translated by Professor Wood as follows[2]:

[1] See footnote 3.

[2] *Vide* the *Yoga-System of Patañjali,* translated by Professor J. H. Wood (*Harvard Oriental Series*) p. 216. In the footnote Professor Wood has noted the significance of this passage as indicative of the knowledge of place value amongst the ancient Hindus. *Vide* also Sarkar, *Hindu Achievement in Exact Science* (London).

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

The Mechanical Investigations of Leonardo Da Vinci. BY I. B. HART. Chicago, The Open Court Publishing Company, 1925. 240+vi pages, 4 plates, 136 figures in text. Price $4.00.

Preliminary to a detailed study of Leonardo's aeronautical investigations, which is the chief object of this volume, the writer devotes a chapter each to a rather minute account of the state of mechanical science during the first half of the fifteenth century, the contemporary scientific influences which bore upon Leonardo during his life time, and the known sources of his amazing knowledge. These are followed by two chapters on Leonardo's mechanics, the first dealing with his contributions to dynamics, the second with his contributions to statics. The substance of the last of these chapters has previously appeared in the *Open Court.*

These five chapters, together with the introductory chapter on the characteristic of Leonardo's manuscripts, cover the first 143 of the 240 pages of the book. This part of the work is admirably done and while it contains little that may not be found in the earlier writers, such as Duhem and Seailles in France and Werner, Grothe, and Feldhaus in Germany, it serves well the purpose of a brief history of the state of progress of the mechanical sciences during the fifteenth century. Some knowledge of the limitations of the investigators of his time is quite indispensable to a fair appraisal of Leonardo's achievements in the field of aeronautics.

The seventh chapter which deals with Leonardo as a pioneer of aviation appeared as a separate paper in the *Journal of the Royal Aeronautical Society.* Here the author brings together and views the passages from Leonardo's various note books which bear on the subject of artificial flight. The treatment cannot be called exhaustive for we notice the omission and lack of reference, among others, to such important utterances relating to flight as the following:[1]

"Dissect the bat, study it carefully, and on this model construct the machine (Ms. F 41 v.).

The bird in its flight without the help of the wind drops half the wing downwards, and thrusts the other half toward the tip backwards; and the part which is moved down prevents the descent of the bird, and that which goes backwards drives the bird forwards.

When the bird raises its wings it brings their extremities near together; and while lowering them it spreads them further apart during the first half of the movement, but after this middle stage, as they continue to descend it brings them together again. (Ms. K 12 v.).

When the bird lowers one of its wings necessity constrains it instantly to extend it, for if it did not do so it would turn right over. (Ms. K 4 v.).

There are many birds which move their wings as swiftly when they raise them as when they let them fall: such as magpies and birds like these.

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[1] The renderings into English are from McCurdy's *Leonardo da Vinci's Notebooks,* New York, 1923.

There are some birds which are in the habit of moving their wings more swiftly when they lower them than when they raise them, and this is seen to be the case with doves and such birds.

There are others which lower their wings more slowly than they raise them, and this is seen with rooks and other birds like these. (Ms. L 58 v., 59 v.)

The speed of birds is checked by the opening and spreading out of the tail. (Ms. L 59 v.)

The thrushes and other small birds are able to make headway against the course of the wind, because they fly in spurts; that is they take a long course below the wind, by dropping in a slanting direction towards the ground, with their wings half closed, and they then open the wings and catch the wind in them with their reverse movement, and so rise to a height; and then they drop again in the same way. (Ms. C.A. 313 r.b.)"

On the other hand there are some unnecessary repetitions as, for instance, the quotation consisting of 13 lines from Ms. K folio 10 v. which appears in full in two places, pp. 161–162 and pp. 178–179 and the shorter quotation from C.A. fol. 161 r.a. which is repeated on successive pages, p. 156, p. 157.

Notwithstanding these shortcomings this chapter offers the most complete treatment of its kind which has appeared up to the present time. Enough is said to show that Leonardo approached the problem of flight in a truly scientific spirit and that he anticipated, not always by very conclusive reasoning, to be sure, many of the accepted principles of aeronautical science. In this as in so many other fields, Leonardo penetrates below the surface of things and lays hold of essentials in a way which, as Hallam remarked long ago, strikes one with something like the "awe of preternatural knowledge."

The final chapter of the book consists of a translation of Leonardo's manuscript known as the *Codice sul Volo degli Uccelli e Varie Altre Materie* (Codex on the Flight of Birds and other Matters), together with a short history of the manuscript itself. This in the reviewer's opinion is perhaps the most valuable part of the book, since it is the first complete translation into English, though not the first publication, since like Chapter VII, this chapter has previously appeared as a separate paper in the *Journal of the Royal Aeronautical Society*. The term "complete" used in the preceding sentence needs, however, some modification. It is complete as to text but not as regards the illustrative figures of the original. Only 51 of the 110 figures of the original manuscript are reproduced and the artistic excellence of the originals has greatly suffered in the reproductions. Of the two facsimile pages, one (Plate vi) is rather imperfect as compared with the original,[1] while the distinctness of both has been impaired by reducing the size from 6″ by 8.5″ of the original to 3.5″ by 5.5″. A slight change in the form and size of the volume would have permitted a complete facsimile full size reproduction of the 26 pages of the manuscript, which would have greatly enhanced the value of the book. Notwithstanding, the volume

[1] The writer is indebted to his friend and colleague, Professor Louis Helmlinge of the Department of Romance Languages of this University, for the use of his facsimile reproduction of the original while writing this review.

will form an indispensable addition to the library of all who are interested in the history of the science of aviation.

R. E. MORITZ.

The Elements of Mechanics. By F. S. CAREY and J. PROUDMAN. London, Longmans, Green and Co., 1925. 314 pages. Price $3.00.

This book by a professor emeritus in the University of Liverpool and a young professor who has made notable advances in the study of the tides is a good example of what can be accomplished when the experience of a great teacher is combined with the brilliance of an ardent investigator. The subject is well presented, one good feature being the frequent use of vectors. Many illustrations are given to assist the reader and numerous examples are worked out at the end of the book to help along any student who is reading the subject for the first time.

Though the work is quite elementary, room is found for a useful chapter on frameworks and an elementary treatment of hydrostatics is included. We fully agree that instruction on these subjects should be given in a course on mechanics.

H. BATEMAN.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) title of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Isis, volume 7, no. 23, March, 1925: "Leibniz, the master builder of mathematical notations" by Florian Cajori, 412–429.

Mathematische Zeitschrift, volume 24, no. 3, December, 1925: "On the representation of functions by trigonometrical integrals" by N. Wiener, 575–616.

The Quarterly Journal of Pure and Applied Mathematics, volume 50, no. 3, October, 1925: "The general form of the suspension bridge catenary" by Ira Freeman, 269–271.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 2910 Harris Park Ave., Austin, Texas.

CLUB TOPICS

By B. H. BROWN, Dartmouth College

THEOREM OF BANG. ISOSCELES TETRAHEDRA

1. Let us state first two lemmas, the proofs of which are immediate.

Lemma 1. *If, between the 6 quantities a, b, c, a', b', c' there exist relations*

$$a+b+c=a+b'+c'=a'+b+c'=a'+b'+c\ ;$$

then

$$a=a'\ ; \qquad b=b'\ ; \qquad c=c'\ .$$

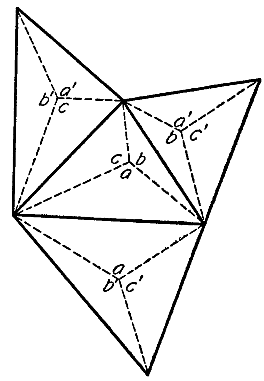
Lemma 2. *If, between the 4 positive quantities, A, B, C, D there exist relations*

$$AB=CD\ ; \qquad AC=BD\ ; \qquad AD=BC\ ;$$

then

$$A=B=C=D\ .$$

Theorem of Bang.[1] *Let a sphere be inscribed in any tetrahedron. Let lines be drawn in the faces connecting the vertices with the points of tangency. The connecting lines in a face form 3 angles which are the same in each face.*



The accompanying figure shows the lines drawn in each face, the tetrahedron being opened out and developed on the plane. It is obvious that the small triangles which share a tetrahedral edge are congruent and the angles of the theorem may then be marked as in the figure. The proof is now complete by application of Lemma 1.

2. *Isosceles Tetrahedra.*[2] The word *isosceles* is applied, with some slight incongruity, to a tetrahedron whose opposite edges are equal, and consequently all of whose faces are congruent triangles.

Theorem. If the faces of a tetrahedron are isoperimetric, the tetrahedron is isosceles. The proof is immediate if we denote pairs of opposite edges by a, a'; b, b'; c, c' and employ Lemma 1.

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[1] Proposed by Bang, *Tidskrift for Math.*, 1897, A, p. 48; proved by Gehrke, *ibid.*, p. 84. The extension of Bang's theorem to escribed spheres and to "attic-spheres" (Dachkugeln, sphères aux combles) has been made by Neuberg, *Jahresbericht*, 1907, p. 345. A dual theorem by Meyer, *Jahresbericht*, 1903, p. 137 may be stated as follows: "*If we circumscribe a sphere to a tetrahedron and at the vertices draw the tangent planes, and cut each tangent plane by the three tetrahedral planes meeting at that vertex, the traces of these last three (in the tangent plane) form three angles, which are the same at each vertex.*" Meyer's theorem is capable of an elegant elementary proof by stereographic projection and inversion. The theorems of Bang and of Meyer are metric specializations of a very general theorem on quadric surfaces proved by Meyer *l.c.*

A somewhat analogous theorem is due to White, *Bull. Am. Math. Soc.*, vol. 14, 1908, p. 220. Suppose a tetrahedron to be such that sums of opposite edges are equal, that is

$$a+a'=b+b'=c+c'.$$

It is then possible to find 4 numbers r_1, r_2, r_3, r_4 such that

$$\begin{aligned} r_1+r_2&=a, & r_3+r_4&=a',\\ r_2+r_3&=b, & r_4+r_1&=b',\\ r_3+r_1&=c, & r_2+r_4&=c'. \end{aligned}$$

Four spheres with radii r_1, r_2, r_3, r_4 and centers at the vertices of the tetrahedron will be mutually tangent. Four spheres have a common orthogonal sphere. This sphere will then be tangent to the *edges*

Theorem. If the faces of a tetrahedron are equiareal, the tetrahedron is isosceles. Proof: Let us represent the areas of the small triangles in the figure by the letters a, b, c; a', b', c'. Then, if the faces of the tetrahedron are equiareal $a=a'$, $b=b'$, $c=c'$ (Lemma 1). Let us denote the tangential distances from the vertices to the inscribed sphere by A, B, C, D. The equality of areas of small triangles gives 3 equations of which

$$\tfrac{1}{2}AB \sin b = \tfrac{1}{2}CD \sin b' ,$$

where b and b' now represent the Bang angles, is typical. Remembering that $b=b'$ etc. (Theorem of Bang), we have

$$AB=CD , \qquad AC=DB \qquad AD=BC ;$$

hence by Lemma 2, $A=B=C=D$, and the theorem follows readily. Incidentally it is now easy to establish the

Theorem. If the inscribed and circumscribed spheres of a tetrahedron are concentric, the tetrahedron is isosceles, and conversely.

We have now employed Lemma 1 in three ways, identifying the letters with (a) angles, (b) lengths of lines, (c) areas of triangles. Can this Lemma be employed in other connections? What further use can be made of Lemma 2? (Trigonometric functions of angles?) Are there analogous lemmas applicable to 4- or to n-dimensional varieties?

CLUB ACTIVITIES

THE JUNIOR MATHEMATICAL CLUB, University of Chicago, Chicago, Ill. [*1924*, 496]

The following papers were presented before the Junior Mathematical Club of the University of Chicago during the year 1924–1925:

October 22, 1924: "Constructions with one instrument only" by Miss Marion Stark.
November 5. "The mathematical library" by Dr. M. I. Logsdon.
November 19. "The postulational method in physics" by Mr. E. S. Akeley.
December 10. "Curve tracing by projective methods" by Mr. R. J. Garver.
January 22, 1925. "On satellites" by Professor K. Laves.
February 5. "Tchebycheff polynomials" by Dr. J. A. Shohat.
February 19. "Finite differences" by Mr. R. W. Barnard.
March 5. "Theories of infinity" by Mr. A. Blake.
April 22. "Semi-invariants of a ruled surface under affine transformations" by Mr. H. A. Simmons.
May 6. "Introduction to hyperbolic functions" by Mr. S. A. Rowland, Jr.

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of the tetrahedron. For any such tetrahedron, if through each point of tangency and through the opposite edge there be passed planes, White has shown that "*These planes form three pairs which intersect the sphere in orthogonal pairs of circles; and that the right angles between these circles at any contact are bisected by the traces of two other planes, each containing four points of contact.*"

[2] The literature in connection with isosceles tetrahedra is extensive. As a starting point we mention: Genty, *Nouvelles Annales*, 1878, 2nd ser., vol. 17, p. 223; Lemoine, *Nouvelles Annales*, 1880, 2nd ser., vol. 19, p. 133; Chefik-Bey, *Nouvelles Annales*, 1880, 2nd ser., vol. 19, p. 403.

May 20. "Development of analytic geometry by means of line coordinates" by Mr. F. S. Nowlan.
June 3. "Affine differential geometry of ruled surfaces" by Mr. H. A. Simmons.

(Report by Mr. Sidney A. Rowland, Jr.)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OREGON, Eugene, Oregon. [*1925*, 454]

During the year 1924–1925 the Mathematics Club of the University of Oregon was under the leadership of the following officers:

President........Marie Ridings
Vice-president....Sylvia Veatch
Secretary........Eula Benson
Treasurer........Roland Humphreys.

The following programs were given:
October 1924. "Magic squares" by Marie Ridings.
November. "History of π" by Sylvia Veatch.
December. "Trisection of an angle, squaring a circle, and doubling a cube" by Roland Humphreys.
January 1925. "Non-euclidean geometry" by Vladimir Rojonsky.
February. "Japanese mathematics" by Eula Benson. "Women in science" by Caroline Felton.
April. "The theory of the gyroscope" by Walter Brattain.
May. Annual picnic and initiation of new members.
June. "The quantum theory" by Leonard Neuman.
The officers elected to serve for the year 1925–1926 are as follows:

President........Eula Benson
Vice-president....Evan Lapham
Secretary........Helen White
Treasurer........Herbert Yearian.

(Report by Miss Eula Benson)

THE GRINNELL COLLEGE MATHEMATICS CLUB, Grinnell, Iowa [*1925*, 91]

The officers of the Grinnell College Mathematics Club for the year 1924–1925 were:

President............Margaret Field
Vice-president..........Ben Morgart
Secretary-treasurer......Ada Grosenbaugh.

The average attendance at the meetings was thirteen. The Club was particularly fortunate in having with them Professor George D. Birkhoff, Harvard Exchange Professor for 1924. The Club endeavored to study at each meeting, one of the constellations prominent at that time.

The following programs were presented:
"Life of Professor Birkhoff" by Dirk Heezon.
"The first works in mathematics in the New World" by Ruth Dougherty.
"The constellation Lyra" by Edith Wier.
"Mathematical recreations" by Mary Pilgrim.
"The constellation Pegasus" by Eugene Woodruff.
"Sexagesimal fractions" by Lela Kaisand.
"The four color problem" by Professor Birkhoff.
"The life of Leibniz" by Harriet Allen.
"The constellation Orion" by Alice Clifton.
"Mathematical fallacies" by Clarke Morris.
"The constellations Cygnus and Pleiades" by Ada Grosenbaugh.
"Rabbi Ben Ezra" by Harold Sweeney.
"Recreational mathematics" by Maude Martens.
"The theory of numbers" by Professor Rusk.

"The life of Archimedes" by Ben Morgart.
"The constellation Gemini" by Barbara Davidson.
"Relation of mathematics to another science, that of shape and form" by Dorothy Wilson.
"Report on the study of high school and college mathematics grades" by Howard Smith.
"Big numbers and little numbers" by Mary Wilson.
"Elementary applications of the theory of probabilities" by Professor McClenon.
"Curves of life" by Ida Iverson.
Summary of the year's work by Margaret Field.

(Report by Miss Ruth Dougherty)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3173. Proposed by A. A. Bennett, Lehigh University.

Let n black balls and n white balls be placed together in an urn. Let these balls be then withdrawn one at a time, none being replaced. Let $P(m)$ denote the probability of there occurring at least one sequence of m consecutive white balls during the complete process of withdrawing all the balls. Let m' denote the maximum value of m for which $P(m) \geq \frac{1}{2}$. Determine m' as a function of n with reference also to its asymptotic character. Generalize to the case of n balls each of more than two colors.

3174. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Construct a triangle given an angle, the sum of the including sides, and the distance of the vertex of the given angle from the orthocenter of the triangle.

NOTE BY THE EDITORS: Consider the case where a linear integral function of the including sides is given.

3175. Proposed by J. B. Reynolds, Lehigh University.

Prove that if the perpendiculars from each of three vertices of a tetrahedron upon the opposite faces meet in a point, the fourth will also pass through that point.

3176. Proposed by J. B. Reynolds, Lehigh University.

Through the vertex, O, of a tetrahedron, $OABC$, are passed three planes perpendicular respectively to OA, OB, and OC. Let BC (extended if necessary) cut the first plane in F, CA cut the second in G, and AB cut the third in H. Prove FGH a straight line.

3177. Proposed by J. L. Riley, Ouachita College, Arkadelphia, Ark.

An elliptic wall having a major axis $2a$ and a minor axis $2b$ stands in the middle of a large field. A horse is tethered to the outside of this wall at the extremity of the minor axis by a rope the length of which is equal to half the periphery of the wall. Find the area of the ground grazed over.

3178. Proposed by B. C. Wong, Berkeley, California.

On OP, the radius vector of any point on the cardioid $\rho=2a(1-\cos\theta)$, as diagonal a rectangle $OMPN$ is constructed with the sides OM and ON bisecting the angles between the radius vector and the axis of the cardioid. Find the loci of M and N and give their constructions without the use of the cardioid.

3179. Proposed by Einar Hille, Princeton University.

The differential equation $y'=y^{2/3}$ has two independent solutions which pass through the origin, namely $y\equiv 0$ and $y=\frac{1}{27}x^3$. Prove that these solutions can be obtained by the method of successive approximations using as first approximation in the former case $y_0=0$, in the latter case $y_0=x^{a_0}$ where a_0 is any positive constant.

3180. Proposed by J. Rosenbaum, Milford, Connecticut.

Given the angles of an inscribed polygon and the radius of circumscribed circle to construct the polygon.

SOLUTIONS

2803. [1920, 31]. Proposed by S. A. Corey, Des Moines, Iowa.

In the November, 1918, number of the *Proceedings of the Edinburgh Mathematical Society* (Vol. 36, part 2, page 103), Professor Whittaker gives the following formula for the solution of algebraic and transcendental equations:

The root of the equation

$$0=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots,$$

which is smallest in absolute value, is given by the series

$$-\frac{a_0}{a_1}-\frac{a_0^2a_2}{a_1\begin{vmatrix}a_1&a_2\\a_0&a_1\end{vmatrix}}-\frac{a_0^3\begin{vmatrix}a_2&a_3\\a_1&a_2\end{vmatrix}}{\begin{vmatrix}a_1&a_2\\a_0&a_1\end{vmatrix}\cdot\begin{vmatrix}a_1&a_2&a_3\\a_0&a_1&a_2\\0&a_0&a_1\end{vmatrix}}-\frac{a_0^4\begin{vmatrix}a_2&a_3&a_4\\a_1&a_2&a_3\\a_0&a_1&a_2\end{vmatrix}}{\begin{vmatrix}a_1&a_2&a_3\\a_0&a_1&a_2\\0&a_0&a_1\end{vmatrix}\begin{vmatrix}a_1&a_2&a_3&a_4\\a_0&a_1&a_2&a_3\\0&a_0&a_1&a_2\\0&0&a_0&a_1\end{vmatrix}}-\cdots.$$

In case of any algebraic equation with imaginary or complex roots the above formula clearly fails. State the conditions under which the formula may be relied upon to give correct results.

Solution and Discussion by Otto Dunkel, Washington University.

The statement of the theorem in this problem is incorrect and a revised statement will be given. The series, when written correctly, is merely a modification of a known process for approximating the roots of an algebraic equation (*1925*, 354–370) and it is not so convenient for computation as this process.

The equation

$$a_0+a_1x+a_2x^2+\cdots+a_rx^r=0,\qquad a_0a_r\neq 0. \tag{1}$$

will be considered where the coefficients a_i may be real or complex. In the denominators of terms of the series occur determinants of order n of the type

$$D(n)=\begin{vmatrix}a_1&a_2&a_3&\cdots&&&&\\a_0&a_1&a_2&\cdots&&&&\\0&a_0&a_1&\cdots&&&&\\0&0&a_0&\cdots&&&&\\ \cdot&\cdot&\cdot&&&&&\\ \cdot&\cdot&\cdot&&&&&\\ \cdot&\cdot&\cdot&&&&&\\0&0&0&\cdots&0&a_0&a_1&a_2\\0&0&0&\cdots&0&0&a_0&a_1\end{vmatrix}, \tag{2}$$

which when developed in terms of the elements of the first row give the recurrent relation

$$D(n)=a_1D(n-1)-a_2a_0D(n-2)+\cdots+(-1)^{r-1}a_ra_0^{r-1}D(n-r)\ , \tag{3}$$
$$D(n)=0\ , \qquad -r+1\leqq n\leqq -1\ , \qquad D(0)=1\ , \qquad D(1)=a_1\ .$$

If we set

$$P(n+r)=(-1)^n\frac{D(n)}{a_0^n}\ , \tag{4}$$

this may be written in the simpler form

$$a_0P(n+r)+a_1P(n+r-1)+a_2P(n+r-2)+\cdots+a_rP(n)=0\ . \tag{5}$$

Then

$$\operatorname*{Limit}_{n=\infty}\frac{P(n)}{P(n+1)}=\frac{1}{c_1}=d_1\ , \tag{6}$$

if the equation

$$a_0y^r+a_1y^{r-1}+a_2y^{r-2}+\cdots+a_r=0 \tag{1'}$$

has a root c_1 whose absolute value is greater than that of any other root, i.e. d_1 is the root of smallest absolute value of the equation (1).[1] The conditions for the existence of the limit (6) will be stated more precisely later. It will now be shown that the series of the problem, when it has a meaning, is merely a modification of (6).

The cofactors of $D(n)$ corresponding to the elements at the extremities of the diagonals are $D(n-1)$, $D(n-1)$, $(-1)^{n-1}a_0^{n-1}$ and $(-1)^{n-1}E(n-1)$, where

$$E(n)=\begin{vmatrix} a_2 & a_3 & \cdots & & & & \\ a_1 & a_2 & \cdots & & & & \\ a_0 & a_1 & \cdots & & & & \\ 0 & a_0 & \cdots & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & \\ 0 & 0 & \cdots & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & \cdots & 0 & a_0 & a_1 & a_2 \end{vmatrix} \tag{7}$$

is a determinant of order n. Hence[2]

$$D(n-2)D(n)=D^2(n-1)-a_0^{n-1}E(n-1)\ . \tag{8}$$

Suppose now that $D(p-1)=0$, $D(n)\neq 0$, $n\geqq p$, then

$$\frac{D(n-1)}{D(n)}-\frac{D(n-2)}{D(n-1)}=\frac{a_0^{n-1}E(n-1)}{D(n-1)D(n)}\ , \qquad n\geqq p+1\ .$$

Summing these relations and multiplying the result by $-a_0$ we have

$$\frac{P(n)}{P(n+1)}=-\frac{a_0D(n-r)}{D(n-r+1)}=-\sum_{m=p}^{m=n-r}\frac{a_0^{m+1}E(m)}{D(m)D(m+1)}. \tag{9}$$

This completes the proof. The series given in the problem can be used only if $p=0$ and the limit (6) exists. An example will be given later where the limit exists and for which p is not zero.

The vanishing of the determinants $D(n)$ indicates in general the existence of imaginary roots in the case where the coefficients a_i are all real. Suppose, for example, $D(p)\neq 0$, $D(p+q+1)\neq 0$, $D(n)=0$, $p+1\leqq n\leqq p+q$, $q<r$. We cannot have $q\geqq r$, for then by (3) all of the $D(n)$'s would be zero contrary to the fact that $D(0)=1$. We have then the following cases:

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[1] See the article cited above.

[2] See Bôcher's *Higher Algebra*, pp. 31-33.

Case I. p odd.

If q is odd there are at least $q+1$ imaginary roots.

If q is even there are at least q or $q+2$ imaginary roots according as $D(p+q+1)$ is greater or less than zero.

Case II. p even.

If q is even there are at least q or $q+2$ imaginary roots according as $D(p)$ is greater or less than zero.

If q is odd and $D(p)D(p+q+1)<0$ there are at least $q+1$ imaginary roots; if $D(p)>0$ and $D(p+q+1)>0$ there are at least $q-1$; if $D(p)<0$ and $D(p+q+1)<0$ there are at least $q+3$.

The proofs of these statements follow easily by methods similar to those used by the writer in the paper[1] "Necessary Conditions for the Reality of all the Roots of an Algebraic Equation."

It is obvious that series similar to (9) may be written for the root of largest absolute value. If the roots have unequal absolute values and $|c_1|>|c_2|>|c_3|>\cdots$, series of a more complicated form may also be written for the products c_1c_2, $c_1c_2c_3$, $c_1c_2c_3c_4$, $\cdots$. If two roots have equal absolute values and the same multiplicities, say $|c_1|>|c_2|>|c_3|=|c_4|>|c_5|>\cdots$, then series may be written for c_1, c_1c_2, $c_1c_2c_3c_4$. If, however, the multiplicity of c_3 is greater than that of c_4 then series may be written for $c_1c_2c_3$ and $c_1c_2c_3c_4$. These series are not as convenient for computation as their equivalents, that is the ratio $P(n+1)/P(n)$ for c_1 and the ratios of certain determinants of the $P(n)$'s in the other cases. The proofs of these statements, as well as the following, are rather long and will not be given here, since the writer expects to present them in a paper which will appear soon. We shall now state the conditions that the limit (6) shall exist or that the infinite series obtained from (9) shall converge.

If $c_1, c_2, c_3, \cdots, c_m$ are distinct roots of (1′) whose absolute values are equal and this absolute value is greater than that of any other remaining root, if any, and if further the multiplicity of c_1 exceeds that of c_2, of c_3, $\cdots$, of c_m, then

$$\operatorname*{Limit}_{n=\infty} \frac{P(n+1)}{P(n)}=c_1 \,.$$

In no other case does a limit exist. When these conditions for a limit are satisfied, and if $D(n)=0$ for some values of n, then it may be shown that for some value of n, say $n=p$, $D(p-1)=0$, $D(n)\neq 0$, $n\geqq p$.

If the coefficients a_i are all real then c_1 must be real in order to satisfy the above requirements for a limit. There can be only one other real root in the set of m, say c_2 where $c_2=-c_1$.

AN EXAMPLE. As an example for which some of the $D(n)$'s are zero we shall take the equation

$$1-x+x^2-x^3-x^4=0 \,, \tag{10}$$

which is a slight modification of a special case of the equations in Problem **3158**, (*1926*, 47). From the theorem of that problem it will be seen that the absolute values of the two imaginary roots lie between those of the two real roots. Either one of the two real roots may then be computed by the series, or, more simply, from the quotient of two successive values of $P(n)$. The difference equation in one case is

$$P(n+4)-P(n+3)+P(n+2)-P(n+1)-P(n)=0 \,, \tag{11}$$
$$P(4)=1 \,, \qquad P(n)=0 \,, \qquad 1\leqq n\leqq 3 \,.$$

Computing a few values we have the adjoined table

n	$P(n)$	n	$P(n)$
1	0	9	3
2	0	10	1
3	0	11	0
4	1	12	4
5	1	13	8
6	0	14	5
7	0	15	1
8	2	16	8

It follows from (4) that $D(2)=D(3)=D(7)=0$. We may use in place of (11) the equation

$$P(n+5)=2P(n+1)+P(n),$$

if we adjoin to the initial values of (11) $P(5)=1$. This simpler equation shows that $P(n)\neq 0$ for $n\geqq 12$ or $D(n)\neq 0$ for $n\geq 8$. Hence in the series (9) for this example $p=8$, and the series will surely converge to the real root of smaller absolute value, but as already stated it would be much simpler to continue this table. The convergence is very slow so that $P(200)/P(201)$ would give an approximation which is correct in only

[1] *Washington University Studies*, vol. 6. Scientific Series, no. 1, pp. 13-19, 1918.

about the first four decimals. To illustrate the computation of all the roots we shall consider the equation

$$x^4-x^3+x^2-x-1=0\,, \tag{10'}$$

whose roots are the reciprocals of the roots of (10). In order to improve the convergence in the case of the larger root we shall reduce the roots by 1.3 and then change their signs. This gives the equation

$$.0491P(n+4)-5.318P(n+3)+7.24P(n+2)-4.2P(n+1)+P(n)=0\,.$$

From this equation we find the successive values of $P(n)$; 0, 0, 0, 1, 108.31, 11583.60, 1238734.513, 132468000.871. Then one root is approximately

$$1.3-\frac{P(7)}{P(8)}=1.3-.00935119803=1.29064880197\,.$$

This is correct up to and including the 8th decimal. For the second real root, after changing its sign, we have the equation

$$P(n+4)=P(n+3)+P(n+2)+P(n+1)+P(n)\,.$$

Here the convergence is more rapid and the computation of the root is fairly easy.

n	$P(n)$	n	$P(n)$	n	$P(n)$	n	$P(n)$
1	0	10	29	19	10671	28	3919944
2	0	11	56	20	20569	29	7555935
3	0	12	108	21	39648	30	14564533
4	1	13	208	22	76424	31	28074040
5	1	14	401	23	147312	32	54114452
6	2	15	773	24	283953	33	104308960
7	4	16	1490	25	547337	34	201061985
8	8	17	2872	26	1055026		
9	15	18	5536	27	2033628.		

From the adjoined table we have as an approximation to this root

$$\frac{P(33)}{P(34)}=.51879006367116$$

in which the last three digits are incorrect. The negative root of (10′) is then $-.5187900637$.

It is also possible to approximate the imaginary roots in the same way. Reducing the roots of (10′) by i in order to make one of the imaginary have a smaller absolute value than that of any other root we obtain the equation

$$P(n+4)=(2-2i)P(n+3)-(5+3i)P(n+2)+(-1+4i)P(n+1)+P(n).$$

n	$P(n)$	n	$P(n)$
1	0	7	$-49-4i$
2	0	8	$-107+160i$
3	0	9	$390+690i$
4	1	10	$3235-82i$
5	$2-2i$	11	$5844-11846i$
6	$-5-11i$	12	$-31682-43645i$

This gives the adjoined table, and from it we have

$$\frac{P(11)}{P(12)}=.114098+.216722i\,.$$

Hence the imaginary roots are approximately $.114098\pm1.216722i$.

A calculation by a second method gave $.114071\pm1.216746i$. This last calculation is probably correct in the figures given. The absolute value of either imaginary root is 1.2220814, and, since this differs very little from the positive root, we see why the original of sequence values for $P(n)$ gave such a slow convergence.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

At Denison University, Professor F. B. WILEY has resumed his work as head of the department of mathematics, after a year's leave of absence spent as exchange professor at Robert College, Constantinople. Professor E. C. RUPP, who was head of the department of mathematics of Shurtleff College last year, has been appointed assistant professor. Mr. DONALD FITCH has been promoted from part-time instructor to full-time instructor.

The book entitled "Algebras and Their Arithmetics" by Professor L. E. DICKSON, which was published two years ago by the University of Chicago Press, is now being translated into German and will soon be published at Zurich, Switzerland.

Professor ARNOLD DRESDEN, University of Wisconsin, was the symposium speaker at the recent meetings of the American Mathematical Society held at the University of Chicago.

On February 1, 1926, an organization meeting was held of the Trustees of the National Research Endowment, recently established by the National Academy of Sciences; Secretary Herbert Hoover was chairman. On this occasion a declaration was made public reciting the purpose of this Endowment, which is to provide adequate funds for research in pure science in this country. Professor OSWALD VEBLEN is a member of the Board of Trustees.

The following appointments to Guggenheim Fellowships are announced: Professor E. P. LANE, of the University of Chicago, for a comparison of American and Italian methods in projective differential geometry; Professor E. B. STOUFFER, of the University of Kansas, for research in projective differential geometry; Professor NORBERT WIENER, of the Massachusetts Institute of Technology, for the study of certain topics forming extensions of the theories of Fourier series and Fourier integrals.

At Columbia University, Professor JULES DRACH, of the chair of analysis applied to geometry at the University of Paris, has been appointed visiting professor for the academic year 1926–27; Professor Drach will lecture in English.

At Stanford University, Professor R. L. GREEN has been made head of a new department of mathematics formed by the combination of the mathematics department and part of the department of applied mathematics; some of the members of the department of applied mathematics have been transferred to the new school of engineering.

The Nobel prize in Physics for 1924 has been awarded to Professor K. M. G. Siegbahn of the University of Upsala for his work in spectrum analysis.

The following additional reports of Summer Sessions to be held in 1926 have been received:

Stanford University. By Dr. KELLEY; Differential equations; By Dr. HAROLD HOTELLING; Theory of functions of a complex variable; By Professors H. F. BLICHFELDT and W. A. MANNING; Theory of groups, Calculus of variations.

Bucknell University, July 6–August 13. In addition to courses in College algebra and Calculus the following advanced courses are given: by Professor H. S. EVERETT: Teaching of junior and senior high school mathematics, Mathematical theory of interest and insurance (second course), Introduction to mathematical philosophy. By Assistant Professor GOLD; Synthetic projective geometry.

The following 27 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1925; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case, if available.

ETHEL L. ANDERTON, Yale, June, Bioche curve pairs.

CLIFFORD BELL, California, December, chemistry, The triangles in and circumscribed to the quartic curves of deficiency zero.

T. L. BENNETT, Illinois, July (date of examination), physics, Mapping by means of linear systems of curves invariant under Cremona involutions.

H. W. BRINKMANN, Harvard, February, Contributions to the theory of Riemann spaces.

P. A. CARIS, Pennsylvania, June, A solution of the quadratic congruence, Modulo p, $p=8n+1$, n odd.

M. G. CARMAN, Illinois, May, theoretical physics and theoretical and applied mechanics, Expansion problems in connection with homogeneous linear q-difference equations.

E. F. COX, Cornell, September, physics, The polynomial solutions of the difference equation $af(x+1)+bf(x)=\varphi(x)$.

H. M. GEHMAN, Pennsylvania, February, Concerning the subsets of plane continuous curves.

V. G. GROVE, Chicago, September, A theory of a general net on a surface.

R. W. Hartley, Pennsylvania, June, Determination of the ternary collineation groups whose coefficients lie in the $NGF(2^n)$.

D. L. Holl, Chicago, September, Viscous fluid motion in eccentric cylinders.

J. W. Hurst, Illinois, August, physics and astronomy, The classification of ternary algebras.

Samuel Jacobson, Yale, June, mathematical physics, Temperature radiation and thermal equilibrium.

Anna M. Lehr, Bryn Mawr, June, physics, The plane quintic with five cusps.

R. G. Lubben, Texas, June, philosophy, The double-elliptic case of the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry.

A. J. Maria, Rice Institute, June, Functions of plurisegments.

R. M. Mathews, Illinois, April, physics, Cubic curves and desmic surfaces.

L. T. Moore, Johns Hopkins, June, zoology, Determination of the type of a tricrunodal quartic from its invariants.

F. S. Nowlan, Chicago, September, Arithmetics of rational division algebras of order nine.

Echo D. Pepper, Chicago, September, Theory of algebras over a quasi-field.

P. G. Robinson, Chicago, June, Surfaces with constant absolute invariants.

Meyer Salkover, Yale, June, mathematical physics, Bateman's extended electrodynamics and the mass of an electron.

H. A. Simmons, Chicago, June, The first and second variations of a double integral for the case of variable limits.

Marcus Skarstedt, California, May, physics, The normal quartic curve of four-space.

V. A. Tan, Chicago, June, Projective properties of the Moutard quadric.

Takashi Terami, California, December, physics, The solution of the differential equation of a vibrating membrane by successive approximations.

L. E. Ward, Harvard, June, Third order boundary value problems and the allied expansions.

The following doctorate with education as major and mathematics as minor subject was conferred:

Lao G. Simons, Columbia, February, Introduction of algebra into American schools in the eighteenth century.

The Mathematical Association was represented by Professor Ethelwynn R. Beckwith at the diamond jubilee of Milwaukee-Downer College on June 12, by Professor E. C. Kiefer at the twenty-fifth anniversary of the founding of the James Millikin University, April 29 to May 1, and by Professor H. B.

Evans at the inauguration of Charles Ezra Beury as president of Temple University on May 7.

Professor H. W. Kuhn has been appointed to the chairmanship of the department of mathematics of Ohio State University, following the resignation of Professor R. D. Bohannan.

It is announced that Professors E. T. Bell, of the University of Washington, and Anna Pell-Wheeler, of Bryn Mawr College, have accepted the invitations of the Council of the American Mathematical Society to give Colloquia at the 1927 Summer Meeting in Madison, Wisconsin, their topics being respectively "Algebraic arithmetic" and "The theory of quadratic forms in infinitely many variables, and applications." The 1927 Summer Meeting of the Mathematical Association of America will be held at Madison in connection with that of the Society.

The sixth International Congress of Philosophy will meet September 13–17, 1926, at Harvard University. In each of four divisions, Metaphysics, Logic and Philosophy of Science, Theory of Values, and History of Philosophy, there will be one general session and four section meetings.

Professor M. I. Pupin, of Columbia University, has received the honorary degree of doctor of laws from the University of California.

Assistant Professor Nina M. Alderton, of Mills College, has been promoted to an associate professorship of mathematics.

Mr. W. C. Arnold, of DePauw University, has been promoted to an assistant professorship of mathematics.

Associate Professor W. H. Berry has been promoted to a full professorship of mathematics at the Brooklyn Polytechnic Institute.

Mr. R. M. Chase has been appointed professor of civil engineering and mathematics at the Alaska Agricultural College and School of Mines, Fairbanks.

Associate Professor T. W. Fravatt, of Pennsylvania State College, has been promoted to a full professorship of mathematics.

Assistant Professor V. G. Grove, of Michigan State College, has been promoted to an associate professorship of mathematics.

Dr. Goldie P. Horton has been promoted to an adjunct professorship of promoted to an associate professorship of mathematics.

Dr. Goldie P. Horton has been promoted to an adjunct professorship of mathematics at the University of Texas.

Associate Professor Emilie N. Martin, of Mount Holyoke College, has been promoted to a full professorship of mathematics.

Dr. R. M. MATHEWS, of the University of Illinois, has been appointed associate professor of mathematics at the University of West Virginia.

Assistant Professor T. A. PIERCE, of the University of Nebraska, has been promoted to an associate professorship of mathematics.

Dr. R. G. PUTNAM has been promoted to an assistant professorship of mathematics at New York University.

Associate Professor W. P. RUSSELL has been promoted to a full professorship of mathematics at Pomona College.

Assistant Professor J. H. TAYLOR, of Lehigh University, has been appointed assistant professor of mathematics at the University of Wisconsin.

Assistant Professor W. L. G. WILLIAMS, of McGill University, has been promoted to an associate professorship of mathematics.

Assistant Professor C. G. YEATON has been promoted to an associate professorship of mathematics at Oberlin College.

Assistant Professor W. C. GRAUSTEIN of Harvard University has been promoted to an associate professorship of mathematics.

Associate Professor A. F. CARPENTER of the University of Washington has been promoted to a full professorship of mathematics.

Associate Professor L. L. SMAIL, of the University of Texas, has been appointed to an associate professorship of mathematics at Lehigh University.

Professor R. A. JOHNSON of Hamlin University has been appointed assistant professor of mathematics at Hunter College.

The following appointsments to instructorships are announced:

Harvard University, Mr. H. B. CURRY, Mr. E. H. CULTER, Mr. H. H. HINRICHSEN, Mr. W. A. JENKINS, Mr. F. C. JONAH, Mr. MORRIS MARDEN, Mr. ROBIN ROBINSON, Mr. GEORGE SANTE, Mr. T. D. SMITH, and Mr. S. B. SOMMERVILLE.

Hunter College, Miss LAURA GUGGENBUHL, Miss MARGARET A. YOUNG, Miss LYLAH KRYDER, and Miss MARGARET LITTANER.

At Rutgers University an *Institute of Mathematics* has been organized as a part of the summer session to be held from June 28 to August 6, 1926. The immediate work of the institute will be in charge of Associate Professor F. D. MURNAGHAM of Johns Hopkins University. There will be associated with him as members of the staff and special lecturers the following: Professor L. P. EISENHART and Mr. ALONZO CHURCH of Princeton University, Professor E. R. HEDRICK of the University of California, Professor H. H. MITCHELL of the University of Pennsylvania, Professor RICHARD MORRIS of Rutgers University, and Professor L. D. HAERTTER of University High School, University of

Minnesota. The advanced courses to be given are as follows: By Professor Murnaghan: Functions of a Complex Variable, Modern Higher Algebra. By Professor Murnaghan and other members of the staff: Topics and Modern Theories in Mathematics, Seminar and Conference Period. The following courses, designed especially for secondary school teachers, are also offered in the Institute: By Mr. Church, Topics in Coordinate Geometry and Calculus; Modern Elementary Geometry. By Professor Haertter, The Teaching of Mathematics in the Junior High School; The Teaching of Mathematics in the Senior High School; The History of Mathematics. Besides the courses in the institute, the usual elementary undergraduate courses in geometry, college algebra, analytical geometry, and the calculus are offered in the regular summer session.

Assistant Professor WALTER DENSTON of Kenyon College was shot on April 8th while out alone shooting at targets. He was a graduate of Christ's College, Cambridge, and a teacher in the Russian Imperial Naval Academy at Kronstadt and in Ashbury College, Ottawa, Canada, before coming to the United States.

Professor F. N. COLE, of Columbia University, for twenty-five years secretary of the American Mathematical Society and editor of the *Bulletin*, died on May 26, 1926, at the age of sixty-four. Death was caused by heart failure brought on by an infected tooth. He was to have retired from teaching on September 20, his next birthday.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of..Dollars, to be known as the..Fund, and to be used

for[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness:.. Signature..

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.
BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.
BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.
Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.
The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.
INDIANA, Purdue University, May, 7-8.
IOWA, Cedar Rapids, April.
KANSAS, Merged in National Meeting.
KENTUCKY, Berea College, May 1.
LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.
MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Baltimore, Md., December 4.
MICHIGAN, Ann Arbor, Mich., April 1.
MINNESOTA, Northfield, Minn., May 22.
MISSOURI, Kansas City, Mo., November.
NEBRASKA, Bethany, Neb., May.
OHIO, Columbus, Ohio, April 2.
ROCKY MOUNTAIN, Colorado College, April, 1927.
SOUTHEASTERN, Atlanta, Ga., March 19-20.
SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.
TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

THE AMERICAN MATHEMATICAL MONTHLY

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THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The fourth annual meeting of the Southeastern Section of the Mathematical Association of America was held at the Philips High School, Birmingham, Alabama, March 20-21, 1925. Friday night a special dinner was given in honor of Professors Oswald Veblen and Archibald Henderson with Professor T. R. Eagles presiding as toastmaster. About forty were present for the dinner and the other sessions of the meeting including the following sixteen members of the Association: S. M. Barton,[1] J. B. Coleman, O. A. Culmer, T. R. Eagles, L. Farley, A. Henderson, C. D. Killebrew, Katherine Meckle, A. B. Morton, Frank Ordway, W. P. Ott, M. T. Peed,[2] W. W. Rankin Jr., D. Rumble, Oswald Veblen.

The following officers were elected for 1925–1926; Chairman, M. T. PEED, Emory University; Vice-chairman, W. P. OTT, University of Alabama; Secretary-treasurer, W. W. RANKIN, Jr., Agnes Scott. Professors A. B. Morton and T. R. Eagles were appointed to act with the secretary-treasurer as a Program Committee. A special committee was appointed to make a study of "Closer correlation of high school and college mathematics" W. W. Rankin, Jr., Chairman; T. R. Eagles, Frank Ordway, W. P. Ott, D. Rumble.

The following papers were presented: (1) "The universe; finite or infinite?" by Professor ARCHIBALD HENDERSON; (2) "Geometry as a branch of physics," by Professor OSWALD VEBLEN; (3) "Projective geometry in the college curriculum," by Professor M. T. PEED; (4) "The practical side of pure mathematics," by Professor S. M. BARTON; (5) "Notations for determinants," by Professor OSWALD VEBLEN; (6) "Honor courses at Cambridge university," (by title) by Professor R. P. STEPHENS; and (7) "A new mode of approach to relativity," by Professor ARCHIBALD HENDERSON. Discussion; "Connecting high school and college mathematics" led by Professor T. R. EAGLES.

Abstracts of the papers are given below, the numbers corresponding to the numbers in the list of titles as above.

1. In this paper were presented the results of some studies, ranging over a period of years, in this fascinating subject. Dr. Henderson recently had the advantage of discussing the subject with Professor Albert Einstein at the University of Berlin. To Professor Einstein is due the notion of the finite, but unbounded universe, which is a logical consequence of his general relativity theory. This paper, which had recently been given by radio under the auspices of the

[1] Professor S. M. Barton died January 5, 1926.

[2] Professor M. T. Peed died August 28, 1925.

National Research Council, traversed the results of investigation up to the present time.

2. (*Condensed quotation*) "Geometry as a branch of mathematics is an abstract science based on unproved propositions whose truth or falsity are indifferent and on undefined terms whose meanings are irrelevant. As a branch of physics, it deals with things which have meaning in terms of human operations and with propositions whose validity within a margin of error is determined by experiment and observation. The abstract treatment of geometry has been found unsuitable for elementary instruction. Therefore we ought frankly to adopt the experimental one. In doing so we should use methods which are as nearly as possible in harmony with those of physical science in general. This means in practice that analytic geometry should be used from the start. The first two axioms of plane geometry would thus be: (1) that coördinates are possible and (2) that if the coördinates are properly chosen, straight lines have first degree equations. These axioms are obtained by generalizing from simple experiments and are justified by the fact that theorems deduced from them by combined analytic and synthetic methods are also verified experimentally. Abstract geometry is a proper subject for a junior or senior course in college."

3. In this paper, attention was called to the changes during the last twenty five years in the college curriculum, in which mathematics has had its share. The development of projective geometry during the last century was remarked. The need of improvement in the preparation of those who are to teach high school mathematics was urged, and projective geometry was indicated as an excellent aid to such improvement. The leading features of the subject were then considered. The principle of duality was explained and its value set forth. It was pointed out that the relation of pole and polar, which, in ordinary geometry, makes little appeal to the average student, is here found to possess an interest quite equal to its importance. The advantage of the modern treatment of parallels was considered, and its merit was made clear. The helpfulness of projective geometry in the study of other branches of mathematics was illustrated and practical applications were mentioned. Two college courses were proposed, using the synthetic and analytic methods, with the former preceding the latter. Finally, the descriptive nature of the subject was shown to be among its most valuable properties.

4. Professor Barton emphasized the fact that his topic was not applied mathematics, but his main point was to show that pure, abstract mathematics was in itself eminently practical. Much practical good grew out of the futile attempts of the ancients to solve the three celebrated problems of antiquity,—apart from other considerations the development of mental power equips man for solving the intricate problems of life. The keen-witted Greeks realized the im-

portance of their discovery of the notion of irrationality. They did not ignore it because it belonged to the thought-world rather than to the material world. The author stressed the absurdity of neglecting the study of pure, abstract principles, for we know not when they may lead to practical results. He cited many instances of practical truths being first made known by pure mathematics and later verified by experiment. In conclusion he expressed the hope that he had shown, in some measure, that pure mathematics is not simply, or mainly, a most useful tool for the astronomer, the engineer, the physicist, the mechanic, and others, but it has in itself a real *practical* side.

5. In his second paper Professor Veblen discussed a notation for odd and even permutations which is suitable for the theory of determinants and for various determinant-like expressions which occur in differential invariants. Notations for a quantity which is ± 1, according to the parity of a permutation, have been in use for a long time in books on algebra. An important advance was made by Ricci and Levi-Civita who, in their absolute differential calculus, introduced quantities $\epsilon_{ij\cdots k}$ which are 0 if there are repetitions in the subscripts and otherwise positive or negative according to parity of permutation. The importance of this was appreciated by certain writers on differential invariants and relativity as, for example, Lipka and Eddington. Another step was made simultaneously by Alexander and Murnaghan who found a set of quantities $\delta^{ij\cdots k}_{\alpha\beta\cdots\gamma}$ which generalize both the ϵ and the Kronecker δ and have decided advantages in flexibility over the previous notations. With the aid of these notations it is possible to make the proofs of theorems on determinants very short and luminous, and also to write a great many formulas explicitly in such a way as to make it easy to see their significance.

7. The customary method, according to Minkowski, of dealing with special relativity, involves the use of the rectangular hyperbola. A more natural mode of approach would be through that of the railroad time-table, and distance-time diagram. Using the result of the Michelson-Morley experiment to secure a new space unit and a new time unit for a set of new axes, no longer at right angles like the original axes, it is possible to set up the Lorentz transformation in an elegant manner, with simple geometrical construction. The method worked out in some detail by Schlesinger of Giessen, has not yet found its way into English and American texts.

W. W. RANKIN, Jr., *Secretary-Treasurer*

THE SECOND ANNUAL MEETING OF LOUISIANA–MISSISSIPPI SECTION

The second annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Hotel Edwards, Jackson, Mississippi, March 20-21, 1925.

All sessions of the Association were well attended, particularly the Friday evening session at which about two hundred were present. Among those attending were the following members of the Association:

A. E. Babbitt, H. E. Buchanan, H. Fox, J. A. Hardin, G. L. Harrell, J. R. Hitt, Anna M. Howe, C. G. Latimer, A. C. Maddox, W. M. Maiden, B. E. Mitchell, I. C. Nichols, R. L. O'Quinn, S. T. Sanders, J. M. Sharp, Nathaniel Smylie, Mary C. Spencer, W. B. Stokes, Robert Torrey, Norma Touchstone, B. M. Walker, S. P. Walker.

Both the Friday evening and the Saturday morning meetings were presided over by the regular chairman of the Section, Dr. B. M. Walker, then professor of mathematics at Mississippi A. &. M College, now president of the institution. Dinner was enjoyed together Friday evening. At the business session Saturday morning, the following officers were elected for 1925–1926:

Chairman, H. E. Buchanan, Tulane University,

Vice-Chairman, H. Fox, A. & M. College of Mississippi,

Secretary-Treasurer, I. C. Nichols, Louisiana State University. The next meeting was set for March, 1926, in New Orleans with Tulane University and Newcomb College.

Before taking up the regular program, Dr. Walker gave a brief discussion of a certain triple infinity of points. The following papers were then presented:

1. "Other worlds than ours," Professor F. R. Moulton of the University of Chicago. An illustrated lecture.

2. "A comparative study of Euclidean and non-Euclidean hypotheses with emphasis on the logical aspects," Professor S. T. Sanders, L. S. U.

3. "Some problems in projective geometry," Dr. Anna M. Howe, Newcomb College of Tulane University.

4. "On a certain binary quadratic congruence," Dr. C. G. Latimer, Tulane University.

5. "Ballistics," Professor Moulton, University of Chicago.

6. "The orthocentric quadrilateral," Professor R. L. O'Quinn, L. S. U.

7. "The mathematical status of the present day freshman," a discussion led by Professor Robert Torrey, University of Mississippi.

8. "Popular fallacies in actuarial science," Mr. A. E. Babbitt, Actuary, Lamar Life Insurance Company.

All papers were well presented and closely followed by those present. Special mention should be made of Dr. Moulton's two lectures, both of which were particularly attractive and profitable.

I. C. NICHOLS, *Secretary-Treasurer.*

EUCLIDEAN INVARIANTS OF SECOND DEGREE CURVES[1]

By C. C. MACDUFFEE, Ohio State University

1. Introduction. Although most works on coördinate geometry attempt to classify second degree curves into types by means of their invariants, the writer knows of no classification which is complete. A complete classification must not only separate the curves into ellipses, parabolas, hyperbolas, parallel lines and intersecting lines, but it must also pick out the imaginary loci from the real, and it must characterize by invariants the parameters in the canonical forms. The semi-axes of the ellipse and hyperbola, the latus rectum of the parabola, the angle between the intersecting lines and the distance between the parallel lines are invariants, and a complete classification of second degree curves must express these invariants in terms of a fundamental system.

The text by Bôcher[2] gives a treatment which is as good as any which the writer has found. Bôcher uses the three well-known invariants and by means of them classifies the second degree curves into types and separates the real curves from the imaginary in every case but one, namely the parallel line case. He also characterizes some of the parameters. He recognizes the fact that the three invariants at hand are insufficient to separate the case of real parallel lines from the case of imaginary parallel lines, and makes no further attempt to effect this separation on the basis of invariants.

Since the three invariants ordinarily used are insufficient completely to classify second degree curves, the idea suggests itself that perhaps there are other invariants or covariants of the second degree polynomial under euclidean transformations than those heretofore recognized. In the present article the Lie theory of continuous groups is brought into play to determine the exact number of functionally independent invariants and covariants. It is found that there are three invariants and two covariants, one other in addition to the given polynomial—a fact which seems to have escaped attention. A certain second degree polynomial Θ is found to serve as the missing covariant, and a complete system of invariants and covariants is exhibited.

This new covariant Θ is found to fill in an interesting manner the gap noticed by Bôcher. Its degree is invariant, and $\Theta = \textit{constant}$ is a necessary and sufficient

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[1] Read before the Ohio Section of the Association April 2, 1926.

[2] M. Bôcher, *Plane Analytic Geometry*, Henry Holt and Co., 1915, pp. 176–188.

condition that we have the parallel line case. This constant is of course an invariant for this case. The lines are real and distinct, real and coincident, or imaginary according as this constant value of Θ is negative, zero, or positive, and the distance between the lines is determined by Θ together with another invariant.

In §7 the general second degree equation is reduced to canonical forms. By using the methods of invariant theory, this reduction is made with a minimum of calculation. The parameters are characterized in terms of invariants.

Since our three invariants and two covariants form a Lie complete system, every invariant magnitude connected with a conic is a function of them, and so is the equation of every covariant curve. In particular, the equation of the director circle is $\Theta=0$. The eccentricity is an irrational invariant, a root of a fourth degree equation whose coefficients are invariants. Simple rational covariants of the hyperbola give the asymptotes and the conjugate hyperbola, while the axes of a central conic are given by a more complicated covariant. These examples serve to show the significance of a complete system of covariants.

The application of these methods to plane cubics and systems of conics, as well as to surfaces in space, should prove interesting and not too difficult.

2. The Lie theory of euclidean invariants.[1] We shall take as the group of euclidean transformations

$$x=x' \cos \omega - y' \sin \omega - \alpha \ ,$$
$$y=x' \sin \omega + y' \cos \omega - \beta \ ,$$

where ω, α, β are real parameters. From these we have

$$\begin{aligned} x' &= x \cos \omega + y \sin \omega + (\alpha \cos \omega + \beta \sin \omega) \ , \\ y' &= -x \sin \omega + y \cos \omega - (\alpha \sin \omega - \beta \cos \omega) \ . \end{aligned} \tag{1}$$

The values $\omega_0=\alpha_0=\beta_0=0$ determine the identical transformation.

Let $f(x, y)$ be a real analytic function. We have $f(x', y')=F(\omega, \alpha, \beta)$, $f(x, y)=F(\omega_0, \alpha_0, \beta_0)$. For sufficiently small values of ω, α, β we have by Taylor's formula

$$F(\omega,\alpha,\beta)=F(\omega_0,\alpha_0,\beta_0)+\sum_{i=1}^{\infty}\frac{1}{i!}\left(\omega\frac{\partial}{\partial\omega_0}+\alpha\frac{\partial}{\partial\alpha_0}+\beta\frac{\partial}{\partial\beta_0}\right)^i F(\omega_0,\alpha_0,\beta_0) \tag{2}$$

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[1] Lie-Scheffers, *Vorlesungen über kontinuierliche Gruppen*, Teubner, 1893, p. 100 and p. 716.

A. Cohen, *An Introduction to the Lie Theory of One-parameter Groups*, Heath, 1911, p. 211 *et seq.*

L. E. Dickson, "Differential equations from the group standpoint," *Annals of Mathematics*, vol. 25, no. 4, June, 1924.

where, as usual, we understand that

$$\left(\omega\frac{\partial}{\partial\omega_0}+\alpha\frac{\partial}{\partial\alpha_0}+\beta\frac{\partial}{\partial\beta_0}\right)F(\omega_0,\alpha_0,\beta_0)=\frac{\partial F(\omega,\alpha,\beta)}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0,}\omega$$
$$+\frac{\partial F(\omega,\alpha,\beta)}{\partial\alpha}\Big]_{\omega_0,\alpha_0,\beta_0}\alpha+\frac{\partial F(\omega,\alpha,\beta)}{\partial\beta}\Big]_{\omega_0,\alpha_0,\beta_0,}\beta\,.$$

Now

$$\frac{\partial F(\omega,\alpha,\beta)}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}=\frac{\partial f(x',y')}{\partial x'}\frac{\partial x'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}+\frac{\partial f(x',y')}{\partial y'}\frac{\partial y'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}$$
$$=\frac{\partial x'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial f(x,y)}{\partial x}+\frac{\partial y'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial f(x,y)}{\partial y}\,.$$

Following Lie, we define the operator

$$U_\omega\equiv\frac{\partial x'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial x}+\frac{\partial y'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial y}=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\,.$$

Similarly we have

$$U_\alpha\equiv\frac{\partial x'}{\partial\alpha}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial x}+\frac{\partial y'}{\partial\alpha}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial y}=\frac{\partial}{\partial x}\,,$$

$$U_\beta\equiv\frac{\partial x'}{\partial\beta}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial x}+\frac{\partial y'}{\partial\beta}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial y}=\frac{\partial}{\partial y}\,.$$

We define the operator

$$U\equiv\omega U_\omega+\alpha U_\alpha+\beta U_\beta,$$

which is completely defined by transformation (1). Evidently we have

$$\left(\omega\frac{\partial}{\partial\omega_0}+\alpha\frac{\partial}{\partial\alpha_0}+\beta\frac{\partial}{\partial\beta_0}\right)F(\omega_0,\alpha_0,\beta_0)=Uf(x,y)\,.$$

In a similar manner we can demonstrate that

$$\left(\omega\frac{\partial}{\partial\omega_0}+\alpha\frac{\partial}{\partial\alpha_0}+\beta\frac{\partial}{\partial\beta_0}\right)^i F(\omega_0,\alpha_0,\beta_0)=U^if(x,y)$$

so that (2) can be written

$$f(x',y')=f(x,y)+Uf(x,y)+\frac{1}{2!}U^2f(x,y)+\frac{1}{3!}U^3f(x,y)+\cdots \tag{3}$$

In particular when we take x' and y' successively for $f(x', y')$ we have group (1) in the form

$$x'=x+Ux+\frac{1}{2!}U^2x+\frac{1}{3!}U^3x+\cdots\,,$$

$$y'=y+Uy+\frac{1}{2!}U^2y+\frac{1}{3!}U^3y+\cdots\,.$$

Thus the operator U completely determines the group (1) for sufficiently small values of ω, α, β. Lie calls U the *symbol* of transformation (1), and U_ω, U_α, U_β the *generators* of (1).

Let $f(x, y)$ be an invariant under transformations (1), *i. e.*, a function such that $f(x', y') \equiv f(x, y)$ identically in ω, α, β. From (3) it is evident that we must have

$$Uf = \omega U_\omega f + \alpha U_\alpha f + \beta U_\beta f \equiv 0 \ ,$$

that is,

$$U_\omega f = 0 \ , \qquad U_\alpha f = 0 \ , \qquad U_\beta f = 0 \ .$$

Since $Uf \equiv 0$ implies $U^i f \equiv 0$, this condition is also sufficient for small values of the parameters.

3. The Lie group for a second degree polynomial. The transformation (1) induces upon the coefficients of the real polynomial

$$\Phi \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c \tag{4}$$

the transformation

$$\begin{aligned}
a' &= c \cos^2\omega + 2h \sin\omega \cos\omega + b \sin^2\omega \ , \\
h' &= -a \sin\omega \cos\omega + h(\cos^2\omega - \sin^2\omega) + b \sin\omega \cos\omega \ , \\
b' &= a \sin^2\omega - 2h \sin\omega \cos\omega + b \cos^2\omega \ , \\
g' &= -a\alpha \cos\omega - h\beta \cos\omega - h\alpha \sin\omega - b\beta \sin\omega + g \cos\omega + f \sin\omega \ , \\
f' &= a\alpha \sin\omega + h\beta \sin\omega - h\alpha \cos\omega - b\beta \cos\omega - g \sin\omega + f \cos\omega \ , \\
c' &= a\alpha^2 + 2h\alpha\beta + b\beta^2 - 2g\alpha - 2f\beta + c \ .
\end{aligned} \tag{5}$$

These transformations together with (1) form a group whose generators are found to be

$$\begin{aligned}
U_\omega &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + 2h\frac{\partial}{\partial a} + (b-a)\frac{\partial}{\partial h} - 2h\frac{\partial}{\partial b} + f\frac{\partial}{\partial g} - g\frac{\partial}{\partial f} \ , \\
U_\alpha &= \frac{\partial}{\partial x} - a\frac{\partial}{\partial g} - h\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial c} \ , \\
U_\beta &= \frac{\partial}{\partial y} - h\frac{\partial}{\partial g} - b\frac{\partial}{\partial f} - 2f\frac{\partial}{\partial c} \ .
\end{aligned} \tag{6}$$

From the Lie theory we know that the differential equations $U_\omega = 0$, $U_\alpha = 0$, $U_\beta = 0$ form a complete system,[1] since they are the generators of a group. The three equations in eight variables have $8-3=5$ functionally independent solutions, which are the 5 invariants and covariants of Φ, and we know that all other invariants and covariants are functions of these.

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[1] Goursat-Hedrick, *Mathematical Analysis*, vol. 2, part 2, Ginn, 1917, p. 267.

If in (6) we delete the terms in x and y, we obtain the generators of the group (5) alone. The corresponding differential equations form a complete system with $6-3=3$ functionally independent solutions, namely the 3 invariants of Φ. Thus a complete system of invariants and covariants of Φ consists of 3 invariants and 2 covariants.

4. A complete system of invariants and covariants. While it is possible to solve the differential equations: $U_\omega=0$, $U_\alpha=0$, $U_\beta=0$ and thus obtain all our invariants and covariants, it is not necessary in this case to do so. The following functions are annihilated by each of the operators U_ω, U_α, U_β and hence are invariants and covariants:

Φ, $\Lambda=a+b$, $\Sigma=ab-h^2$, $\Delta=abc-af^2-bg^2-ch^2+2fgh$, $\Theta=(ab-h^2)(x^2+y^2)+2(bg-hf)x+2(af-hg)y+ac+bc-g^2-f^2$.

If we can now show that these 5 functions are functionally independent, we shall have shown that they form a Lie complete system of invariants and covariants. In particular they are independent when $b=g=c=0$, for in this case the Jacobian

$$\frac{\partial(\Lambda,\Sigma,\Delta,\Phi,\Theta)}{\partial(x,y,a,f,h)}$$

does not vanish identically. We will spare the reader the calculation.

5. Invariance of the degree of a covariant. The covariants Φ and Θ may be of degree 2, degree 1, degree 0, or vanish identically. The degree of each covariant is an arithmetic invariant. It is obvious, since transformations (1) are linear, that the degree of a covariant cannot be raised. It follows then that it cannot be lowered, for this would imply that it could be raised by the inverse transformation which again is linear.

6. Geometric invariants. If we multiply (4) through by a real number $k\neq 0$, we do not alter the locus of $\Phi=0$. The invariants and covariants are changed, however, becoming

$$k\Lambda,\ k^2\Sigma,\ k^3\Delta,\ k\Phi,\ k^2\Theta\ . \tag{7}$$

The invariants cannot, then, measure geometric magnitudes, for they do not always have the same values for congruent curves. We shall call them *algebraic* invariants and covariants of *weights* 1, 2, 3, 1, 2, respectively.

It is easy to form invariants, *e. g.* $\frac{\Sigma}{\Lambda^2}$, $\frac{\Delta}{\Lambda^3}$, $\frac{\Delta^2}{\Sigma^3}$, which actually maintain a fixed value for all congruent curves. These we shall call *geometric* invariants. We see that there are just two independent geometric invariants.

The invariants (7) may be used in classifying curves, however, for since $k\neq 0$ their vanishing or non-vanishing is an invariant property of the curve. Moreover, the signs of invariants of even weight, as Σ and $\Lambda\Delta$, are unalterable and

may be used in separating types of curves. It is impossible to classify curves according to the sign of Δ.[1]

7. Reduction of the second degree equation. We have seen that the degree of Φ is invariant. If Φ is of degree 1, the locus of $\Phi=0$ is a straight line. If Φ is of degree 0, there is no locus. If Φ has no degree (vanishes identically), the locus is the entire plane. We shall henceforth assume that Φ is of degree 2, *i. e.*, that $(a, h, b) \neq (0, 0, 0)$. This is true when and only when the invariant $\Lambda^2-2\Sigma=a^2+b^2+2h^2$ is positive.

From (5) we see that

$$h'=2h \cos 2\omega-(a-b) \sin 2\omega \ .$$

It is evidently possible under all circumstances to choose ω so that $h'=0$. If $a-b\neq 0$, ω is any value satisfying

$$\tan 2\omega=\frac{2h}{a-b} \ .$$

If $a-b=0$, $\omega=\pi/4$ will make $h'=0$ irrespective of whether also $h=0$ or not. Thus every equation $\Phi=0$ can be reduced (dropping primes) to the form

$$ax^2+by^2+2gx+2fy+c=0 \ , \qquad (a,b)\neq(0, 0) \ . \tag{8}$$

We now divide our analysis into cases.

Case I. $\Sigma\neq 0$. Since $h=0$, we now have $\Sigma=ab\neq 0$. If we apply to (8) the transformation

$$x=x'-\frac{g}{a} \ , \qquad y=y'-\frac{f}{b} \ ,$$

we obtain

$$ax^2+by^2+c=0 \ , \qquad ab\neq 0 \ . \tag{9}$$

If we set $h=g=f=0$ in Δ, we find that $\Delta=abc$, so that $\Delta=0$ when and only when $c=0$.

Case I A. $\Sigma\neq 0$, $\Delta\neq 0$. We have

$$ax^2+by^2+c=0 \ , \qquad abc\neq 0 \ ,$$

which is readily reducible to the generic canonical form

$$\frac{x^2}{\xi}+\frac{y^2}{\eta}=1 \ , \qquad \xi\eta\neq 0 \ . \tag{10}$$

[1] A certain well-known American text proposes to do exactly this.

The parameters ξ and η are irrational geometric invariants. We have $a=\eta$, $b=\xi$, $c=-\xi\eta$, $h=f=g=0$, and hence

$$\Lambda=\xi+\eta\ , \qquad \Sigma=\xi\eta\ , \qquad \Delta=-\xi^2\eta^2\ . \tag{11}$$

The quotient $-\Sigma^2/\Delta=1$ is an algebraic invariant of weight 1, so we may characterize ξ and η by geometric invariants as follows:

$$\xi+\eta=\frac{-\Lambda\Delta}{\Sigma^2}\ , \qquad \xi\eta=\frac{\Delta^2}{\Sigma^3}\ .$$

Hence ξ and η are the roots of

$$\Sigma^3u^2+\Lambda\Sigma\Delta u+\Delta^2=0 \tag{12}$$

Each coefficient is of weight 6, so ξ and η are geometric invariants. The values are both real, for the discriminant

$$\Delta^2(\Lambda^2-4\Sigma)=\Delta^2[(a-b)^2+4h^2]$$

can never be negative.

This is as far as we need to go, but it is customary to subdivide (10) into three cases according as ξ and η are both positive, both negative, or opposite in sign. Since $\xi\eta$ has the sign of Σ, they are opposite in sign when and only when $\Sigma<0$, and we have the hyperbola. When $\Sigma>0$ and $\Lambda\Delta<0$, both roots of (12) are positive and we have the ellipse. When $\Sigma>0$ and $\Lambda\Delta>0$, there is no locus. We cannot have $\Lambda\Delta=0$ when $\Sigma>0$.

Case I B. $\Sigma\neq 0, \Delta=0$. Now (9) assumes the form

$$ax^2+by^2=0\ , \qquad ab\neq 0\ . \tag{13}$$

The coefficients a and b are algebraic invariants of weight 1, the roots of the quadratic equation

$$u^2-\Lambda u+\Sigma=0 \tag{14}$$

Since only the ratio $a:b$ is of geometrical significance, we shall consider the symmetrical equation (13) as our canonical form rather than divide through by a or b.

The discriminant $\Lambda^2-4\Sigma=(a-b)^2$ of (14) is never negative, so the roots are real. When $\Sigma<0$ the roots are opposite in sign and (13) consists of two intersecting lines. When $\Sigma>0$ the roots are like in sign and the graph consists of a single point.

If we denote by θ one of the angles between the lines (13) when $\Sigma<0$, we find that $\tan\theta$ is a geometric invariant defined by the equation

$$\Lambda^2\tan^2\theta+4\Sigma=0\ .$$

Case II. $\Sigma=0$. Since $h=0$, we now have $\Sigma=ab=0$ and either $a\neq 0$, $b=0$ or $a=0$, $b\neq 0$. The second alternative can be reduced to the first by the transformation $x=y'$, $y=-x'$. Hence (8) can be written

$$x^2+2gx+2fy+c=0 .$$

The transformation $x=x'-g$, $y=y'$ gives

$$x^2+2fy+c=0 . \tag{15}$$

Let us set $b=h=g=0$, $a=1$ in Δ. We have $\Delta=-f^2$, so that $\Delta=0$ when and only when $f=0$.

Case II A. $\Sigma=0$, $\Delta\neq 0$. We now have $f\neq 0$ so that the transformation

$$x=x' , \qquad y=y'-\frac{c}{2f} ,$$

is valid, reducing (15) to the canonical form

$$x^2=4py , \qquad p\neq 0 . \tag{16}$$

The parameter p in (16) is an irrational invariant. We have for (16) $a=1$, $f=-2p$ and all other coefficients 0, so that $\Lambda=1$, $\Delta=-4p^2$. Since $\Lambda\neq 0$ for the canonical form, we know that it is different from zero for the entire case. Similarly we know that $\Delta/\Lambda^3<0$ for the entire case. Thus we may take as p either one of the real non-zero geometric invariants defined by the equation

$$4\Lambda^3p^2+\Delta=0 .$$

Case II B. $\Sigma=\Delta=0$. We have $f=0$ so that (15) assumes the canonical form

$$x^2+c=0 . \tag{17}$$

For (17) we have $\Lambda=1$, $\Sigma=\Delta=0$. The invariants are therefore insufficient to characterize this case. But the covariant Θ meets our needs in a peculiar and interesting manner. We have for (17) $\Theta=c$. Since the degree of Θ is invariant we see that Θ reduces to a constant whenever $\Sigma=\Delta=0$. Conversely, since

$$\Delta=c(ab-h^2)-f(af-hg)-g(bg-hf) ,$$

it follows that when Θ reduces to a constant, $\Sigma=\Delta=0$. Thus the condition that Θ reduce to a constant characterizes this case.[1]

Evidently (17) represents coincident lines when and only when $\Theta\equiv 0$, parallel lines when and only when $\Theta=$ *constant* <0, and no locus when and only when $\Theta=$ *constant* >0.

The distance d between the lines (17) is given by $d^2=-4\Theta/\Lambda^2$.

[1] *Cf.* B. Niewenglowski, *Cours de géométrie analytique*, Gauthier-Villars, 1894, vol. 1, p. 282. Niewenglowski obtains the constant term of Θ and recognizes that it is invariant when $\Sigma=\Delta=0$, and uses it to obtain the distance between the parallel lines.

8. Further invariants. Every geometric magnitude which is intrinsic to a curve—*i.e.*, is not dependent upon the particular coördinates used—is obviously an invariant under transformation of coördinates. A magnitude which is defined for all curves of a canonical set of the family $\Phi=0$ is given by a formula which is an invariant function of the coefficients of Φ.

We shall give two illustrations.

The area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is πab. We have for this canonical form $\Sigma=a^2b^2$, $\Delta=-a^4b^4$. Hence $\Delta^2/\Sigma^3=a^2b^2$ is a geometric invariant, and so for every ellipse the area is $\pi\Delta\Sigma^{-3/2}$. The formula is based upon the above canonical form and has no interpretation for curves not reducible to this form.

The eccentricity e of the ellipses and hyperbolas (10) as customarily defined is given by

$$e^2=\frac{\xi-\eta}{\xi}\,,\qquad \xi>0\,,\qquad \xi\geqq\eta\,.$$

Hence $e^2-2=-(\xi+\eta)/\xi$, so that

$$\frac{e^4}{(e^2-2)^2}=\frac{(\eta-\xi)^2}{(\eta+\xi)^2}\,.$$

But from (11),

$$(\eta-\xi)^2=(\eta+\xi)^2-4\xi\eta=\frac{\Lambda^2\Delta^2}{\Sigma^4}-\frac{4\Sigma\Delta^2}{\Sigma^4}\,,$$

$$\frac{e^4}{(e^2-2)^2}=\frac{\Lambda^2-4\Sigma}{\Lambda^2}\,.$$

Therefore e is a root of the equation

$$\Lambda^2e^4=(\Lambda^2-4\Sigma)(e^2-2)^2\,,$$

which is equivalent to[1]

$$(e^2-1)^2\Sigma+(e^2-1)(\Lambda^2-2\Sigma)+\Sigma=0\,.$$

9. Covariant curves. The equation of every covariant curve can be written in terms of the five fundamental covariants. Our simplest example is the locus of all points from which orthogonal tangents can be drawn to the conic. The equation of this locus is $\Theta=0$. For central conics (10) it is the director circle,[2] for the parabola it is the directrix.

A covariant curve of the hyperbola

$$b^2x^2-a^2y^2-a^2b^2=0$$

is evidently the conjugate hyperbola

$$b^2x^2-a^2y^2+a^2b^2=0\,.$$

For this case

$$\Phi=b^2x^2-a^2y^2-a^2b^2\,,\qquad \Sigma=-a^2b^2\,,\qquad \Delta=a^4b^4\,.$$

Hence $\Delta/\Sigma=-a^2b^2$ is of the same weight as Φ, and

$$\Phi-\frac{2\Delta}{\Sigma}=0$$

is the equation of the conjugate hyperbola. Since these expressions are covariants, we know that the hyperbola conjugate to an arbitrary hyperbola $\Phi=0$ has the equation[3]

$$\Sigma\Phi-2\Delta=0\,.$$

[1] See S. L. Loney, *The Elements of Coördinate Geometry*, Macmillan, 1897, p. 344. Loney obtains many interesting results without calling attention to their invariant nature.

[2] Loney, p. 365.

[3] Loney, p. 329.

The asymptotes of a hyperbola evidently constitute a covariant curve. By an argument similar to the above, we conclude that their equation is given by

$$\Sigma\Phi-\Delta=0.$$

To find the equations of the axes of a central conic in covariant form is a problem of somewhat more complexity. For the canonical form (10) we have

$$\Phi=\eta x^2+\xi y^2-\xi\eta\ ,\qquad \Theta=\xi\eta(x^2+y^2)-\xi\eta^2-\xi^2\eta\ ,$$
$$\Lambda=\xi+\eta\ ,\qquad \Sigma=\xi\eta\ ,\qquad \Delta=-\xi^2\eta^2\ .$$

Hence

$$\eta x^2+\xi y^2=\Phi-\Delta/\Sigma\equiv G$$

is a covariant of weight 1, and

$$\eta^2x^2+\xi^2y^2=\Lambda\Phi-\Theta\equiv H$$

is of weight 2. We find from these equations that

$$x^2y^2=\frac{\xi\eta G^2-\eta GH-\xi GH+H^2}{-\xi\eta(\xi-\eta)^2}=\frac{\Sigma G^2-\Lambda GH+H^2}{-\Sigma[\Lambda^2-4\Sigma]}\equiv M\ .$$

For the canonical form (10) the axes are the lines $x=0$ and $y=0$. For the canonical form the covariant M is a perfect square and its square root factors into linear factors. If we transform back from (10) to (4), the lines $x=0$ and $y=0$ are transformed into the axes and the properties of M of factoring are not destroyed. If therefore we form M from the original coefficients and variables it will be a perfect square, and this square root will factor into linear factors which, equated to zero, will give the axes of the conic.

For central conics the factor Σ in the denominator is not zero. The factor $\Lambda^2-4\Sigma$ tells us that no determinate axes will be obtained when $\Lambda^2-4\Sigma=0$. This was to be expected, since the invariant $\Lambda^2-4\Sigma$ vanishes when and only when $\Phi=0$ is a circle.

Let it be required to find the equations of the axes of the hyperbola $2xy-1=0$. We have

$$\Phi=2xy-1\ ,\qquad \Theta=-x^2-y^2\ ,\qquad \Lambda=0\ ,\qquad \Sigma=-1\ ,\qquad \Delta=1\ ,$$
$$G=2xy\ ,\qquad H=x^2+y^2\ ,\qquad M=\tfrac{1}{2}(x^2-y^2)^2\ .$$

The axes are therefore $x+y=0$ and $x-y=0$.

THE ELEMENTARY CHARACTER OF CERTAIN MULTIPLE INTEGRALS CONNECTED WITH FIGURES BOUNDED BY PLANES AND SPHERES[1]

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1. Introduction. In the classical problem of calculating the area, or volume, cut out of a sphere by a square prism with one diameter of the sphere as an axis, one encounters certain integrals, which, although reducible to elementary integrals by appropriate substitutions, at first sight appear to be elliptic integrals of a non-elementary character. This leads to the question as to how far one can go in raising problems about figures formed of spheres and planes without meeting other than elementary integrals. In this paper we shall investigate this question, and, by fairly simple methods shall justify the following conclusions. *For any surface or volume obtained from a figure formed entirely of spheres and planes, the content, coördinates of the center of gravity, and moment of*

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inertia about any axis are elementary functions of the parameters determining the figure. The parameters referred to here are either the coefficients of the equations of the planes and spheres, or the coördinates of a set of points sufficient to determine the figure, or a group of lengths and angles of this character. The statement just made does not necessarily imply the elementary character of the integrals used to obtain the contents, moments, and moments of inertia, since these integrals are definite. Our methods of proof, however, show that there is *at least one method of so setting up these integrals that all the intermediate indefinite integrals will be elementary.* One other property of our solution may be noticed. No exponential or logarithmic terms appear, as all the quantities referred to in our theorem involve only algebraic, trigonometric, or inverse trigonometric functions and their combinations.

2. The dissection of surfaces. We shall reduce the treatment of our problem in the general case to that for a small number of simpler figures, into which every figure of the type we are considering may be decomposed. We begin with the case of surfaces, for which we have:

THEOREM I. *Any surface obtained from a figure composed entirely of spheres and planes may be decomposed into a number of plane right triangles, spherical right triangles, circular sectors on a plane and circular sectors on a sphere.*

For, the plane parts of such a figure are curvilinear polygons, whose sides are straight lines or arcs of circles. To break up such a curvilinear polygon, we select any point O in its plane. We join this with each vertex of the polygon by a straight line, and also draw all the tangents to the circular sides of our polygon which pass through O. We mark in the intersections of all the lines just drawn with the polygon, and regard them as vertices in our later steps. Since we are only considering figures made up of a finite number of spheres and planes, the number of vertices is finite. Taking them in order on the curvilinear polygon, we denote them by $A_1, A_2, \cdots A_n$. Our original figure may now be looked upon as the sum of the figures $O\,A_i\,A_{i+1}$, taken with proper signs.[1] Each such figure is either a triangle, which is reduced to two right triangles by drawing in an altitude, or it is bounded by the circular arc A_iA_{i+1} and the straight lines OA_i, OA_{i+1}. In the latter case, we join B_i, the center of the circle containing arc A_iA_{i+1}, with A_i and A_{i+1}, and also draw in $\overline{A_iA}_{i+1}$, the chord of arc $\widehat{A_iA}_{i+1}$. We then have:

$$O\,A_iA_{i+1} = O\,\overline{A_iA}_{i+1} \pm (B_i\,A_iA_{i+1} - B_i\,\overline{A_iA}_{i+1}).$$

The triangles OA_iA_{i+1} and $B_iA_iA_{i+1}$ are divided by altitudes, and $B_i\widehat{A_iA}_{i+1}$ is a sector. Thus we have shown how to break up all the plane parts of our figure into right triangles and sectors.

[1] By considering our areas as positive or negative in accordance with the sense of rotation along the boundary, we could eliminate the indeterminacy of sign here, as later. We have used positive areas, with indeterminate signs instead of this convention in the interest of simplicity.

For the spherical parts of our figure, we may use a similar method of reduction, with the straight lines and circles replaced by great circles and small circles respectively. We thus reduce these parts, or curvilinear polygons on a sphere, $A_1, A_2, \cdots A_n$ to a set of sectors on a sphere, $B_i\widehat{A_iA}_{i+1}$, and a number of right spherical triangles.

3. The dissection of volumes. For the reduction of volumes, we have an analogous theorem.

THEOREM II. *Any volume obtained from a figure composed entirely of spheres and planes may be decomposed into a number of pyramids, with right triangles as bases, cones with circular sectors as bases, spherical pyramids with right spherical triangles as bases and spherical cones with circular sectors on the sphere as bases.*

To prove this, we select any point in space O. We join this by planes and conical surfaces with vertex at O to all the straight line segments or circular arcs in which two of the plane or spherical surfaces intersect. We also draw tangent cones from O to the spheres of the figure. These surfaces enable us to regard any volume obtained from our figure as the sum, with suitable signs, of a number of conical figures, each of which has a base either in a single plane, or on a single sphere.

For any such cone in which the base is plane, we apply the reduction of section 2 to the base, reducing it to right triangles and circular sectors. On joining the boundaries of these figures to the vertex, we reduce this cone to a number of pyramids with right triangles as bases, and a number of cones with circular sectors as bases.

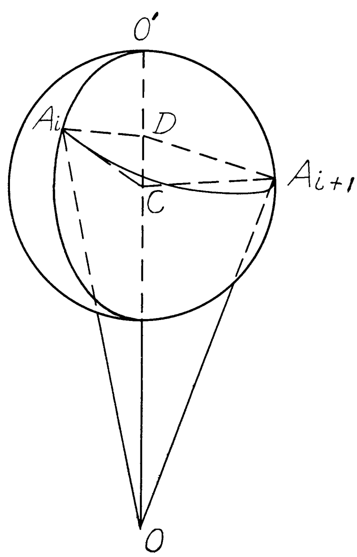

FIG. 1.

For a conical figure in which the base is spherical, we proceed as follows. The base is a spherical curvilinear polygon, on a single sphere, say with center at C. Join OC (Fig. 1) and prolong it to intersect the sphere at O'. Either intersection may be used. Now join O' with each of the vertices of the curvilinear polygon by great circles on the sphere. The tangent great circles to the sides which pass through O' are also drawn. We regard all the points of intersection of these great circles with the polygon as vertices, and, taking them in order on the polygon, denote them by $A_1, A_2 \cdots A_n$. By connecting the arcs of the great circles A_iO' with O by planes, which is possible since O, being on the prolonged radius $O'C$, lies in the planes of these circles, we may reduce the conical figures to a sum, with proper signs, of

conical figures $O\text{-}O'A_iA_{i+1}$. These figures have spherical bases, two plane faces, and one curved one. To reduce them, we draw in the plane of the arc A_iA_{i+1} to cut OO' at D, and also the conical surface with vertex C, and base arc A_iA_{i+1}. We then see that

$$O\text{-}O'A_iA_{i+1}=O\text{-}DA_iA_{i+1}\pm(C\text{-}O'A_iA_{i+1}-C\text{-}DA_iA_{i+1}).$$

In case A_iA_{i+1} happens to be a great circle, D coincides with C, and the last term vanishes. In general, $O\text{-}DA_iA_{i+1}$ and $C\text{-}DA_iA_{i+1}$ are cones, whose reduction we have already considered, while $C\text{-}O'A_iA_{i+1}$ is a spherical cone. It may be reduced by first decomposing its base into right spherical triangles and sectors on a sphere by the method of section 2, and then joining the lines so introduced to C by planes and conical surfaces.

We have thus reduced our volume to figures of the types mentioned in the theorem.

4. Areas. When a figure compcsed of planes and spheres is specified by parameters which are coördinates of points, coefficients in equations of surfaces, or lengths and angles, parameters of all these types may be calculated in terms of them for the simpler figures obtained by dissection mentioned in theorem I. The calculation will obviously introduce at most algebraic and trigonometric functions, or their inverses. But, from elementary geometry, we have as the area of a triangle of base b and altitude h, $\frac{1}{2}bh$; of a sector of radius r and angle A, $\frac{1}{2}r^2A$, of a right spherical triangle with angles A, B, $\pi/2$ on a sphere of radius r, $(A+B-\pi/2)r^2$ and of a sector on a sphere with angle A, side R, on a sphere of radius r, $Ar^2(1-\cos R)$. Hence, from theorem I and the additive character of areas, we have:

THEOREM III. *Any area in a figure composed of spheres and planes is an elementary function of the paramteers of the figure.*

5. Volumes. For volumes we note that the parameters for the figures mentioned in theorem II may be computed in terms of those of the original figure. For the simpler figures, we have, from elementary geometry, as the volume of a cone or pyramid of base B and height h, $(1/3)Bh$, and as that of a spherical cone or pyramid with base of area B on a sphere of radius r, $(1/3)Br$. In each case, B may be computed in terms of the parameters by theorem III, and, since volumes are additive, we have:

THEOREM IV. *Any volume in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

6. Centers of gravity of surfaces. Since the coördinates of the center of gravity of a composite area are easily computed in terms of the areas and coordinates of the centers of gravity of the parts, and the question of areas has been disposed of in theorem III, we need merely show that the coördinates of the center of gravity of the simple figures mentioned in theorem I are elementary

functions. For the plane figures, we readily find by integration that the center of gravity of a triangle is at the median point, while for a circular sector of angle A and radius r, it is on the radius which bisects it at a distance $\dfrac{4\,r\sin(A/2)}{3A}$ from the center.

For spherical figures, we first prove a general result.[1] If z denotes the distance to a diametral plane, S refers to the spherical surface, S_z to the projection of this surface on this diametral plane, θ the angle made by the tangent plane to the sphere and this plane at any point, or by the radius and ordinate at the point, since these are respectively normal to the two planes, and R is the radius of the sphere, we have:

$$A\bar{z}=\iint z\,dS = \iint (z/\cos\theta)\,dS\cos\theta = \iint R\,dS_z = RA_z\,.$$

This shows that the coördinates of the center of gravity of a figure on a sphere with respect to axes through its center, and hence those with respect to any axes, may be computed in terms of the areas of the projections of this figure on the coördinate planes. The figures we are concerned with are spherical right triangles and sectors on a sphere. These will project into figures bounded by straight lines and arcs of ellipses. These figures, by the method of section 2, may be decomposed into triangles and elliptic sectors, and the areas of the latter found from those of the circular sectors of which they are the orthogonal projections by introducing the proper scale factor. As the computation we have outlined nowhere introduces anything beyond elementary functions, we have demonstrated:

THEOREM V. *The coördinates of the center of gravity of any area in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

7. Centers of gravity of volumes. From theorem IV, and the method of finding centers of gravity of composite volumes, we see that we have merely to dispose of the centers of gravity of the solids mentioned in theorem II. These are easily thrown back to the result of the preceding section. For, a cone or pyramid may be thought of as made up of elements parallel to, and similar to, the base. The centers of gravity of these elements are all on the line joining the vertex with the center of gravity of the base, and since the masses of the elements are proportional to the square of the distances from the vertex to them, measured along this line, our problem is identical with that of finding the center of gravity of a rod coincident with this line whose density varies as the

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[1] *Cf.* P. Appell, *Traité de Mécanique Rationnelle,* Paris, 1919, 4th edition, problem 33, p. 171.

square of the distance from one end. By integration, we easily find that for such a rod the center of gravity is 3/4 of the length of the rod from this end.

For a spherical cone or pyramid, we divide into spherical layers concentric with, and similar to the base, and apply the preceding argument, using instead of distance from the vertex to the element, distance to its center of gravity, which is proportional to it.

Thus we find, for a cone, pyramid, spherical cone or spherical pyramid, the center of gravity is on the line joining the vertex to the center of gravity of the base, and is 3/4 the distance from the vertex to this point. In view of theorem V, this proves

THEOREM VI. *The coördinates of the center of gravity of any volume in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

8. **Moments of inertia of surfaces.** The moment of inertia of a composite figure about any axis is simply the sum of the moments of inertia of its several parts about that axis. This reduces our problem to the consideration of the moment of inertia of a figure of the type mentioned in theorem I about any axis. But, from the relation between the mass, distance between an axis and a parallel axis through the center of gravity, and the moments of inertia about these axes, we see that in view of theorems III and V, we may pass from the moment of inertia about any axis to that about a parallel axis through the center of gravity, and thence to any parallel axis. Thus we need only compute the moments of inertia of our standard figures about axes through a single point, for example, the origin. But for these moments of inertia, we need merely to know the integrals and products of inertia: I_x, I_y, I_z, I_{xy}, I_{yz}, I_{xz}, where $I_x=\int x^2dm$, $I_{xy}=\int xy\, dm$, and the other quantities are defined similarly.

For the plane figures, we may compute the results directly by the ordinary methods. We find, for a right triangle with sides a and b taken as x and y axis respectively:

$$I_x=Ma^2/6\ , \qquad I_y=Mb^2/6\ , \qquad I_{xy}=Mab/12\ , \qquad I_z=I_{xz}=I_{yz}=0\ .$$

For a circular sector of radius r and angle A, taking the radius bisecting it as the x axis, and the perpendicular to it through the vertex in the plane of the sector as the y axis, we find:

$$I_x=\frac{Mr^2}{4}\left(1+\frac{\sin A}{A}\right), \qquad I_y=\frac{Mr^2}{4}\left(1-\frac{\sin A}{A}\right), \qquad I_{xy}=I_z=I_{xz}=I_{yz}=0\,.$$

For spherical figures, we take the origin at the center of the sphere on which our triangle or sector lies, and examine the integrals of inertia. Let R be the radius of the sphere, S refer to the spherical surface, and S_z refer to the projection on the xy plane. Let θ denote the angle made by the tangent plane to the

sphere at any point with the xy plane, *i. e.* the angle made by the radius to this point and the ordinate, respectively normal to these two planes.

We have:

$$I_z = \int\!\!\int z^2 dS = \int\!\!\int z(z/\cos\theta)dS\cos\theta = \int\!\!\int zRdS_z = R\int\!\!\int zdS_z.$$

This last integral may be interpreted as the volume bounded by A, A_z, and a cylindrical surface with elements parallel to the z axis, and computed from this interpretation. Since A is a spherical triangle, or sector on a sphere, its projection A_z is a figure bounded by three elliptic arcs, which may in particular be straight lines or circles. We draw in the chords of these arcs and planes through them parallel to the z axis. These divide our original volume into a portion whose base is a triangle, and three parts whose bases are segments of ellipses. The portion with triangular base is formed entirely of planes and spheres, and hence its volume may be computed by theorem IV. For the other pieces, we use a further dissection. Let the base of our figure (Fig. 2) be bounded by the straight line $B_z C_z$, the projection of the arc BEC, and the elliptic arc $B_zD_zC_z$, the projection of the arc BDC. D and E are arbitrary points, inserted to facilitate reference to the arcs. Since arc BDC was one of the arcs bounding A, it is a circular arc. We draw in its plane, which cuts the plane of BEC in chord BC. The volume bounded by the planes BDC and BEC and the sphere may be computed by theorem IV. The volume of the remainder of our figure may be computed by noting that, if plane BDC were parallel to the z axis, its volume would be zero, and if it is not parallel to the z axis, its equation may be solved for z and written in the form $z=ax+by+c$. Thus the volume of the figure bounded by BDC, $B_zD_zC_z$ and the plane and cylinder is:

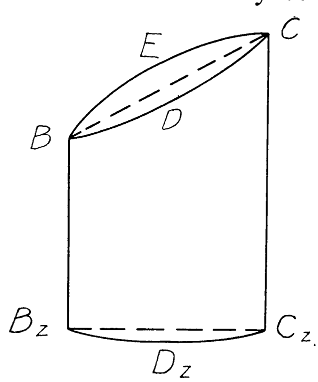

FIG. 2

$$\int\!\!\int z\,dS_z = a\int\!\!\int x\,dS_z + b\int\!\!\int y\,dS_z + c\int\!\!\int dS_z$$
$$= aA'\bar{x} + bA'\bar{y} + cA'.$$

Here A' is the area $B_zD_zC_z$, which may be computed by finding the area BDC by theorem III, and multiplying by the cosine of the inclination of its plane, or $1/\sqrt{1+a^2+b^2}$, and $\bar{x}$, $\bar{y}$ are the x and y coördinates of the center of gravity of $B_zD_zC_z$, which are equal to those of BDC, from which it is projected, and hence may be computed by theorem V. Thus I_z, and similarly I_x and I_y may be computed.

For the products of inertia, we have:

$$I_{xz} = \iint xz\,dS = \iint x(z/\cos\theta)\,dS\cos\theta = \iint xR\,dS_z = RA_z\bar{x}\ .$$

But, S_z is a plane figure bounded by straight lines and arcs of ellipses. Hence its area, A_z, and the x coördinate of its center of gravity, $\bar{x}$, may be computed by the method of section **6**. The remaining products of inertia, I_{xy} and I_{yz} may be computed similarly.

Thus we have now proved

THEOREM VII. *The moment of inertia about any axis of any area in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

9. Moments of inertia of volumes. From the discussion of section 8, we see that if we compute the moments of inertia of the solids mentioned in theorem II about an arbitrary axis passing through a single point, we shall have solved the general problem. These solids are cones, pyramids, spherical cones and spherical pyramids, and the moment of inertia of such a figure about any axis through its vertex is readily thrown back to theorem VII.

For, as in section **7**, we may divide such a figure into plane or spherical layers similar to the base, with center of similitude at the vertex. Hence the radii of gyration of these sections for any given axis through the vertex will be proportional to the distances they cut off on any line through the vertex. But their masses are proportional to the squares of these distances. Thus their moments of inertia are proportional to the fourth power of these distances, and consequently, if the moment of inertia of the surface which is at the base is I_s, the moment of inertia of the volume is:

$$I = \int_0^r I_s \frac{x^4}{r^4}\,dx = \frac{rI_s}{5}\ .$$

Since x must be measured normal to the layers, r is the radius of the sphere in the case of the spherical figures, and the altitude in the case of the cones and pyramids.

We have thus shown that, for a cone or pyramid, the moment of inertia about any axis through the vertex is 1/5 the product of the altitude by the moment of inertia of the base about this axis, while for a spherical cone or spherical pyramid, it is 1/5 the product of the radius of the sphere by the moment of inertia of the base about this axis. Recalling theorems IV, VI and the relation connecting moments of inertia about parallel axes, we see that this combined with theorem VII, proves

THEOREM VIII. *The moment of inertia about any axis of any volume in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

10. Summary and further results. The essential content of the theorems we have stated is the fact that, for any surface or volume in a figure formed entirely of spheres and planes, the content, coördinates of the center of gravity and moment of inertia about any axis are elementary functions of the parameters determining the figure. Certain additional conclusions may be drawn from our proofs. Since the geometrical dissections and treatment of the fundamental integrals is capable of translation into analytic operations, we have not only shown that the final definite integrals are elementary, but that there is at least one way of handling the original multiple integrals throughout which all the intermediate indefinite integrals will be elementary.

We also note that the final results in all cases involve only algebraic, trigonometric, and inverse trigonometric functions. In fact, while we have applied our results to multiple integrals, and used a few integrals in our proofs, these are all of simple character, and could easily be replaced by the circumlocutions of elementary geometry. The spirit of our discussion is synthetic, and it would, after slight modification, become an elementary treatment of contents, centers of gravity and moments of inertia for the figures considered.

It is also to be observed that in proving our theorems we have in each case given an explicit method of computing the final result. In specific applications the process is, of course, generally considerably shorter than that which we have outlined. If the reader has any doubt of the practical value of these methods, it is suggested that he apply them to the figure formed by a sphere cut by a square prism with one diameter of the sphere as an axis, mentioned in the introduction, and try to check his results by direct multiple integration.

The class of figures to which our methods apply may be extended somewhat by orthogonal projection. For plane figures, areas are merely changed by a constant factor, centers of gravity project directly, and for a suitable choice of coördinate axes, the integrals and product of inertia, I_x, I_y, I_{xy}, are unchanged, multiplied by the square of the scale factor, and multiplied by the scale factor respectively. By first carrying out the dissection, and then projecting sectors with one elliptic arc into circular sectors, we obtain:

THEOREM IX. *For any plane figure bounded by straight lines and arcs of ellipses or circles, the area, coördinates of the center of gravity, and moment of inertia about any axis are elementary functions of the parameters determining the figure.*

We have used special cases of this theorem in sections 6 and 8.

We may use analogous considerations in three dimensional space, using simple three dimensional strains instead of orthogonal projection. Such strains change

volumes, centers of gravity, and integrals of inertia in a simple manner. We shall not attempt to characterize any general class of figures involving ellipsoids which may be transformed by simple strains into figures involving spheres, confining ourselves to figures with a single ellipsoid, which can obviously be so transformed. Consequently we formulate:

THEOREM X. *For any volume in a figure composed of a single ellipsoid and a number of planes, the volume, coördinates of the center of gravity, and moment of inertia about any axis are elementary functions of the parameters of the figure.*

Our original discussion of figures involving spheres and planes can probably be carried out for figures of more dimensions involving hyperspheres and hyperplanes. The question of finding other fairly wide classes of figures in three space for which the problems we have treated lead to elementary integrals also suggests itself.

ON THE ORIGIN OF THE TERM "ROOT"

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Starting from the investigation of a philological problem the writer came, with the kind help of Professor David Eugene Smith,[1] to some results which may interest students of mathematics, and which he therefore wishes to submit to the readers of this MONTHLY.

The term "root" has its origin in the Arabic. "Latin works translated from the Arabic have *radix* for a common term, while those inherited from the Roman civilization have *latus*."[2] *Radix* ("root") is the Arabic *jadhr*, while *latus* (Greek, πλευρά, *pleura*, meaning "rib" or "side") is the side of a geometric square.

It is certainly rather strange that such a term as "root" should be used in this connection. It suggests that if the basic number is a root, the square might be a bush, and so on up in a kind of a mathematical garden.

The Chinese, indeed, do use the word *kun* to mean root, grass, and shrub, and the Hindus also use the word *mûla* for the root of a plant, but this was very likely due to the Arabic influence, which is so often seen in China and which may have spread into India by way of China. It is, however, a fact that thus far we have no satisfactory explanation as to why this botanical term should have found place in the theory of numbers. Considered geometrically we can see why the line represented the basic number, and why the square represented its

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[1] Professor Smith has been most helpful in discussing with me the mathematical side of the question and giving me his advice and suggestions. In addition to this he has placed at my disposal his private library, which is especially rich in old and rare Arabic books and manuscripts, without which this paper could not have been written.

[2] See Smith, *History of Mathematics*, vol. II, p. 150.

second power, and the cube its third power. We can also see why a race of geometers should have used πλευρά, πλευρόν (*pleura*, *pleuron*) for the first power, and why they would not use such a term as ῥίζα[1] (*riza*, "root") in this connection. It is also clear that the Latin writers, whose theoretical mathematics came wholly from the Greeks, would naturally use *latus* ("side") for the same purpose. Therefore the problem before us, as already set forth clearly by Professor Smith,[2] is to find out whether the medieval Latin authors were correct in their translation of the Arabic word *jadhr* by *radix* ("root").

The writer's doubts about the matter were aroused by the fact that the Hebrew scholars of the earlier Middle Ages did not commonly translate the word *jadhr* by *shoresh* ("root") but by *gader* which means "wall," "side," "limit," or "boundary."[3] For example, Abraham bar Chiia (*c.* 1100), commonly known as Savasorda, in his Hebrew geometry, never uses *shoresh* ("root") but always *gader*. Ibn Ezra seems to be the first who introduced into Hebrew, in his *Sefer ha-Mispar* ("Book of Arithmetics") the term "root," using both the traditional word *gader* and the later form *shoresh*. Even in the first line of his *Sefer ha-Echad* he says *shoresh wiyesod* ("root and basis"), showing that he knew the Arabic *jadhr*, was doubtful how to translate it, and thought it best to give the two meanings, *shoresh* ("root") and *yesod* ("basis"). In his Latin translation of Savasorda's work (*Liber Embadorum*, ed. Curtze, Leipzig, 1902). Plato of Tivoli, in his early definitions (p. 18, §§ 19, 20) uses *latus* for *gader*, but later (p. 32, § 8; p. 34, §§ 9, 10; p. 91, § 20; p. 92, §§ 23, 24; p. 98, 4) he uses *radix* instead. The corresponding Hebrew text, *Hibbûr ha-Meshiḥah we ha Tisboreth* (Berlin, 1912), always uses the word *gader* (p. 25, § 46; pp. 26, 27, §§ 46, 47; p. 51, §§ 91, 93, 93a, *et passim*).

These doubts as to the correct translation from the Arabic were strengthened by the statement of Professor Smith that the idea of "root" had its origin solely in the Arabic writings. The writer therefore started out to investigate the real meaning of the word *jadhr*, not depending upon lexicons or the Latin versions of the thirteenth century, but seeking the meaning as it appears in the manuscripts of the old Arabic mathematicians themselves.

Mohammed ibn Mûsâ al-Khowârizmî[4] (*c.* 825), is the oldest Arabic mathematician of much prominence, and it was his *'Ilm al-jabr wa'l muqabalah* that gave to Europe both the name and the foundation principles of algebra. His

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[1] See also Tropfke, *Geschichte der Elementar-Mathematik*, vol. I, p. 214, note 864, and Ruska, *Zur, arabischen Algebra*, p. 68, note 1.

[2] *History*, II, 150.

[3] See Rosin, *Magazin für die Wissenschaft des Judentums*, V, pp. 47 *seq.*, and Steinschneider, *Ibn Ezra* pp. 94 and 110; reprinted also in his *Gesammelte Schriften*, (ed. Marx-Malter), p. 475.

[4] Smith, *History*, I, 170.

chapter *Bâb al Misâcha* ("on geometry")[1] begins as follows: "Know that the meaning of the expression 'one in one' is an 'area'; and its meaning is one cubit (in length) in one cubit (in breadth). And every roof[2] of equal sides and angles which has one cubit for each side is a (square) unit. But if it has two cubits for every side and has equal sides and angles, then the whole roof is four times the roof which has one cubit in one cubit. . . . . And in every square roof of equal sides one of its sides (multiplied) in a square unit is its *jadhr*; or if the same be multiplied in two (square units) then it is like two of its *jadhrs*, whether this roof be small or great." Here we have a clear definition of the *jadhr*. It is a side multiplied with a square unit. *Jadhr* not only means "root" but also[3] "basis," "foundation," "lowest part." Mohammed ibn Mûsâ was a clear thinker. He knew that we can never get a square measure by multiplying one side by one side, in the ordinary meaning of multiplication, since the side has only one dimension. Therefore he begins his chapter on areas by introducing the new notion of a square unit. Then he says that in order to get the area of any figure we must first multiply the one side by the square unit; this is then the basis to be multiplied by the other side. We do not multiply one side by one side, but one *jadhr*, representing the square basis, by one side representing the number. The same definition and idea is also to be found elsewhere in his algebra.[4] It is also found in the old Hebrew geometry, the *Mishnath ha-Middoth* ("Theory of Measures"),[5] as follows: "What is a quadrangle equal as to sides and angles? Ten from every side. Multiply length on breadth, and the product is the area, namely hundred. The one side is its one *'iqqar*, its two sides are its two *'iqqars*, and so three and four." Now this last sentence has no sense as it stands. But *'iqqar* is just the Hebrew word for *jadhr*, and its meaning is "root," "basis," or "lowest part." Considering the fact that there are striking parallels between *Mishnath ha-Middoth* and al Khowârizmî, and that the latter was

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[1] See the Rosen edition, London, 1831; Arabic text, p. 50 *seq.*; English text, p. 70 *seq.* Rosen's translation is faulty in certain places and it has been corrected by the present writer. In the study of the Arabic text he has been greatly helped by having access to the manuscript in Professor Smith's library described in this MONTHLY for 1925, p. 395.

[2] This is the original meaning of *satch*, later used for surface. It corresponds to the Hebrew *gag* (Greek στέγη, *stegê*, "roof," "surface"); see *Mishnath ha—Middoth*, ed. Hermann Schapira, p. 14, note 3.

[3] See Lane, *Arabic-English Dictionary*.

[4] Arabic text, p. 11, line 11; and still more clearly in p. 13, line 9 seq., where it reads: "and one of its sides multiplied into the (square) unit is its *jadhr* . . . . and we make the other side, namely *hz* (three), and this is the number of its *jadhrs*."

[5] The author is not known, but he may have lived even as early as the beginning of the Christian era. See Smith, *History*, I, 174; M. Steinschneider, *Festschrift Zunz* (Berlin, 1864); H. Shapiro, *Abhandlungen zur Geschichte der Mathematik*, III, 3.

possibly influenced by his Hebrew predecessors,[1] we may venture to correct the quoted passage in his algebra to read as follows: "And its one side into one (square unit) is its one *'iqqar*, and into two (square units) is its two *'iqqars*," and so on.

These passages in the *Mishnath ha-Middoth* and al-Khowârizmî have always puzzled historians. Rosen (p. 71, lines 17 and 19), Aristide Marre in his French translation (Rome, 1866, quoted by Shapiro, p. 25, note 3), Shapiro himself (*loc. cit.*), and Ruska (*loc. cit.*, pp. 104–105) fail to give any explanation of them, and Ruska confesses frankly that he can find no meaning in them.

The explanation here given, however, is perfectly clear, and is based upon the meaning of *jadhr* and *'iqqar* in two of the earliest Arabic and Hebrew mathematical texts in which the terms appear. The term does not mean "root," but "square basis," that by whose multiplication we get the square area. This was the reason why *jadhr* was used by later writers, such as Omar Khayyam, as the basic number of a square number. The latter did not know anything more about the original meaning of *jadhr*, and he used the word[2] loosely for *ḍil'* (Greek, πλευρά), which means "rib" or "side."[3] But he,[4] like al-Khowârizmî, still knew that it was a concrete number in opposition to an abstract number.

By the time of Beḥâ Eddîn (*c.* 1600) the original meaning was entirely forgotten. In his *Kholâsat al-Ḥisâb* ("Essence of Arithmetic")[5] he says: "What is multiplied into itself is called *jadhr* in arithmetic, *dil'* ("side") in geometry, and *shai'* ("thing," "cause") in algebra." He certainly understood by *jadhr* the abstract "root." In this misunderstood form the term *jadhr* found its way into medieval Latin as *radix*.

Professor Smith is right in calling attention to the different sources of the two terms *latus* and *radix*. The fact suggests the two leading paths by which ancient science found its way into medieval Europe. The one led from Greek to Latin and thence to other western languages. This route has been quite thoroughly explored. The other road may be characterized as the Greek-Arabic-Latin-modern-language one, and the study of this pathway has only just begun.

Many other terms than *radix* may be based upon an imperfect understanding of the Arabic sources. On the other hand thousands and thousands of Arabic

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[1] Perhaps even by this work, since the date is so uncertain, or perhaps the two were influenced by some common source which is now unknown. Such a state of affairs is found in the Latin translations from the Arabic versions of Euclid. See the argument of Shapiro in the preface to his edition.

[2] See the Arabic text, p. 5, lines 9, 10, and at the bottom. After p. 20 he uses only *ḍil'*.

[3] Arabic text, p. 4, lines 3, 8. [4] Arabic text, p. 3, line 10.

[5] Nesselmann ed., Berlin, 1843; Arabic text, p. 15; German transl., p. 14. There are two Arabic manuscripts of this important work in Professor Smith's library, and these the author has consulted in the preparation of this article.

manuscripts on mathematics, astronomy, medicine, and the other sciences are still reposing unread by modern Arabists, in the great libraries of Europe. The fact that a number of very important ones are in Professor Smith's library goes to show that abundant source material is likely to be found even in the New World,—for example, in the remarkable collection now in the Jewish Theological Seminary in New York. A thorough study of the scientific manuscripts in Arabic is of the greatest importance for the history of human knowledge and of the foundations of our modern civilization. Our history should be no longer a history of wars and massacres but a history of the progress of the human spirit, of human culture, of human rights, and of human liberties. The present interest in the history of science is evidence that this new conception of the story of the race is beginning to be realized.

QUESTIONS AND DISCUSSIONS

Edited by Tomlinson Fort, Hunter College, Park Ave. & 68th St., New York, N. Y., and H. E. Buchanan, Tulane University, New Orleans, La.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

REPLIES TO QUESTIONS

46 [*1922*, 210]. A geometrical construction will often become impracticable in special cases where some of the construction points are imaginary, although the final result is real. Is there any system according to which a construction failing in this way may be replaced by one that will work?

For example, let a line l_1 cut a circle c_1 in P_1 and Q_1, and let a line l_2 cut a circle c_2 in P_2 and Q_2. If the four points are real, the intersection of P_1P_2 with Q_1Q_2 and of P_1Q_2 with P_2Q_1 can be found directly. If the four points are imaginary, the intersections will still be real, and there ought to be a simple way of getting them as the points common to a line and a circle.

I. Reply by P. H. Daus, University of California, Southern Branch.

Problems of the type suggested here are essentially solved in several texts on projective geometry. The interested reader will find such a discussion in chapter XII of Dowling's *Projective Geometry*. In view of the special problem proposed it is interesting to develop the ideas entirely from the standpoint of circles. Proofs of the results stated below will be found in texts on college geometry, for example, Durell's *Plane Geometry for Advanced Students*, chapters X and XIII.

If A and B are two points such that the tangents from them to two circles are equal, then the line AB contains all such points and is called the radical axis of the circles. All circles which have the same radical axis have their centers on a line perpendicular to the radical axis, and are said to be coaxal. Coaxal systems may be of two types. A system of the first type is such that all the circles have two real points L_1 and L_2 in common; and that of the second, is such

the four points $ADEF$ are harmonic. Similarly for the points A, D', E', F'. The two harmonic ranges having the point A in common, the lines EE', FF' intersect on the line DD', so that the pencil of lines $X(ADEF)$ is perspective both to the involution of conjugate points on s with respect to (C), and to the involution of conjugate points on s' with respect to (C'). Hence any pair of lines projecting from X a pair of conjugate points of the involution on s will meet s' in a pair of conjugate points with respect to (C'). Similarly for X'.

DISCUSSIONS

I. A Curious Algebraic Function.

By Frank Schlesinger, Yale University Observatory

In the discussion of measures (x) of the positions of star images on certain photographic plates that extend from $-X$ to $+X$ an empirical term of the form Px^n was introduced and a least-squares solution was carried out to secure the best agreement between the measured positions and those known in advance, n being assumed and P being the unknown. The star images are uniformly distributed on the plate. The question arose as to the degree of completeness with which Px^n represents the empirical term if its true form is Kx^{n+2}. In that case the least-squares solution has made a minimum the sum of the squares of the quantities

$$Kx^{n+2}-Px^n.$$

If we square, integrate with respect to x from $-X$ to $+X$, differentiate with respect to P and set the derivative equal to zero we obtain

$$P=K\frac{2n+1}{2n+3}X^2$$

and the difference between the two terms is

$$Kx^{n+2}-K\frac{2n+1}{2n+3}X^2x^n .$$

This has a (numerical) maximum value at

$$x=X\left(\frac{n}{n+2}\cdot\frac{2n+1}{2n+3}\right)^{\frac{1}{2}}.$$

At the edge of the plate $(x=\pm X)$ occurs another maximum, of course not in the algebraic sense. The ratio of the two maxima is

$$\left(\frac{2n+1}{n+2}\right)^{(n+2)/2}\left(\frac{n}{2n+3}\right)^{n/2}.$$

If we call this function $f(n)$ and study it for values of $n \geqq 1$, we observe the following curious property: $f(1)=0.44721$, $f(2)=0.44643$, $f(3)=0.44631$, $f(4)=0.446281$, $f(5)=0.446270$, $f(6)=0.446266$, $f(\infty)=0.44626032=2e^{-3/2}$.

We see then that, for all values of n (integral or fractional) not less than unity, this function has the same value within less than 0.001; the value for $n=2$ differs from the limit by only one part in 2600.

II. Quadrangles with the Brocard Property

By P. S. Wagner, Johns Hopkins University

The Brocard points and angle of a triangle, and also of special quadrangles, have been taken up and discussed in detail.[1] It is the purpose of this paper to take up this question for the general quadrangle.

If the convex quadrangle a, b, c, d is such that the angles formed by joining an internal point x to the vertices, *i.e.*, xab, xbc, xcd, xda, are all equal to the same angle θ, and if we denote the lines xa, xb, xc, xd by r_1, r_2, r_3, r_4, respectively, and the angles dxa, axb, bxc, cxd by $\alpha, \beta, \gamma, \delta$, respectively, then

$$\begin{aligned} \frac{r_1}{r_2} &= \frac{\sin(\beta+\theta)}{\sin\theta}; \qquad & \frac{r_2}{r_3} &= \frac{\sin(\gamma+\theta)}{\sin\theta} \\ \frac{r_3}{r_4} &= \frac{\sin(\delta+\theta)}{\sin\theta}; \qquad & \frac{r_4}{r_1} &= \frac{\sin(\alpha+\theta)}{\sin\theta} \end{aligned} \tag{1}$$

from which, by multiplication,

$$(\sin\alpha\cot\theta+\cos\alpha)(\sin\beta\cot\theta+\cos\beta)(\sin\gamma\cot\theta+\cos\gamma)(\sin\delta\cot\theta+\cos\delta)=1. \tag{2}$$

If we let $\cot\theta=\pm i$, we see that the equation is satisfied. Hence the quartic (2), which is

$$(\cot\theta+\cot\alpha)(\cot\theta+\cot\beta)(\cot\theta+\cot\gamma)(\cot\theta+\cot\delta)=\csc\alpha\csc\beta\csc\gamma\csc\delta, \tag{3}$$

is identically

$$(\cot^2\theta+1)(\cot^2\theta+A\cot\theta+B)=0 \tag{4}$$

so that, if we write (3) as

$$\cot^4\theta+S_1\cot^3\theta+S_2\cot^2\theta+S_3\cot\theta+S_4\csc\alpha\csc\beta\csc\gamma\csc\delta=0 \tag{5}$$

we have,

$$A=S_1 : B+1=S_2 : A=S_3 : B=S_4\csc\alpha\csc\beta\csc\gamma\csc\delta\ .$$

Hence the general condition on the angles of the quadrangle is that

$$1-S_2+S_4=\csc\alpha\csc\beta\csc\gamma\csc\delta \tag{6}$$

and since now

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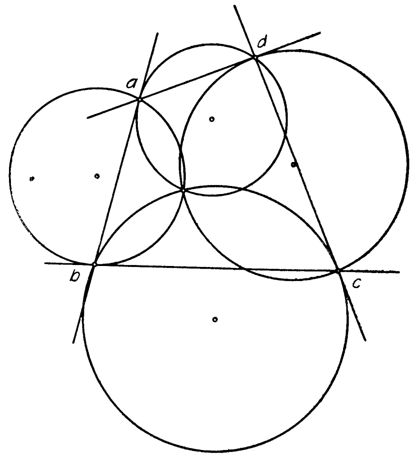
[1] See *e. g.* Casey: *A Sequel to Euclid*, 6th ed., p. 172; Casey: *A Treatise on the Analytic Geometry of the Point, Line, Circle and Conic Sections*, p. 64; Coolidge: *A Treatise on the Circle and the Sphere*, p. 61.

$$\cot^2\theta + A\cot\theta + B = 0 \text{ becomes} \tag{7}$$

$$\cot^2\theta + S_1\cot\theta + S_2 - 1 = 0 \tag{8}$$

we have $\cot\theta$ in terms of the cotangents of the angles of the quadrangle.

If a, b, c, d are the complex names of the vertices of our quadrangle, we have



$$\frac{x-a}{b-a} = \rho t : \frac{\bar{x}-\bar{a}}{\bar{b}-\bar{a}} = \rho/t \tag{9}$$

so that

$$\begin{aligned}
(x-a)(\bar{a}-\bar{b}) &= t^2(a-b)(\bar{x}-\bar{a})\\
(x-b)(\bar{b}-\bar{c}) &= t^2(b-c)(\bar{x}-\bar{b})\\
(x-c)(\bar{c}-\bar{d}) &= t^2(c-d)(\bar{x}-\bar{c})\\
(x-d)(\bar{d}-\bar{a}) &= t^2(d-a)(\bar{x}-\bar{d})\ .
\end{aligned} \tag{10}$$

The condition that these four lines pass through the same point x is the vanishing of

$$\begin{vmatrix}
a(\bar{a}-\bar{b}), & \bar{a}(a-b), & (a-b), & (\bar{a}-\bar{b})\\
b(\bar{b}-\bar{c}), & \bar{b}(b-c), & (b-c), & (\bar{b}-\bar{c})\\
c(\bar{c}-\bar{d}), & \bar{c}(c-d), & (c-d), & (\bar{c}-\bar{d})\\
d(\bar{d}-\bar{a}), & \bar{d}(d-a), & (d-a), & (\bar{d}-\bar{a})
\end{vmatrix}. \tag{11}$$

This, then, is the general condition on the quadrangle *abcd* that it shall have at least one Brocard Point. However, the symmetry of the determinant shows that if there is one there must be two of these points; the other one being derived from the negative sense, *i.e.* *adcb*, of the quadrangle. In each case, we have the same Brocard angle.

It is also of interest to note that if we impose the condition that the quadrangle shall be cyclic we may replace $\bar{a}$ by $1/a$, etc., in (11). This general condition then reduces to the condition that the four vertices a, b, c, d are harmonically separated, *i.e.* the quadrangle is Casey's[1] harmonic quadrangle

The method of locating these points geometrically, when they exist, is a direct extension of the one used by Coolidge[1] for the case of the triangle. Name the vertices of the quadrangle, in positive order, consecutively a, b, c, d, beginning arbitrarily. Then draw a circle through a touching bc at b, a circle through b touching cd at c, a circle through c touching da at d, and a circle through d touching ab at a. (Figure.) These four circles will all meet in a point which is

[1] *Loc. cit. ante.*

[1] *Loc. cit. ante.*

one of the Brocard points if there is one. The proof of this is analogous to that for the triangle. To locate the other Brocard point start at any vertex, name the consecutive vertices a, b, c, d in the negative or clockwise order and repeat the above constructions.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be **sent.**

REVIEWS

Mathematics of Finance. By LLOYD SMAIL. New York, McGraw-Hill Book Company, 1925. 310 pages. Price $3.50.

During the past few years several books on the subject of mathematics of finance have been written. In the present volume Professor Smail has attacked the subject more vigorously than have most writers.

The work is intended as a practical text book for students in colleges of business administration with the minimum requirements in algebra. The author has added a number of sections in small print in which he has discussed the subject in a more theoretical manner. For students interested in the theory of the subject chapters XIV and XV should furnish sufficient material to carry on the work. In chapter XIV the general theory of interest and discounts is discussed with the aid of the calculus and higher analysis. In chapter XV the theory of varying annuities is presented by means of finite differences.

The book is divided into three parts; Part I deals with simple and compound interest, discounts, equation of payments, annuities, depreciation, valuation of bonds, building and loan associations.

Part II discusses the general theory of probability, the mortality table, life annuities, and life insurance. However, after such an excellent presentation of the subject in part I, one wonders why the author did not go more into detail in part II. A few more solutions of problems and illustrations would have been in keeping with the former section.

Part III gives a brief discussion of logarithms, progressions, and the binomial theorem.

Throughout the book we have a clear, concise treatment of the subject. Numerous applications and illustrations have been given and a variety of problems follow each chapter. Chapter IX is given to a summary of the most important formulas and an analysis of problems in simple and compound interest, discounts, and annuities. A general summary of all formulas is included with the tables. The tables have been chosen with great care and should answer any purpose for which they are needed.

The arrangement of the material is logical and is adaptable to any ordinary requirement of the student or instructor. On the whole this should be an acceptable text for colleges of business administration.

A. W. Richeson.

ARTICLES IN CURRENT PERIODICALS.

Bulletin of the American Mathematical Society, volume 31, nos. 9–10, November-December, 1925; "On the Number of Elements of a Group which have a Power in a given Conjugate Set" by Louis Weisner, 492–496; "Note on the Construction of Tables of Linear Forms" by D. N. Lehmer, 497–498; "Criteria that any Number of Real Points in n-Space shall lie in an $(n\text{-}k)$-Space" by H. S. Uhler, 499–502; "Characteristic Parameter Values for an Integral Equation" by W. A. Hurwitz, 503–508; "On the Polynomial of the Best Approximation to a Given Continuous Function" by J. Shohat, 509–514; "Resolvent Sextics of Quintic Equations" by L. E. Dickson, 515–522; "Improper Double Integrals" by C. A. Shook, 524–530; "The Fundamental Region for a Fuchsian Function" by L. R. Ford, 531–539; "Some Phases of Descriptive Geometry" by W. H. Roever, 540–550.

Journal of Mathematics and Physics (Massachusetts Institute of Technology), volume 5, no. 1, December, 1925: "The Elementary Theory of Almost Periodic Functions of Two Variables" by Philip Franklin, 40–54; "Adjoint and Inverse Determinants and Matrices" by L. H. Rice, 55–64.

Proceedings of the Edinburgh Mathematical Society, volume 43, part 1, May, 1925: "Some Axi-Symmetric Determinants with Integers for Elements" by J. J. Nassau, 64–69. Part 2, November, 1925: "On the Rearrangement of Terms in a Complex Series" by S. Beatty, 85–91.

Science, new series, volume 63, no. 1625, February 19, 1926; "The Earliest Known American Arithmetic" by Louis C. Karpinski, 193–195.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB TOPICS

I. La Courbe du Diable.

By B. H. Brown, Dartmouth College.

Ever since the middle of the eighteenth century[1] the equation

$$y^4-x^4-96a^2y^2+100a^2x^2=0 \tag{1}$$

has been extensively employed to test the abilities of students in curve tracing. The origin of the name *courbe du diable* we do not know, but in all probability the curve was summarily christened by some exasperated youth who felt strongly and expressed himself thus forcibly on the subject. Frost[2] in his excellent book says that this equation "deserves to be considered carefully as to the points which we have at present discussed," (symmetry, location, singular

[1] Cramer, *Introduction à l'Analyse des lignes courbes*, 1750, p. 19.

[2] Frost, *Curve Tracing*, 4th ed., London, 1918, pp. 26, 27, plate II, fig. 15.

points and tangents thereat, asymptotes, etc.). Lacroix,[3] Laurent,[4] Moigno,[5] Briot and Bouquet,[6] Niewenglowski,[7] and others have included this problem in their treatises. Loria[8] merely mentions it as a worthwhile example, but Teixeira[9] gives a detailed discussion (with a very inaccurate figure).

The actual tracing of the curve requires no greater ability than the Continental teachers have always expected of their students. The slope in the form

$$y' = \frac{x(x^2-50a^2)}{y(y^2-48a^2)},$$

gives a good deal of information. The polar form

$$\rho^2 = 96a^2 + 4a^2 \frac{\cos^2\theta}{\cos 2\theta}$$

gives more. With the information thus gleaned we can certainly trace the curve better than Teixeira does.

But it is a little surprising that none of the authors to whose works we have access notice that (1) is a very slight variation from the degenerate locus

$$(y+x)(y-x)(y^2+x^2-100a^2)=0 . \tag{2}$$

We see easily that $(\pm\sqrt{50}a, \pm\sqrt{50}a)$ are singular points of (2) but not of (1); and we may now discuss how the branches of our devil-curve have broken away from each other at these points. This point of view furnishes us an excellent idea of the shape of the curve.

II. 1926 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By W. C. EELLS, Whitman College, Walla Walla, Washington

Last year in this MONTHLY (*1925*, 258) a list of important events in the history of mathematics for which the year 1925 was a centennial year was published. It was suggested that program committees of college and high school mathematics clubs, teachers of courses in the history of mathematics, and others might find these events of interest as suggestive of timely topics for programs and special investigations.

Below is given a similar list of events which occurred, according to one or more of the standard historians of mathematics, a whole number of centuries

[3] *Traité élémentaire de calcul différentiel et intégral*, 1837, p. 158.

[4] *Traité d'Analyse*, vol. 2, p. 185.

[5] *Leçons de calcul différentiel et de calcul intégral*, etc., vol. 1, Paris, 1840, p. 222.

[6] *Géométrie analytique*, 2nd ed., p. 197.

[7] *Cours de géométrie analytique*, vol. 2, Paris, 1895, p. 73.

[8] *Ebene Kurven*, vol. 1, p. 101.

[9] *Traité des courbes spéciales*, Fr. ed. 1908, vol. 1, p. 296.

preceding 1926, accompanied in some cases by suggestions of closely-related topics.

B.C.

274. Approximate date of birth of Eratosthenes. Geography and geodesy of the ancients; prime numbers and "sieve" of Eratosthenes; the Julian calendar.

A.D.

526. Death of Boethius: Medieval geometry and arithmetic; mathematics among the Romans.

626. Wang Hs'iao-T'ung ordered to conduct examinations on the calendar. Solution of numerical cubic equations in China. (See Mikami: *The Development of Mathematics in China and Japan*, Leipzig, 1913. pp.53–56.)

826. Birth of Tabit ibn Qorra. Development of mathematics at Bagdad; Arabian translations of Euclid; magic squares.

1526. Death of Ferro. European solution of cubic equations.

1526. Introduction of present symbol for square root by Rudolph. Development of algebraic symbolism.

1626. Death of Cataldi. Continued fractions.

1626. Death of Francis Bacon. Experimental foundation of applied mathematics.

1626. Death of Gunter. Origin of trigonometric functions (cosine and cotangent); computation of tables of logarithmic functions; "Gunter's chain" and land surveying.

1626. Death of Snell. Computation of π; polar triangles; "loxodrome" and navigation.

1626. Publication of Girard's treatise on trigonometry, containing first use of abbreviations: sin, tan, and sec.

1726. Death of Nicolaus Bernoulli. The Bernoulli family; mathematics at St. Petersburg.

1726. Establishment of Hollis professorship of mathematics at Harvard College, first endowed professorship of mathematics in America. Early mathematics at Harvard and other colonial universities.

1826. Publication of Davies' *Descriptive Geometry*. Mathematics at West Point.

1826. Publication of Warren Colburn's *Arithmetic Upon the Inductive Method of Instruction*. Colburn's influence on American elementary mathematics.

1826. First publication of the *Journal für die reine und angewandte Mathematik*, or "Crelle's Journal." Mathematical periodicals.

1826. Birth of H. J. S. Smith. Development of mathematics at Oxford; number theory. (For biography see Macfarlane's *Ten British Mathematicians.*)

1826. Birth of Riemann. Riemann surfaces; elliptic non-Euclidean geometry.

1826. Publication of Abel's memoir on "Abelian Functions."

1826. Presentation of Lobachevsky's first paper on foundations of geometry at University of Kasan. Hyperbolic non-Euclidean geometry.

CLUB ACTIVITIES

THE MATHEMATICS CLUB OF BUCKNELL UNIVERSITY, Lewisburg, Pa. [*1924*, 203]

Meetings held during the year 1924–1925 were as follows:

October 13, 1924. "As the crow flies," short cuts in arithmetic, by Professor H. S. Everett.

January 19, 1925. "Number systems" by Professor J. S. Gold.

April 20. "Some moments and I" by Professor C. A. Lindemann.

June 1. Social meeting at the home of Professor Gold.

(Report by Professor Everett)

PI MU EPSILON, Bucknell University, Lewisburg, Pa.

The Bucknell University chapter of Pi Mu Epsilon, a national scholarship fraternity for students of mathematics, was founded March 5, 1925. The following officers were elected for the remainder of the year:

Director.........Professor H. S. Everett
Vice-director.....M. F. Decker '25
Secretary........Hulda J. Baxter '25
Treasurer........W. I. Miller '26
Librarian........Catherine S. Baxter '25.

Meetings held during the remainder of the year were as follows:

March 18, 1925. Business meeting.

April 28. Business meeting.

May 6. Initiation and banquet at the Cameron House.

May 8. Lecture "The total solar eclipse of 1925" by Professor J. A. Miller of Swarthmore College, illustrated by slides and a Pathé reel.

May 26. Lawn party at the home of Professor Everett.

(Report by Professor Everett)

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I. [*1925*, 202]

The Committee on Program of the Mathematics Club of Brown University for the year 1925-1926 consists of Professor R. E. Gilman, Professor C. R. Adams, Winifred F. Pine '26, and Agnes A. Duffy '27. The Committee on Arrangements consists of Mr. F. C. Jonah, Mary V. Kenny '26, Hazel M. Gilbert '27, Leslie T. Fagan '26, Harvey J. Ollsen '28.

The following program is announced:

November 10, 1925. "Women as mathematicians" by Edith S. Remington '26. "Mathematica Gothica" by Joseph A. Yates '27. "The cell of the honey bee" by B. W. Shaw '28.

December 1. "Fermat" by Mildred V. Mott '27. "Graphic solution of cubic and biquadratic equations" by Jacob Goodman '26.

January 12, 1926. "Aspects of analysis situs" by Professor Marston Morse.

February 23. "Archimedes" by Nellie C. Morton '27. "Descartes Geometry of 1637" by Harvey J. Ollsen '28. "Geometrical constructions with compasses alone" by Winifred F. Pine '26.

March 23. "Too little mathematics and too much" by Professor E. B. Wilson, Harvard School of Public Health.

April 27. "History of arithmetic" by Lucy G. Russell '26. "Sun dials" by Arthur R. Tebbutt '27.

May. Picnic.

(Report by Mr. Sauté)

The Ibis Club, Women's College of Delaware, Newark, Delaware.

The officers for the year 1924–1925 were:

President	Mary L. Marvel '26
Vice-president	Virginia Chapman '26
Secretary-treasurer	Geraldine Messick '27
Chairman of program committee	Madeline Winthrup '25
Faculty Adviser	Professor C. J. Rees.

The Ibis Club holds regular meetings every two weeks. The general subject for the year 1924–1925 was "The history of mathematics."

The calendar for the year was as follows:

December 22, 1924. "History of mathematics in Delaware College" by Dr. Harter.

January 16, 1925. "Origin of mathematical symbols" by Tacy Hurst.

February 19. "Nichomachus of Gerasa" by Florence Wilson.

March 5. "The Rhind papyrus" by Frances Goldstein.

March 19. "Isaac Newton" by Francis Ecbert.

April 2. "Felix Klein" by Sara Slaughter.

April 16. "The school mystic and its miraculous master" by Merrel Pyle.

May 3. Annual banquet. Speaker from the University of Pennsylvania.

May 21. "Tartaglia and Cardan" by Mary L. Marvel.

At each meeting the president introduced a mathematical recreation which followed the discussion of the prepared lectures.

(Report by Miss Marvel)

The Mathematics Club of Hood College, Frederick, Maryland [*1924*, 399]

The officers for 1924–1925 were the following:

President	Irene Sauserman '25
Secretary-treasurer	Lulu Brant '26
Reporter	Evelyn Ditto '26.

The program for the year was as follows:

October 13, 1924. Initiation of new members. Address of welcome by Irene Sauserman '25, president.

November 10. "The theory of relativity" by Evelyn P. Wiggin, instructor.

December 8. Debate. Resolved that the subject matter of the course in mathematics in the elementary and secondary schools of the United States is better adapted to the needs of the pupil than the corresponding courses in foreign countries.

Affirmative: Flora Morton '25, Catherine Hess '26, Lottie Yohn '27.

Negative: Beatrice Hood '26, Effie Holstein '26, Louise Wright '26.

January 12, 1925. "The aims and value of mathematics" by Ruth Reiver '25. "Cancellation of nines" by Hilda Rickard '25.

February 9. "Eclipses" by Professor Lillian O. Brown.

March 9. "Flatland" by Mary H. Moomau '25.

March 30. "The Calendar" by Grace Brown '25.

April 11. "Napier" by Dorothy Kieny '27. "Drawing animals by means of coordinates" by Anna L. Schaidt '27.

May 10. Annual picnic. (Report by Professor Packer)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3181. Proposed by C. K. Robbins, Purdue University.

Solve each of the following differential equations:

$$(1)\quad (p-ay)^a=c(p-by)^b\ ;$$
$$(2)\quad p-ay=ce^{ay/(p-ay)}\ ;$$
$$(3)\quad p^2-2apy+(a^2+b^2)y^2e^{(2a/b)\tan^{-1}(p-ay)/by}=c\ ,$$

where $p=dy/dx$ and a, b, c are given constants.

3182. Proposed by D. H. Lehmer, University of California.

Prove the following two theorems and show how they may be used to advantage in finding the factors of R.

THEOREM 1. Let R be a non-square integer of the form $8n+k$ and represent by $2^\alpha(2m+1)$ any even denominator of a complete quotient occurring in the expansion of $\sqrt{R}$ in a continued fraction. Then, if $K=1$, $\alpha\geqq 3$; if $K=4$ or 0, $\alpha\geqq 2$; if $K=5$, $\alpha\geqq 2$; and if $K=2$, 3, 6, or 7, $\alpha=1$.

THEOREM 2. If R contains a square factor, C^2, then every multiple of C appearing as a denominator of a complete quotient must also contain a factor C^2.

3183. Proposed by Nathan Altshiller-Court, University of Oklahoma.

One of two given circles is fixed, the other rolls on a fixed straight line. Find (1) the locus of the trace of the radical axis of the two circles on their line of centers; (2) the envelope of the radical axis; (3) the locus of the limiting points of the two circles.

3184. Proposed by Otto Dunkel, Washington University.

Reduce to a product the determinant

$$\begin{vmatrix} \cdots & 1 & n & \cdots & n^{p_i} & \cdots \\ \cdots & a_i & (n+1)a_i & \cdots & (n+1)^{p_i}a_i & \cdots \\ \cdots & a_i^2 & (n+2)a_i^2 & \cdots & (n+2)^{p_i}a_i^2 & \cdots \\ & \vdots & \vdots & & \vdots & \\ \cdots & a_i^{r-1} & (n+r-1)a_i^{r-1} & \cdots & (n+r-1)^{p_i}a_i^{r-1} & \cdots \end{vmatrix},$$

where the ith group of p_i+1 consecutive columns are indicated, where there are k such groups such that $p_1+p_2+\cdots+p_k+k=r$ and where n is any number, which may, in fact, have a value in any group of columns different from that in any other group of columns without affecting the result.

3185. Proposed by B. C. Wong, Berkeley, California.

Prove geometrically that the tangent to the cardioid $\rho=2a(1-\cos\theta)$ and the bisector of the vectorial angle of the point of contact meet on the cissoid $\rho=2a\sin\theta\tan\theta$.

3186. Proposed by A. A. Bennett, Lehigh University.

Let the twelve letters, $a, b, c, \ldots k, l$, denote in some unknown order, the twelve distinct residue classes taken modulo 12. Assume that the complete multiplication of 144 products, giving each product as a letter of the set, is known, but nothing further is given concerning the sum or difference of two letters. Show that each residue class is completely identifiable. Determine whether or not a like theorem holds for the case when the modulus is 16, 18, or 24.

SOLUTIONS

501 [1916, 341]. Proposed by R. P. Baker, University of Iowa.

Find the minimum amount of lumber one inch thick required to pack a gross of spheres three inches in diameter in a rectangular box.

Solution by Norman Anning, University of Michigan.

Assume that the spheres are laid in r layers, each layer containing q rows and each row containing p spheres and let the rows and layers be "staggered" so that, in places remote from the boundary, each sphere touches 12 others. That this arrangement of uniform spheres yields maximum economy of space is well known. It is assumed moreover that $p\geqq 2$, $q\geqq 2$, $r\geqq 2$.

Consider four equal spheres each of which touches the other three; their centers will be the vertices of a regular tetrahedron. Let a be the edge of this tetrahedron; b, the altitude of any face; c, the altitude of the tetrahedron. Then a is the distance between adjacent spheres in a row, b is the distance between adjacent rows in a layer, and c is the distance between adjacent layers. When $a=3$, $b=a(\sqrt{3}/2)=2.59808$, $c=a(\sqrt{6}/3)=2.44949$.

A study of plan and elevation reveals the fact that, in order to allow for the staggering mentioned above, the inside dimensions of a box must be $a+(p-1)a+(a/2)$, $a+(q-1)b+(b/3)$, $a+(r-1)c$. Our problem becomes the following: to choose positive integers p, q, r, subject to the relation $pqr\geqq 144$, such that $(3p+1.5+2)(2.59808q+1.26795+2)(2.44949r+0.55051+2)-(3p+1.5)(2.59808q+1.26795)(2.44949r+0.55051)$ shall be a minimum. The sign "$\geqq$" is used above rather than "$=$" because there is no *a priori* reason why a box designed for 150 spheres should not require less material than one designed for just 144.

It is found by trial that the minimum occurs when we put $p=4$, $q=6$, $r=6$. The minimum amount of lumber is 1571.3 cubic inches.

By-product: when $p=5$, $q=5$, $r=6$, the amount of wood required is 1600.5 cubic inches; thus a 2% increase in the amount of material used permits a 4% increase in the capacity of the box.

2694 [1918, 170]. Proposed by N. P. Pandya, Sojitra, India.

Find the locus of the centroid of a triangle, whose vertex lies on a given parabola, whose base of given length is a segment of a given straight line of unlimited length, and one of whose base angles is known.

Solution by Harry Langman, Brooklyn, New York.

Let the given side $BA=c$ lie upon the fixed line L, and let the variable triangle have the fixed angle BAP. Take L with the positive sense BA as the x axis and a line parallel to AP as the y axis. If the coördinates of P, a point on the curve, are (x, y) and the coördinates of the centroid of BAP are (x', y') then

$$x=x'+\frac{c}{3}, \qquad y=3y'.$$

If the given conic has the equation

$$lx^2+2mxy+ny^2+2l'x+2m'y+n'=0\ ,$$

it is an ellipse, hyperbola, or parabola according as $ln-m^2$ is positive, negative, or zero. This property is unchanged by the above substitution, and hence the locus is the same kind of conic as the given conic. These facts also follow directly from the form of the transformation by considering its geometric significance.

2770 [1919, 171]. Proposed by A. M. Harding, University of Arkansas.

Solve the differential equation

$$\frac{d^3x}{dt^3}-2(\mu t+\lambda)\frac{dx}{dt}+(1-\mu)x=0$$

where μ and λ are constants.

SOLUTION BY R. H. SCIOBERETI, Berkeley, California.

This differential equation may be replaced by the following system

$$\frac{dx}{dt}=y\ ;\qquad \frac{dy}{dt}=z\ ;\qquad \frac{dz}{dt}=2(\mu t+\lambda)y+(\mu-1)x\ . \tag{1}$$

For $\mu=0$, it reduces to

$$\frac{dx}{y}=\frac{dy}{z}=\frac{dz}{2\lambda y-x}=dt\ , \tag{2}$$

whose general integral is

$$\left.\begin{aligned} x&=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}\\ y&=C_1\lambda_1e^{\lambda_1 t}+C_2\lambda_2e^{\lambda_2 t}+C_3\lambda_3e^{\lambda_3 t}\\ z&=C_1\lambda_1^2e^{\lambda_1 t}+C_2\lambda_2^2e^{\lambda_2 t}+C_3\lambda_3^2e^{\lambda_3 t}\end{aligned}\right\}, \tag{3}$$

where C_1, C_2, C_3 are three arbitrary constants and λ_1, λ_2, λ_3 the three roots of the equation $\alpha^3-2\lambda\alpha+1=0$, if these roots are all distinct. If we now consider the integral which reduces to x_0, y_0, z_0 for $t=0$, then the three constants C_i will be determined by the three equations:

$$x_0=\overset{0}{C_1}+\overset{0}{C_2}+\overset{0}{C_3}\ ;\qquad y_0=\lambda_1\overset{0}{C_1}+\lambda_2\overset{0}{C_2}+\lambda_3\overset{0}{C_3}\ ;\qquad z_0=\lambda_1^2\overset{0}{C_1}+\lambda_2^2\overset{0}{C_2}+\lambda_3^2\overset{0}{C_3}\ .$$

It will be possible to vary these arbitrary constants in such a way that equations (3) will be a solution of system (1). These new variables $C_i(t)$ must however satisfy the two following conditions: for $t=0$, they must reduce to $\overset{0}{C_1}$, $\overset{0}{C_2}$, $\overset{0}{C_3}$ respectively, and satisfy the differential equations obtained by the substitution of (3) in the original system (1); we thus find:

$$\frac{\partial x}{\partial t}+\sum_i\frac{\partial x}{\partial C_i}\frac{dC_i}{dt}=\sum_i C_i\lambda_ie^{\lambda_i t}$$

$$\frac{\partial y}{\partial t}+\sum_i\frac{\partial y}{\partial C_i}\frac{dC_i}{dt}=\sum_i C_i\lambda_i^2e^{\lambda_i t}$$

$$\frac{\partial z}{dt}+\sum_i\frac{\partial z}{\partial C_i}\frac{dC_i}{dt}=(2\mu t+\lambda)\sum_i C_i\lambda_ie^{\lambda_i t}+(\mu-1)\sum_i C_ie^{\lambda_i t}\ ,$$

which by virtue of (3) reduce to the following:

$$\sum_i e^{\lambda_i t}\frac{dC_i}{dt}=0\ ,\qquad \sum_i e^{\lambda_i t}\lambda_i\frac{dC_i}{dt}=0\ ,\qquad \sum_i e^{\lambda_i t}\lambda_i^2\frac{dC_i}{dt}=\mu\sum_i C_ie^{\lambda_i t}(2\lambda_it+1)\ .$$

Solving these with respect to the derivatives, we shall obtain

$$\frac{dC_k}{dt}=\beta_k\mu\, e^{-\lambda_k t}\sum_i C_ie^{\lambda_i t}(2\lambda_it+1)\ ,\qquad (k=1,2,3) \tag{4}$$

where the β_k have the following constant values:

$$\beta_1=\frac{1}{(\lambda_1-\lambda_2)((\lambda_1-\lambda_3)}\,,\quad \beta_2=\frac{1}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}\,,\quad \beta_3=\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\,,$$

The system of differential equations (4) has the following properties:
i) for $\mu=0$, its integral reduces to $C_k=C_k^0$,
ii) the functions in the right members having finite partial derivatives with respect to C_i and μ satisfy Lipschitz's condition,
iii) the coefficients of the C_i are integral functions of t and of the parameter μ.
Consequently this system may be integrated by Picard's method of successive approximations, the integrals being themselves integral functions of μ.
From (4) we derive the three integral equations

$$C_k=C_k^0+\beta_k\mu\int_0^t[e^{-\lambda_k t}\sum_i C_i e^{\lambda_i t}(2\lambda_i t+1)]dt \qquad (k=1,2,3)$$

in the right members of which the particular solutions $C_k=\overset{0}{C_k}$ are substituted as a first approximation; hence

$$C_k^1=C_k^0+\beta_k\mu\int_0^t[e^{-\lambda_k t}\sum_i C_i^0 e^{\lambda_i t}(2\lambda_i t+1)]dt\ ;$$

for a second approximation we put C_k^1 under the integral sign and repeat this operation n times, obtaining in this manner the following expression:

$$C_k^n=C_k^0+\beta_k\mu\int_0^t[e^{-\lambda_k t}\sum_i C_i^{n-1}e^{\lambda_i t}(2\lambda_i t+1)]dt$$

Now when we let n approach ∞, C_k^n will converge toward limits $C_k(t,\mu)$ which form the integral of system (4), reducing to C_k^0 for $t=0$.

REMARK.—It is also possible to derive the integrals of (1), which will take given initial values by a direct substitution in (1) of series of the form

$$\left.\begin{aligned} x&=x_0(t)+\sum_k x_k(t)\mu^k\\ y&=y_0(t)+\sum_k y_k(t)\mu^k\\ z&=z_0(t)+\sum_k z_k(t)\mu^k \end{aligned}\right\}. \qquad (5)$$

The functions $x_0(t)$, $y_0(t)$ and $z_0(t)$ must take the given initial values for $t=0$, while $x_k(t)$, $y_k(t)$, $z_k(t)$, $(k\geqq 1)$ must vanish for $t=0$. The coefficients of (5) are determined by systems of linear differential equations. Furthermore, since the coefficients of (1) are also linear in μ, an auxiliary system of the form

$$\frac{dX}{dt}=\frac{dY}{dt}=\frac{dZ}{dt}=(A+B\mu)(X+Y+Z)$$

may be chosen in such a way that it will be dominant for the given system, the quantities A and B being positive constants. The integrals of this auxiliary system which reduce to x_0, y_0, z_0 for $t=0$, are integral functions of the parameter μ, and consequently the integrals of the proposed system (1) will also be integral functions of μ. (*cf.* H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 58 *et seq.*)

NOTE: If the roots of the equation $\alpha^3-2\lambda\alpha+1=0$ are not all distinct, equations (3) must be replaced by a different set and the solution must be modified accordingly.

2784 [1919, 366]. Proposed by T. H. Gronwall, New York City.

Show that all solutions in integers of $y^2=1+x+x^2+x^3+x^4$ are given by

$$x=-1,\ \ 0,\ \ 3\ ;$$
$$y=\pm 1, \pm 1, \pm 11\ .$$

SOLUTION BY A. A. BENNETT, Lehigh University.

The given equation may be written in two ways

$$\left[x^2+\frac{x}{2}+\frac{\sqrt{5}-1}{4}\right]^2=y^2-\frac{(5-2\sqrt{5})}{4}\left[x+\frac{3+\sqrt{5}}{2}\right]^2, \qquad (1$$

$$\left[x^2+\frac{x}{2}+1\right]^2=y^2+\frac{5x^2}{4}. \qquad (2)$$

Hence if x and y have any real values satisfying the given equation we must have

$$x^2+\frac{x}{2}+\frac{\sqrt{5}-1}{4}\leqq |y|\leqq x^2+\frac{x}{2}+1, \qquad (3)$$

and hence

$$|y|=x^2+\frac{x+a}{2}, \qquad 0<a\leqq 2,$$

If x and y are required to be integers, a must be 2 if x is even, and in this case (2) shows that $x=0$. If x is odd then a must be unity. In this case, since

$$y^2=\left[x^2+\frac{x+1}{2}\right]^2-\frac{(x-3)(x+1)}{4},$$

we must have $x=3$ or $x=-1$. This gives the results stated in the problem.

2873. [1921, 36–1925, 520]. Proposed by D. H. Richert, Bethel College, Newton, Kan.

At B is the enemy's battery. At M_1 a battery is to be placed to silence B. Listening posts are installed at M_1, M_2, M_3, all provided with stop-watches. From the maps at hand, the three sides of the triangle $M_1M_2M_3$ are known. B is not visible from any one of the points M_1, M_2, M_3. The sound of a gun fired at B reaches M_1 at the time T, and M_2 at the time $T+\tau_1$ sec., and it reaches M_3 at the time $T+\tau_2$ sec. How far is B from M_1?

SOLUTION BY NORMAN ANNING, University of Michigan.

Take as unit of distance the distance travelled by sound in air in one second at the time and place in question and take as zero of time the instant at which the gun was fired. Then $BM_1=T$; $BM_2=T+\tau_1$; $BM_3=T+\tau_2$. It is assumed that B is in the plane of the triangle $M_1M_2M_3$ and that the known distances M_2M_3, M_3M_1, M_1M_2 are respectively t_1, t_2, t_3.

Connecting the six distances of the four coplanar points there exists the identical relation:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & T^2 & (T+\tau_1)^2 & (T+\tau_2)^2 \\ 1 & T^2 & 0 & t_3^2 & t_2^2 \\ 1 & (T+\tau_1)^2 & t_3^2 & 0 & t_1^2 \\ 1 & (T+\tau_2)^2 & t_2^2 & t_1^2 & 0 \end{vmatrix}=0.$$

From this equation, the value of T can be found in terms of the quantities τ_1, τ_2, t_1, t_2, t_3, which are all known.

Lest the enemy should get the range of M_1 while the above equation is being solved it is better to proceed as follows:

By means of the string construction, describe on the map an arc of the hyperbola in which P moves when $PM_2-PM_1=\tau_1$; likewise an arc of the hyperbola for which $PM_3-PM_1=\tau_2$; and, as a check, an arc of the hyperbola for which $PM_3-PM_2=\tau_2-\tau_1$. Under ideal conditions of observation and measurement, these three curves will be concurrent at the map position of B. Under fire, they will be nearly concurrent and will indicate the approximate position of B. Give to an artillery officer at M_1 the map position of B; the required result will follow.

Salmon states (*Conic Sections*, p. 134) that the identity quoted above is due to Cayley.

NOTE BY OTTO DUNKEL. A geometrical construction may be made to obtain the position of B. The same notation as above will be used. Draw the circles with centers M_2, M_3, and radii r_1, r_2; then B is the center of a circle tangent externally to the circles at P_2, P_3, respectively. It is easily shown that P_2P_3 passes through a center of similitude S of the circles M_2 and M_3; and that if Q_2 and Q_3 are the other two points of intersection, M_2Q_2, M_3P_3 are parallel as also M_2P_2, M_3Q_3. Let the tangents from S to the circles M_2, M_3, B be SN_2, SN_3, SB_1; then it may be shown that $SB_1^2=SN_2.SN_3$. Let the line SM_1 cut the circle B again at M, then a circle C passed through N_2, N_3, and M_1 cuts SM_1 in M, which may be thus found. Let the radical axis of C and M_2 cut SM_1 in R_2, and, similarly, let the radical axis of C and M_3 cut SM_1 in R_3. Then the tangent from R_2 to M_2 touches the circle at P_2, likewise the tangent from R_3 to M_3 touches at P_3. We have now merely to draw M_2P_2 and M_3P_3 in order to locate B as their intersection.

2938. [1921, 467]. Proposed by C. F. Gummer, Queen's University.

If $a, b, \ldots, i$ are real numbers $\geqq 0$, and if $\begin{vmatrix} a^r & b^r & c^r \\ d^r & e^r & f^r \\ g^r & h^r & i^r \end{vmatrix}$ is equal to zero for five real values of r other than zero, prove that the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ has either two rows or two columns proportional, or a single row or column of zeros.

SOLUTION BY OTTO DUNKEL, Washington University.

Set the first determinant of the problem equal to $D(r)$, then the second determinant is $D(1)$. If $D(1)$ has a column or row of zeros the theorem is true. Suppose next that it has a column or row with two zeros and the third element is not zero, then by interchange of columns and rows this column or row may be made the first one with the non-vanishing element in the upper left hand corner, i.e. $a\neq 0$. Hence in this case $D(r)=a^r[(ei)^r-(hf)^r]$, and since it is zero for a value of $r\neq 0$, $ei-hf=0$. In this case the theorem is again true.

We may now assume that no two elements of a column or row are zero. If only one element of $D(1)$ is zero we may suppose it to be a. If two elements are zero we may suppose that one of these elements is a, and in this case no one of the elements b, c, d, g can be zero. Hence the second element must be e, f, h, or i. By suitable changes this zero element can be brought to the position of e, and so we shall suppose it to be e. In this case neither f nor h can be zero. Finally if there is a third zero element it must be i if the other two zeros are placed as before. With this understanding of the position of the zeros

$$D(r)=A_1^r+A_2^r+A_3^r-A_4^r-A_5^r-A_6^r \tag{1}$$

where

$$\begin{aligned} A_1&=aei\ , & A_4&=ahf\ , \\ A_2&=dhc\ , & A_5&=dbi\ , \\ A_3&=gbf\ , & A_6&=gec\ . \end{aligned} \tag{2}$$

We have the following cases to consider: I. no one of the six terms (2) is zero; II. the only terms of (2) which are zero are A_1 and A_4; III. the only zero terms are A_1, A_4, and A_6; IV. the only zero terms are A_1, A_4, A_5, A_6. This last case must be excluded since A_2 and A_3 have the same sign and $D(r)$ cannot vanish.

It will be convenient now to prove the

LEMMA. *If $A_1, A_2, \cdots, A_k$ are real quantities greater than zero, and $c_1, c_2, \cdots, c_k$ are also real and no one of them is zero, and if*

$$c_1A_1^x+c_2A_2^x+\cdot\cdot\cdot+c_kA_k^x \tag{3}$$

vanishes for k finite values of x, then $A_1=A_2=\cdots=A_k$.

PROOF. Suppose that (3) is zero for k values of x, then the derivative of

$$c_1+c_2\left(\frac{A_2}{A_1}\right)^x+\cdots+c_k\left(\frac{A_k}{A_1}\right)^x$$

vanishes for $k-1$ values of x. Thus

$$c_2\log\left(\frac{A_2}{A_1}\right)+c_3\log\left(\frac{A_3}{A_1}\right)\left(\frac{A_3}{A_2}\right)^x+\cdots+c_k\log\left(\frac{A_k}{A_1}\right)\left(\frac{A_k}{A_2}\right)^x$$

vanishes for $k-1$ values of x. Then by the same reasoning

$$c_3\log\left(\frac{A_3}{A_1}\right)\log\left(\frac{A_3}{A_2}\right)+c_4\log\left(\frac{A_4}{A_1}\right)\log\left(\frac{A_4}{A_2}\right)\left(\frac{A_4}{A_3}\right)^x+\cdots+c_k\log\left(\frac{A_k}{A_1}\right)\log\left(\frac{A_k}{A_2}\right)\left(\frac{A_k}{A_3}\right)^x$$

vanishes for $k-2$ values of x. Finally

$$c_k\log\left(\frac{A_k}{A_1}\right)\log\left(\frac{A_k}{A_2}\right)\cdots\log\left(\frac{A_k}{A_{k-1}}\right)$$

vanishes for one value of x. Hence at least one pair of the A's are equal. The terms of (3) having equal A's can be combined and we have an expression of form (3) with fewer than k terms which vanishes for k distinct values of x. By repetition of this process we finally find that, if (3) vanishes for k distinct values of x, the A's must all be equal.

Now in case I, $D(0)=0$ and hence $D(r)$ vanishes for six values of r and it follows from the lemma that all of the A's in (2) are equal. This says that $a(ei-hf)=0$ etc. and hence the second and third columns are proportional to the first. In case II we have

$$A_2^r+A_3^r-A_5^r-A_6^r$$

is zero for four values of r, and hence $a=0$, $bf-ec=hc-bi=0$. Thus the last two columns are proportional. In case III we have $A_2^r+A_3^r-A_5^r$ which by the lemma can vanish for three values of r only if $A_2=A_3=A_5=0$, which is contrary to hypothesis. This completes the proof of the theorem of the problem.

3131. [1925, 204]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Find the locus of the mid-point of the segment intercepted by two fixed tangents to a given conic on a variable tangent to the same conic.

SOLUTION BY T. W. EDMONDSON, New York University

Taking the two fixed tangents as axes of coördinates, the equation of the conic may be written

$$g^2x^2+2chxy+f^2y^2+2cgx+2cfy+c^2=0\ .$$

Let the variable tangent be $lx+my+n=0$. Then the tangential equation of the conic is

$$\begin{vmatrix} g^2 & ch & cg & l \\ ch & f^2 & cf & m \\ cg & cf & c^2 & n \\ l & m & n & 0 \end{vmatrix}=0\ ,$$

which reduces to

$$(fg+ch)n^2-2cgmn-2cfnl+2c^2lm=0\ ,$$

if we remove the factor $(ch-fg)$, which is not zero since the conic is assumed to be non-degenerate.

The coördinates of the mid-point of the segment intercepted on the variable tangent are

$$x=-\frac{n}{2l}\ ,\ y=-\frac{n}{2m}$$

whence

$$l=-\frac{n}{2x}\,, \qquad m=-\frac{n}{2y}\,.$$

Substituting these values in the tangential equation, we find that the equation of the locus of the mid-point is

$$2(fg+ch)xy+2cgx+2cfy+c^2=0,$$

a hyperbola whose asymptotes are parallel to the two fixed tangents.

Also solved by MICHAEL GOLDBERG, M. S. KNEBELMAN, F. H. LOUD, and AUGUST SÖRENSEN.

3132. [1925, 204]. Proposed by J. W. Clawson, Ursinus College.

The trilinear coördinates of the circumcenter of a triangle are: $\cos A$, $\cos B$, $\cos C$; that is, the distances of this point from the sides of the triangle are proportional to $\cos A$, $\cos B$, and $\cos C$ respectively. The points ($\sin A$, $\sin B$, $\sin C$), ($\sec A$, $\sec B$, $\sec C$), and ($\csc A$, $\csc B$, $\csc C$) are also well-known points (symmedian, orthocenter, and centroid) which can be located by simple geometrical constructions.

Can the points ($\tan A$, $\tan B$, $\tan C$) and ($\cot A$, $\cot B$, $\cot C$) be found by simple geometrical constructions? Have the points been named?

SOLUTION BY MICHAEL GOLDBERG, Philadelphia, Penn.

The point ($\tan A$, $\tan B$, $\tan C$) may be constructed as follows: Draw a line parallel to AB at a distance $\tan C$ from it. To obtain $\tan C$ lay off along one side of the angle C an arbitrary unit length and erect a perpendicular at its end to meet the other side. Draw a line parallel to BC at a distance $\tan A$ intersecting the first parallel at D; similarly, a parallel to AC intersecting the first parallel at E. The desired point is the intersection of AE and BD.

The construction for ($\cot A$, $\cot B$, $\cot C$) is similar. I am not aware that these points have special names.

3135 [1925, 261]. Proposed by N. P. Pandya, Amreli, Kathiawad, India.

Form the equation of lowest degree with rational coefficients whose roots are $\sin A$, $\sin 4A$, $\sin 7A$, and $\sin 10A$, where $13A=\pi$.

SOLUTION BY A. G. CLARK, Colorado Agricultural College.

The twelve imaginary roots of unity satisfy the equation

$$x^{12}+x^{11}+\cdots+x+1=0\,. \tag{1}$$

Setting $y=x+x^{-1}$ we obtain the equation

$$y^6+y^5-5y^4-4y^3+6y^2+3y-1=0\,, \tag{2}$$

whose roots are $2\cos 2nA$, $n=1, 2, 3, 4, 5, 6$, $A=\pi/13$. Since $2\cos 2nA=2-4\sin^2 nA$, if we reduce the roots of (2) by 2, and divide the roots of the resulting equation by -4, we obtain the equation for $\sin^2 nA$. The equation $4096z^6-13312z^5+16640z^4-9984z^3+2912z^2-364z+13=0$, is satisfied by $\sin^2 A$, $\sin^2 4A$, $\sin^2 7A=\sin^2 6A$, $\sin^2 10A=\sin^2 3A$, $\sin^2 2A$, $\sin^2 5A$.

NOTE BY OTTO DUNKEL, Washington University.

By a method given by Gauss the equation (2) may be solved by solving first the cubic

$$x^3+x^2-4x+1=0\,, \tag{4}$$

all of whose roots are real, and then by solving quadratics of the form

$$x^2-x_1x+x_3=0\,,$$

where x_1 is the smaller positive root and x_3 the negative root of (4) [see Burnside and Panton's *Theory of Equations*, vol. 1, ex. 15, page 101].

The equation (2) may also be reduced to two cubics

$$\begin{aligned} 2y^3+(1-\sqrt{13})\, y^2-2y-3+\sqrt{13}=0\ , \\ 2y^3+(1+\sqrt{13})\, y^2-2y-3-\sqrt{13}=0\ . \end{aligned} \tag{5}$$

This may be verified by taking the product of the left sides of the two equations. The roots of the first are $2\cos 2A$, $-2\cos 5A$, $2\cos 6A$; the roots of the second are $2\cos 4A$, $-2\cos 3A$, $-2\cos A$.

It should be observed that in the above solution it has not been shown that there is no equation of lower degree having the desired roots.

Also solved by MICHAEL GOLDBERG.

3136 [1925, 261]. Proposed by Paul Capron, U. S. Naval Academy.

If the sines of the half sides of a spherical triangle are h, k, l, and the cosines of the halves of the corresponding opposite angles are H, K, L, and if P and p are the polar distances of the circumscribed and inscribed circles then

$$\sin P=\frac{hkl}{2\sqrt{\sigma(\sigma-h)(\sigma-k)(\sigma-l)}}\ , \qquad \cos p=\frac{HKL}{2\sqrt{\Sigma(\Sigma-H)(\Sigma-K)(\Sigma-L)}}\ ,$$

where

$$\sigma=(h+k+l)/2 \quad \text{and} \quad \Sigma=(H+K+L)/2\ .$$

SOLUTION BY R. H. SCIOBERETI, Berkeley, California.

Let us start from the two fundamental formulae

$$\tan P=\sqrt{\frac{\sin S}{\sin(A-S)\sin(B-S)\sin(C-S)}}=\frac{\sin S}{N} \tag{1}$$

and

$$\cos P=\sqrt{\frac{\tan(A-S)\tan(B-S)\tan(C-S)}{\tan(A-S)+\tan(B-S)+\tan(C-S)}} \tag{2}$$

where $2S$ represents the spherical excess, and

$$N=\sqrt{\sin S\sin(A-S)\sin(B-S)\sin(C-S)},\quad \text{(norm of the angles)}$$

$$n=\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)},\quad \text{(norm of the sides)},$$

$$s=(a+b+c)/2\ .$$

We shall express all the trigonometric functions of the preceding angles by means of the trigonometric functions of the sides. From

$$\tan\tfrac{1}{2}a=\sqrt{\frac{\sin S\ \sin(A-S)}{\sin(B-S)\sin(C-S)}}\ , \qquad \tan\tfrac{1}{2}b=\sqrt{\frac{\sin S\ \sin(B-S)}{\sin(C-S)\sin(A-S)}}\ , \text{ and}$$

$$\tan\tfrac{1}{2}c=\sqrt{\frac{\sin S\sin(C-S)}{\sin(A-S)\sin(B-S)}}\ ,$$

we obtain by multiplication

$$\tan\tfrac{1}{2}a\cdot\tan\tfrac{1}{2}b\cdot\tan\tfrac{1}{2}c=\frac{\sin^2 S}{N}\ ,$$

and by comparison with equation (1)

$$\tan P=\frac{\tan\tfrac{1}{2}a\cdot\tan\tfrac{1}{2}b\cdot\tan\tfrac{1}{2}c}{\sin S}.$$

Now from Cagnoli's formula giving the spherical excess as a function of the sides

$$\sin S=\frac{n}{2\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c} \tag{4}$$

we easily derive the following expressions:

$$\left.\begin{aligned} \sin(A-S) &= \frac{n}{2\cos\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c} \\ \cos(A-S) &= \frac{\sin^2\frac{1}{2}b+\sin^2\frac{1}{2}c-\sin^2\frac{1}{2}a}{2\cos\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c} \end{aligned}\right\}$$

and similar relations for $(B-S)$, $(C-S)$; hence

$$\tan(A-S)=\frac{n}{\sin^2\frac{1}{2}b+\sin^2\frac{1}{2}c-\sin^2\frac{1}{2}a}=\frac{n}{k^2+l^2-h^2},$$

$$\tan(B-S)=\frac{n}{l^2+h^2-k^2}, \quad \tan(C-S)=\frac{n}{h^2+k^2-l^2}.$$

Substituting these values in equation (2), we shall have

$$\cos P=\sqrt{\frac{\dfrac{n^3}{(k^2+l^2-h^2)(l^2+h^2-k^2)(h^2+k^2-l^2)}}{n\left[\dfrac{1}{k^2+l^2-h^2}+\dfrac{1}{l^2+h^2-k^2}+\dfrac{1}{h^2+k^2-l^2}\right]}}, \tag{5}$$

$$\tan P=\frac{2\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c}{n}=\frac{2hkl}{n}. \tag{6}$$

Finally if we multiply (5) by (6), we shall obtain the required formula

$$\sin P=\frac{2hkl}{n}\cdot\frac{n\left[(h^2+k^2-l^2)(k^2+l^2-h^2)(l^2+h^2-k^2)\right]^{-1/2}}{\left[\dfrac{1}{k^2+l^2-h^2}+\dfrac{1}{l^2+h^2-k^2}+\dfrac{1}{h^2+k^2-l^2}\right]^{1/2}},$$

$$\sin P=2\,hkl\left[(h^2+k^2-l^2)(l^2+h^2-k^2)+(k^2+l^2-h^2)(h^2+k^2-l^2)+(l^2+h^2-k^2)(k^2+l^2-h^2)\right]^{-1/2}.$$

The second factor may be written after a few simple transformations

$$\sqrt{4h^2k^2-(h^2+k^2-l^2)^2}=\sqrt{(h+k+l)(k+l-h)(l+h-k)(h+k-l)};$$

hence
$$\sin P=\frac{hkl}{2\sqrt{\sigma(\sigma-h)(\sigma-k)(\sigma-l)}}.$$

NOTE BY OTTO DUNKEL, Washington University.

The two formulae of the problem are consequences of a relation which may be stated thus: If x, y, z are three angles such that the sine of one angle is equal to the sine of the sum of the other two, then

$$4s(s-\sin x)(s-\sin y)(s-\sin z)=\sin^2 x\sin^2 y\sin^2 z, \tag{1}$$

where $2s=\sin x+\sin y+\sin z$.

If O is the center of the circumscribing circle of the triangle ABC and if A', B', C' are the mid-points of the sides, then, if we set $\angle BOA'=x$, $\angle COB'=y$, $\angle AOC'=z$, either the sum of two of the angles is equal to the third or $x+y+z=180°$. Now from the right triangles such as BOA' we have $\sin x=h/\sin P$, and then $s=\sigma/\sin P$, $s-\sin x=(\sigma-h)/\sin P$. The first formula follows by substituting these in (1).

If O is the center of the inscribed circle and A', B', C' are the feet of perpendiculars from O to the sides, we may set $x=\angle BOA'$, $y=\angle COB'$, $z=\angle AOC'$. (These letters have not, of course, the same meanings as in the preceding paragraph.) Here $x+y+z=180°$, and from one of the right triangles such as BOA' we have $\sin x=K/\cos p$, $s=\Sigma/\cos p$, $s-\sin x=(\Sigma-K)/\cos p$, and the second formula follows by substitution in (1). In either case the expression under the radical is easily seen to be positive and the positive sign is to be taken for the radical.

Also solved by J. A. BULLARD and A. PELLETIER.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

At its meeting in April, 1926, the National Academy of Sciences elected as foreign associates Professor JACQUES HADAMARD of the Ecole Polytechnique and the Sorbonne, and Professor MAX PLANCK, director of the Institute of Theoretical Physics of the University of Berlin. Professor OSWALD VEBLEN was elected to the Council of the Academy for a period of three years.

Harvard University has awarded from its Milton Fund for Research a sum of money to Professor E. B. WILSON for expenses connected with investigations of stellar statistics to be pursued in accordance with biometric methods involving partial correlation.

At Princeton University, Professor J. W. ALEXANDER has been promoted to an associate professorship, and Dr. T. Y. THOMAS has been appointed to an assistant professorship of mathematics.

Professor E. T. BELL, of the University of Washington, has been appointed professor of mathematics at the California Institute of Technology.

Associate Professor J. D. BOND, of the University of Tennessee, has been promoted to a full professorship of mathematics.

Dr. LAURA BRANT, of Vassar College, has been appointed professor of physics and mathematics at Judson College, Marion, Ala.

Assistant Professor THOMAS BUCK, of the University of California, has been promoted to an associate professorship of mathematics.

Assistant Professor G. M. CONWELL, of the New York State College for Teachers, has been promoted to a full professorship of mathematics.

Dr. L. M. GRAVES has been appointed assistant professor of mathematics at the University of Chicago.

Assistant Professor GERTRUDE A. HERR, of Iowa State Teachers College, has been promoted to an associate professorship of mathematics.

Dr. JEWELL C. HUGHES, of the University of Arkansas, has been promoted to an assistant professorship of mathematics.

Assistant Professor GLENN JAMES has been projoted to an associate professorship of mathemathics at the Southern Branch of the University of California.

Associate Professor H. E. JORDAN, of the University of Kansas, has been promoted to an associate professorship of mathematics.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7–8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Baltimore, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

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MEETING OF THE SOUTHERN CALIFORNIA SECTION

The third regular meeting of the Southern California Section of the Mathematical Association of America was held at the California Institute of Technology, in Pasadena, on Saturday, March 13, 1926, Professor Harry Bateman presiding.

There were fifty present, including the following twenty-five members of the Association: O. W. Albert, E. E. Allen, H. Bateman, F. P. Brackett, J. R. Campbell, M. Collier, M. E. Conn, P. H. Daus, H. H. Gaver, H. E. Glazier, E. R. Hedrick, H. C. Hicks, G. H. Hunt, W. E. Mason, G. F. McEwen, W. A. Newlin, B. Podolsky, W. P. Russell, H. M. Showman, M. Skarstedt, D. V. Steed, H. C. Van Buskirk, L. E. Wear, Clyde Wolfe, E. R. Worthington.

The following officers were elected for the coming year: Professor H. C. WILLETT, chairman, Professor W. P. RUSSELL, Vice-chairman, Professors E. E. ALLEN and L. E. WEAR, program committee.

The following papers were presented. Short abstracts appear below.

1. "Fibonacci series," by Mr. MORGAN WARD, California Institute of Technology (by invitation).

2. "Function space," by Professor E. R. Hedrick, University of California, Southern Branch.

3. "An example of periodic analysis of river flow," by BORIS PODOLSKY, University of Southern California.

4. "Certain link motions connected with conformal transformations," by Mr. H. C. HICKS, California Institute of Technology.

5. "On imaginary elements in construction problems," by Professor P. H. DAUS, University of California, Southern Branch.

6. "A demonstrative talk on lightning phenomena," by Professor R. W. SORENSON, California Institute of Technology (by invitation).

1. Mr. Ward considered the residues modulo p of the terms of a Fibonacci series and the cycles formed for different moduli. He discussed the generalization to series formed from the difference equation $x_n = px_{n-1} + qx_{n-2}$ and some of the resulting properties.

2. Professor Hedrick discussed the fundamental concepts of function space as developed by Hilbert, Schmidt, and Jackson and showed the relation between these ideas and our ordinary concepts of least squares, standard deviation, vector cross product and others.

3. Mr. Podolsky discussed the analysis of the flow of Owens River, by means of the apparatus he described at a previous meeting.

4. Mr. Hicks described a linkage devised to illustrate the transformation $(\zeta+c)^n/(\zeta-c)^n=(z+c)/(z-c)$, and in particular for the value of $n=2$. If the point ζ moves along a straight line then z describes an airfoil section. This problem arose in connection with the design of aeroplanes.

5. Professor Daus discussed the solution of a problem reprinted in the December issue of the MONTHLY. He described the relations existing between conjugate coaxal systems of circles and imaginary points of a line, and the use of such systems to solve the type of construction problem suggested.

6. The Section was fortunate in having Professor R. W. Sorenson, of the California Institute of Technology to address it on the subject of lightning phenomena. After a very interesting talk, the meeting adjourned to the high tension laboratory, where a series of illustrative experiments had been arranged.

P. H. DAUS, *Secretary-Treasurer*

DECEMBER MEETING OF MARYLAND-VIRGINIA-D. C. SECTION

The eighteenth regular meeting of the Maryland-Virginia-District of Columbia Section was held at George Washington University, Washington, D.C., on December 5, 1925. The chairman, W. D. Lambert, presided at both sessions and the Washington members entertained those attending the meeting at luncheon.

The attendance was 94 and included the following 51 members: O. S. Adams, R. N. Ashmun, Clara L. Bacon, Sarah Beall, W. W. Bigelow, G. A. Bingley, R. F. Borden, C. C. Bramble, J. A. Bullard, G. R. Clements, A. Cohen, A. Dillingham, Jessie B. Edmondson, H. English, J. T. Erwin, P. J. Federico, H. Gwinner, W. M. Hamilton, H. L. Hodgkins, L. S. Hulburt, W. D. Lambert, A. E. Landry, E. A. LeLacheur, Florence P. Lewis, Eugenie M. Morenus, F. Morley, F. D. Murnaghan, C. A. Nelson, G. A. O'Donnell, B. C. Patterson, E. C. Phillips, G. Y. Rainich, O. Ramler, C. H. Rawlins, J. N. Rice, A. W. Richeson, Susan V. Richmond, H. A. Robinson, R. E. Root, G. A. Ross, W. F. Shenton, C. M. Sparrow, T. H. Taliaferro, Evelyn R. Thompson, Marian M. Torrey, J. Tyler, C. E. VanOrstrand, P. S. Wagner, W. J. Wallis, Elizabeth W. Wilson, E. W. Woolard.

At the morning session the following papers were presented:

(1) "The use of a certain general relation for the rapid interpolation of solar rectangular coordinates and similar functions," by Mr. J. E. WILLIS, U. S. Naval Observatory, introduced by Mr. W. M. Hamilton.

(2) "Twenty years' trial of statistical methods in meteorology," by Mr. E. W. WOOLARD, U. S. Weather Bureau.

(3) "Three homographic maps," by Professor FRANK MORLEY, Johns Hopkins University.

(4) "Freshman mathematics for non-technical students", by Professor R. F. BORDEN, George Washington University.

The afternoon session was devoted to relativity. At the invitation of the section Dr. PAUL R. HEYL of the Bureau of Standards spoke on physical aspects of relativity and Professor F. D. MURNAGHAN of Johns Hopkins University on mathematical aspects. Dr. G. Y. RAINICH of Johns Hopkins University led the discussion which followed.

Abstracts of the papers:

1. Mr. Willis gave formulae for the rapid interpolation in the case of certain species of functions where the values of their several members can be represented by the equations

$$F_i(x_n) = M \cdot F_i(x_o) + N \cdot F_i(x_1) .$$

The F's may be any functions whose second derivatives are respectively proportional to them at a given value of the argument. In the computation, $M = m + \Delta m$ and $N = n + \Delta n$. In particular, formulae were given for the cases in which the F's were the rectangular coordinates of a body in orbital motion, $\cos x$ and $\sin x$, or $\cosh x$ and $\sinh x$.

2. Mr. Woolard gave a discussion of the more successful attempts thus far made to apply both graphical and numerical statistical methods to long range forecasting and to the elucidation of the relations between weather at different times and places over the globe. It was concluded that statistical methods have so far been relatively unproductive in meteorology; and that the outlook for the future, particularly as regards the weather problems of the temperate zone, is not hopeful.

3. Professor Morley stated that the homography, $axy + bx + cy + d = 0$ is considered as a correspondence between the points of two maps, M_1, M_2, lying face up in a plane. There are two fixed points and a measure. For three maps, M_1, M_2, M_3, we have 3 homographies whose resultant is identity, 3 pairs of fixed points and 3 measures. On investigating the question as to whether the fixed points determine the 3 homographies, it is found that in general they do.

4. Professor Borden raised the question of what is a desirable course for freshmen who enter with only one year of algebra and take only one year of mathematics in college. He considered it more desirable to give a fairly wide range of ideas and methods with simple applications to clarify them, than to emphasize manipulation of intricate exercises in a narrower field of topics. A suggested course included the usual algebra and trigonometry with em-

phasis on graphical representation, and some elements of the calculus worked in where they could be used to advantage.

5. Dr. Heyl's subject was "The present experimental status of the theory of relativity." The theory of relativity was not founded solely upon the negative result of the Michelson-Morley experiment but upon the joint testimony of a number of different experiments of the same nature. In consequence, if it is now found that the Michelson-Morley experiment gives a positive result we must suspend judgment until these experiments are repeated under the same conditions which apparently altered the result of the Michelson-Morley experiment, namely at a great altitude.

There are six different lines of experiment to be considered. The three confirmatory tests originally suggested by Einstein have now all given a result favorable to his theory. The last of these came into line only recently, that is the shift of the rays of the spectrum, which has lately been confirmed very satisfactorily by means of the companion star of Sirius. The Michelson-Morley experiment appears at present to give a result unfavorable to Einstein while the other two experiments are favorable to Einstein but must be repeated at different altitudes before judgment is reached.

6. In discussing the mathematical aspects of the theory of relativity, Professor Murnaghan called attention to the serious difficulties which confront the student who is introduced to the subject in its historical order of development. In the "special" theory it is perfectly easy to develop the time idea since the assumption as to the constancy of the velocity of light is made. In the general theory no such *a priori* discussion of time can be made and we are forced to an *a posteriori* position. In remarking on Silberstein's discussion of Professor Miller's recent experiments, the speaker pointed out that there is at present an important problem of somewhat the same type outstanding in hydrodynamics; namely, the viscous layer on a body in motion in a fluid. It is entirely possible that a discussion of one of these problems may help to clarify the other although they arise in fields apparently so remote.

J. A. BULLARD, *Secretary-Treasurer*

A NEW METHOD FOR DETERMINING A SERIES SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT OR VARIABLE COEFFICIENTS[1]

By W. O. PENNELL, Southwestern Bell Telephone Co., St. Louis, Mo.

1. Introduction. This paper describes a method for determining a power series solution of linear differential equations with constant or variable coefficients by a method of operational division. Operational methods for the solution of linear equations with constant coefficients are well known having been developed by Oliver Heaviside and many others.[2] The method outlined in this paper is a general method which is applicable not only to equations with constant coefficients but also to equations with variable coefficients, and includes Heaviside's method as a special case. So far as the present investigation has developed it appears that this method of solution is, in general, applicable to all types of linear differential equations. Cases may arise where the method involves the solution of an algebraic equation of degree higher than the second for which only a numerical approximation can be obtained.

The method, in general, consists of replacing the differential notation d^n/dx^n by p^n in the differential equation and then obtaining the value for the dependent variable by an algebraic solution which is called the operational solution. The series solution is then obtained by a method of operational division which is the key to the process.

2. Notation. In this paper the following notations will be used: $p(\ )$ denotes $d/dx(\ )$, $p^n(\ )$ denotes $d^n/dx^n(\ )$, $1/p(\ )$ denotes $\int(\ )dx$. $1/p^2(\ )$ denotes $\int\int(\ )\,dxdx$ etc. For example, px^n is interpreted as nx^{n-1} $1/p$ is x, x^n/p is $x^{n+1}/(n+1)$, $(1/p)e^x$ is e^x, etc. Attention is called to the fact that in this definition of $1/p$ the constant of integration is omitted.

The symbol $\backsimeq$ will be used to denote operational equivalence. For example, $(1/p)\backsimeq x$, $(1/p^n)\backsimeq(x^n/n!)$. Equations of operational equivalence are not subject to the laws of ordinary algebraic equations.

3. Linear equations with variable coefficients. *Simple operational division.* The operational solution of the differential equation is first obtained

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[1] Read at the Kansas City meeting of the Association, December 31, 1925. This paper has been confined to ordinary linear differential equations. The methods are also applicable to partial linear differential equations. The operational division described, or an adaptation from it, is applicable to many non-linear differential equations. The writer wishes to acknowledge his indebtedness to Mr. H. R. Fritz who has read the paper and made helpful suggestions; also to Professor W. H. Roever, of Washington University, for many helpful criticisms and comments.

[2] The underlying principles of Heaviside's work as well as those involved here are symbolic in character. A fairly complete bibliography by Mr. Eugene Stephens, of the literature on symbolic methods, is found in *Washington University Studies, volume XII, Scientific Series no. 2*, pp. 137–52, 1925.

and then the algebraic solution is found by a process of operational division. This can perhaps best be made clear by a simple example.

Take the equation

$$\frac{dy}{dx}+x^{1/2}y=1\,,\quad \text{or}\quad y\eqsim\frac{1}{p+x^{1/2}}\,\cdot \tag{1}$$

(1) is the operational solution. To obtain the actual solution, divide the right hand member of (1) by long division as follows:

$$\begin{array}{l|l|l}
p+x^{1/2} & 1 & \frac{1}{p}-\frac{x^{3/2}}{p}+\frac{2}{5}\frac{x^3}{p}-\frac{2x^{9/2}}{4.5p}+\cdots \\
 & \underline{1+x^{3/2}} & x-\frac{2}{5}x^{5/2}+\frac{2.x^4}{4.5}-\frac{2^2x^{11/2}}{4.5.11}+\cdots \\
 & -x^{3/2} & \\
 & \underline{-x^{3/2}-\frac{2}{5}x^3} & \\
 & \frac{2}{5}x^3 & \\
 & \underline{\frac{2}{5}x^3+\frac{2x^{9/2}}{4.5}} & \\
 & -\frac{2x^{9/2}}{4.5} &
\end{array}$$

Two lines are given in the quotient the upper being the operational terms obtained by division and the lower the algebraic equivalents which are used when multiplying the second term in the divisor.

$$y=x-\tfrac{2}{5}x^{5/2}+\frac{2x^4}{4\,.5}-\frac{2^2x^{11/2}}{4\,.5\,.11}+\cdots$$

is a solution, *i. e.*, a particular integral. The complete primitive is obtained by adding to this the solution of

$$\frac{dy}{dx}+x^{1/2}y=0 \quad \text{which is}\quad y=ce^{(-2/3)x^{3/2}}.$$

Hence the complete primitive is

$$y=ce^{(-2/3)x^{3/2}}+x-\tfrac{2}{5}x^{5/2}+\frac{2x^4}{4\,.5}-\frac{2^2x^{11/2}}{4\,.5\,.11}+\cdots$$

In the above equation (1) the complementary function can also be obtained in an operational manner by putting $py+x^{1/2}y\eqsim cp$, where c is a constant. This gives

$$y\eqsim\frac{cp}{p+x^{1/2}}\,\cdot$$

The result by operational division is the same as above, $y=ce^{(-2/3)x^{3/2}}$.

In general, the complementary function will be obtained by substituting for zero some quantity which is equivalent to zero and using such quantity

as a dividend, the coefficient of y being the divisor in operational division. For example, cp in the above illustration is equal to zero because $cp=c(d/dx)1=0$. The details for obtaining these quantities will be given later.

In the operational division the terms of the divisor were arranged in descending powers of p. If we arrange them in ascending powers, the division is as follows:

$$
\begin{array}{l|l|l}
x^{1/2}+p & 1 & x^{-1/2}+\frac{1}{2}x^{-2}+\frac{1}{2}.2x^{-7/2}+\frac{1}{2}.2.\frac{7}{2}x^{-5}+\cdots \\
 & 1-\frac{1}{2}x^{-3/2} & \\
 & \quad\frac{1}{2}x^{-3/2} & \\
 & \quad\frac{1}{2}x^{-3/2}-\frac{1}{2}.2x^{-3} & \\
 & \qquad\frac{1}{2}.2x^{-3} & \\
 & \qquad\frac{1}{2}.2x^{-3}-\frac{1}{2}.2.\frac{7}{2}x^{-9/2} & \\
 & \qquad\quad\frac{1}{2}.2.\frac{7}{2}x^{-9/2} &
\end{array}
$$

$$y=x^{-1/2}+\frac{1}{2}x^{-2}+\frac{1}{2}.2x^{-7/2}+\frac{1}{2}.2.\frac{7}{2}x^{-5}+\cdots$$

The above series is divergent.

Usually this operational division will give two formal solutions, a convergent and a divergent series depending upon the arrangement of terms in the divisor and dividend, whether in descending or ascending powers of p.

2. *Equations having two particular integrals.* The above statement regarding two solutions, one convergent and the other divergent, is not universally true. In some cases two solutions are obtained, each convergent, by dividing with terms arranged in ascending or in descending powers of p. These two solutions arise in case there are two particular integrals for the equation. An example will illustrate operational division under such conditions.

Given $xp^2y+y=x^2$, or $y \approx \dfrac{x^2}{xp^2+1}$.

$$
\begin{array}{l|l|l}
xp^2+1 & x^2 & \left(\frac{x^2}{x}\right)\frac{1}{p^2}-\left(\frac{x^3}{x(3)!}\right)\frac{1}{p^2}+\left(\frac{x^4}{x4.3.(3)!}\right)\frac{1}{p^2}-\left(\frac{x^5}{x.5.4.4.3.(3)!}\right)\frac{1}{p^2}+\cdots \\
 & x^2+\frac{x^3}{(3)!} & \frac{x^3}{3!}-\frac{x^4}{4.3(3)!}+\frac{x^5}{5.4.4.3.(3)!}-\frac{x^6}{6.5.5.4.4.3.(3)!}+\cdots \\
 & \quad-\frac{x^3}{(3)!} & \\
 & \quad-\frac{x^3}{(3)!}-\frac{x^4}{4.3(3)!} & \\
 & \qquad\frac{x^4}{4.3(3)!} & \\
 & \qquad\frac{x^4}{4.3(3)!}+\frac{x^5}{5.4.4.3.(3)!} & \\
 & \qquad\quad-\frac{x^5}{5.4.4.3.(3)!} &
\end{array}
$$

If we divide the other way:

$$
\begin{array}{l|l|l}
1+xp^2 & x^2 & x^2-2x \\
 & x^2+2x & \\
 & -2x & \\
 & -2x-0 &
\end{array}
$$

Hence $y=x^2-2x$ is a solution and we have obtained two particular integrals. The complementary functions are obtained by solving

$$p^2y+x^{-1}y=Ap, \tag{1}$$

and

$$p^2y+x^{-1}y=Bp^2. \tag{2}$$

From (1),

$$y \Bumpeq \frac{Ap}{p^2+x^{-1}}, \tag{3}$$

$$y = A\left[x-\frac{x^2}{1.2}+\frac{x^3}{1.2^2.3}-\frac{x^4}{1.2^2.3^2.4}+\cdots\right]$$

From (2),

$$y \Bumpeq \frac{Bp^2}{p^2+x^{-1}}, \tag{4}$$

$$\begin{aligned} y = -B\log x&\left[x-\frac{x^2}{1.2}+\frac{x^3}{1.2^2.3}-\frac{x^4}{1.2^2.3^2.4}+\cdots\right]\\ &+B\left[1+x-x^2(\tfrac{1}{4}+1)+x^3\left(\frac{1}{2^2.3}+\frac{1}{2^2.3^2}+\frac{1}{2.3}\right)-\cdots\right]. \end{aligned} \tag{5}$$

Hence a complete solution is:

$$\begin{aligned} y = -x+\frac{x^2}{2}+\frac{x^3}{1.2^2.3}-\frac{x^4}{1.2^2.3^2.4}+\frac{x^5}{1.2^2.3^2.4^2.5}-\frac{x^6}{1.2^2.3^2.4^2.5^2.6}+\cdots\\ +\left[A-B\log x\right]\left[x-\frac{x^2}{1.2}+\frac{x^3}{1.2^2.3}-\frac{x^4}{1.2^2.3^2.4}+\cdots\right]\\ +B\left[1+x-x^2(\tfrac{1}{4}+1)+x^3\left(\frac{1}{2^2.3}+\frac{1}{2^2.3^2}+\frac{1}{2.3}\right)-\cdots\right]. \end{aligned}$$

3a. *Solutions involving a generating function.* In some cases, by division a geometric series is obtained and the answer is the generating function of such series.

An example is the equation $-np^2y+xp^3y=x$. By dividing,

$$y \Bumpeq \frac{x}{-np^2+xp^3}=\frac{-x^3}{6n}-\frac{x^3}{6n^2}-\frac{x^3}{6n^3}-\cdots=\frac{-\dfrac{x^3}{6n}}{1-\dfrac{1}{n}}=\frac{x^3}{6(1-n)}.$$

If we reverse the terms in the divisor,

$$y \Bumpeq \frac{x}{xp^3-np^2}=\frac{x^3}{6}+\frac{nx^3}{6}+\frac{n^2x^3}{6}+\cdots=\frac{\dfrac{x^3}{6}}{1-n}=\frac{x^3}{6(1-n)}.$$

The generating function $\dfrac{x^3}{6(1-n)}$ is the same in each case and satisfies the equation.

3b. *Case where the generating function is infinite.* Sometimes the remainder from an operational division is the same as the dividend as shown by the following example:

$$xp^3y-p^2y=12x,\quad y \Bumpeq \frac{12x}{xp^3-p^2}, \tag{1}$$

$$\begin{array}{r|r|l} xp^3-p^2 & 12x & \dfrac{12x}{xp^3} \\ & \underline{12x-12x} & \\ & 12x & 2x^3 \end{array}$$

The remainder $12x$ is the same as the dividend and if the division were continued the quotient would be $2x^3+2x^3+2x^3+\cdots$ or infinity.

In all such cases the answer is $k\psi(x)\log x$ where $\psi(x)$ is the first term of the quotient as obtained by straight operational division and k is a constant the value of which is found by substituting $k\psi(x)\log x$ in the differential equation and solving for k.

For example, as applied to the above, the answer is $2kx^3\log x$. Substituting this in the differential equation (1) we find $k=1$ so that $y=2x^3\log x$ is the particular integral.

4. *Application to equations with more than two terms containing y.* The method is applicable to equations with more than two terms containing y.

For example take the equation $p^2y+xpy+y=x$.
The operational solution is

$$y \risingdotseq \frac{x}{p^2+xp+1}.$$

In the process of division all terms in the divisor and dividend should be arranged in descending powers of p. In the above case the result of the division is

$$y=\frac{x^3}{(3)!}-\frac{4x^5}{(5)!}+\frac{6.4x^7}{(7)!}-\frac{8.6.4x^9}{(9)!}+\cdots$$

5. *Compound operational division.* In some examples the simple operational division such as has been described fails[1] and a more involved division is necessary which I have called compound division.

An example is $xp^2y+py+xy=x^2$.
Since x is $1/p$, xp^2 is of the degree 1 in p; that is, the first two terms are of the same degree. It will simplify the division to use p as the first term in the divisor. Where the degree of p is the same in two terms generally the term without x should be used as the first term in the divisor.

$$\begin{array}{l|l|l}
p+xp^2+x & x^2 & \dfrac{x^2}{p}-\dfrac{x^4}{9p}+\dfrac{x^6}{5^2.\,3^2\,p}-\cdots \\
 & x^2+2x^2+\dfrac{x^4}{9} & \dfrac{x^3}{3}-\dfrac{x^5}{5.3^2}+\dfrac{x^7}{5^2.\,3^2.\,7}-\cdots \\
 & \overline{-2x^2-\dfrac{x^4}{9}} & \dfrac{x^3}{9}-\dfrac{x^5}{5^2.\,3^2}+\dfrac{x^7}{3^2.\,5^2.\,7^2}-\cdots \\
 & -\dfrac{x^4}{9}-\dfrac{4x^4}{9}-\dfrac{x^6}{5^2.\,3^2} & \\
 & \overline{\dfrac{4x^4}{9}+\dfrac{x^6}{5^2\cdot 3^2}} & \\
 & +\dfrac{x^6}{5^2.\,3^2}+\dfrac{6x^6}{5^2\cdot 3^2}+\dfrac{x^8}{3^2.\,5^2.\,7^2} & \\
 & \overline{-\dfrac{6x^6}{5^2.3^2}-\cdots} &
\end{array}$$

[1] When the ratio of the first terms of two successive dividends is a constant, the simple operational division leads to a series of infinite series. The compound division substitutes for each of these infinite series its generating function.

The particular integral is

$$y = \frac{x^3}{3^2} - \frac{x^5}{3^2\cdot 5^2} + \frac{x^7}{3^2\cdot 5^2\cdot 7^2} - \cdots.$$

If the division is performed as usual the first term of the quotient is $x^3/3$ and the first remainder is $-2x^2-x^4/3$ (not $-2x^2-x^4/9$ as shown). The division can then be expressed as follows:

$$\frac{x^2}{p+xp^2+x} \Bumpeq \frac{x^3}{3} - \frac{2x^2}{p+xp^2+x} - \frac{\dfrac{x^4}{3}}{p+xp^2+x},$$

$$\frac{x^2}{p+xp^2+x} \Bumpeq \frac{x^3}{9} - \frac{\dfrac{x^4}{9}}{p+xp^2+x}.$$

The first term of the quotient is then $x^3/9$ and not $x^3/3$. $x^3/9$ is then written as the first term in the third line of the quotient. The first remainder is $-x^4/9$ and not $-2x^2-x^4/3$. The terms $2x^2$ and $-2x^2$ and the similar terms shown in bold face type in the example are, in practice, marked with a circle and disregarded and the division continues.

The process is performed mentally, the multiplication to arrive at the term in the third line of the quotient being obtained by subtracting from one the ratio in question and taking the reciprocal of the result. For example the ratio of $-2x^2$ to x^2 is -2, and $1-(-2)=3$. The multiplier to apply to the first term of the second line of the quotient is $1/3$. Similarly the ratio of $4x^4/9$ to $-x^4/9$ is -4. $1-(-4)=5$ and $1/5$ is the multiplier to apply to the second term of the second line of the quotient, etc.

One complementary function is given by

$$y \Bumpeq \frac{c_1 p}{p+xp^2+x}.$$

$$\begin{array}{r|l|l}
p+xp^2+x & p & \dfrac{p}{p} - \dfrac{x}{p} + \dfrac{x^3}{2^2p} - \dfrac{x^5}{2^2.\,4^2\,p} + \cdots \\
& p+x & \\
& -x & 1 - \dfrac{x^2}{2} + \dfrac{x^4}{2^2.\,4} - \dfrac{x^6}{2^2.\,4^2.\,6} + \cdots \\
& -x\ \mathbf{-x} - \dfrac{x^3}{2^2} & \\
& \mathbf{x} + \dfrac{x^3}{2^2} & -\dfrac{x^2}{2^2} + \dfrac{x^4}{2^2.4^2} - \dfrac{x^6}{2^2.4^2.6^2} + \cdots
\end{array}$$

$$\begin{array}{r}
\dfrac{x^3}{2^2} + \dfrac{\mathbf{3}}{\mathbf{4}}\mathbf{x^3} + \dfrac{x^5}{2^2.\,4^2} \\
\hline
-\dfrac{\mathbf{3^3}}{\mathbf{4}}\mathbf{x^3} - \dfrac{x^5}{2^2.\,4^2} \\
-\dfrac{x^5}{2^2.\,4^2} - \dfrac{\mathbf{5x^5}}{\mathbf{2^2.4^2}} \\
\hline
+\dfrac{\mathbf{5x^5}}{\mathbf{2^2.4^2}} + \cdots
\end{array}$$

Hence

$$y = \frac{x^3}{3^2} - \frac{x^5}{3^2.\,5^2} + \frac{x^7}{3^2.5^2.7^2} - \cdots$$
$$+ C_1\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.\,4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots\right].$$

This is not the complete primitive since it contains only one constant. The remainder of the complementary function is obtained by substituting e^t for x in the equation which then reduces to

$$p^2y+e^{2t}y=0$$

The solution is obtained by operational division of $y \simeq \dfrac{C_2p}{p^2+e^{2t}}$ and is $C_2\log x\left[1-\dfrac{x^2}{2^2}+\dfrac{x^4}{2^2.4^2}-\cdots\right]$

$$+C_2\left[\frac{x^2}{2^2}-\frac{x^4}{2^2.4^2}(1+\tfrac{1}{2})+\frac{x^6}{2^2.4^2.6^2}(1+\tfrac{1}{2}+\tfrac{1}{3})-\cdots\right].$$

The change of variable makes the division somewhat simpler. The same answer can be obtained by solving

$$xp^2y+py+xy=x^{-1}-x^{-1} \quad \text{or} \quad y \simeq \frac{x^{-1}-x^{-1}}{xp^2+p+x^2}.$$

6. *Rule for obtaining the complementary function.* Let x denote the independent and y the dependent variable. In the differential equation substitute x^c for y and collect the coefficients of each of the various powers of x. Equate to zero that coefficient which contains c to the highest power and solve for c. Suppose there are n roots $c_1, c_2, \cdots, c_n$ then the first terms of the several complementary functions are $k_1x^{c_1}, k_2x^{c_2}, \cdots, k_nx^{c_n}$ where k_1, k_2, etc. are the constants of integration.

The complementary functions are obtained by operational division as follows. The divisor in each case is composed of the operational coefficients of the dependent variable in the differential equation. The first terms in the divisor are the operational coefficients of the dependent variable of those terms which when x^c was substituted for y gave the equation in c which was used.

Let $\psi(x)p^s$ denote the first term in the divisor. Then the successive dividends are

$k_1(\psi(x)p^sx^{c_1}-\psi(x)p^sx^{c_1})$, $k_2(\psi(x)p^sx^{c_2}-\psi(x)p^sx^{c_2})$, $\cdots k_n(\psi(x)p^sx^{c_n}-\psi(x)p^sx^{c_n})$

If any one of the expressions $k_r[\psi(x)p^sx^{c_r}]$ is equal to zero then $-k_r[\psi(x)p^sx^{c_r}]$ may be omitted from the dividend. For example if $c_r=0$ the dividend is $k_r[\psi(x)p^s]$ since $x^0=1$. Suppose $s=2$ and $c_r=1$. Then the dividend is $k_r\psi(x)p^2x=k_r\psi(x)ppx=k_r\psi(x)p$ since $px=1$.

If the equation in c has n equal roots each equal to c_n, then the first terms of the corresponding complementary function are

$$k_1x^{c_n}, \qquad k_2x^{c_n}(\log x), \qquad k_3x^{c_n}(\log x)^2, \cdots, k_nx^{c_n}(\log x)^{n-1}.$$

The complementary functions are then obtained as above except these values are used in place of x^{c_1}, x^{c_2}, etc.

The classical method of obtaining the complementary functions is to equate the terms containing the dependent variable to zero and solve the equation. It will be seen that the above procedure is similar as it equates the dependent variable to expressions which are equal to zero and then obtains the solutions by operational division.

Example

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}-2x^{-2}y=0.$$

Substituting x^c, for y we get

$$c(c-1)x^{c-2}+acx^{c-1}-2x^{c-2}=0.$$

Equating the coefficient of x^{c-2} to zero we get $c^2-c-2=0$, $c=2$ or -1.

Hence the exponent of x in the first term of one complementary function must be 2, and in the other complementary function -1. To obtain this put

$$p^2y+apy-2x^{-2}y=A-A\ , \tag{1}$$

and

$$p^2y+apy-2x^{-2}y=2Bx^{-3}-2Bx^{-3}. \tag{2}$$

The solutions are obtained by operational division:

$$y \risingdotseq \frac{A-A}{p^2+ap-2x^{-2}}=A\left[\frac{x^2}{2}-\frac{ax^3}{4}+\frac{3a^2x^4}{5.4.2}-\frac{a^3x^5}{5.4.3}+\cdots\right],$$

and

$$\risingdotseq \frac{2Bx^{-3}-2Bx^{-3}}{p^2+ap-2x^{-2}}=B\left[-x^{-1}-\frac{a}{2}\right].$$

Example

$$(2x+x^3)\frac{d^2y}{dx^2}-\frac{dy}{dx}-6xy=0,$$

The index equation is $2c^2-3c=0$, $c=0$ or $3/2$. Hence we put

$$-py+2xp^2y+x^3p^2y-6xy=Ap, \tag{1}$$

and

$$-py+2xp^2y+x^3p^2y-6xy=(x^{1/2}-x^{1/2})B. \tag{2}$$

Note that p was chosen in equation (1) because the first term in the division

$$y \risingdotseq \frac{Ap}{-p+2xp^2+x^3p^2-6x}$$

is a constant, that is, it contains x with exponent 0. And $x^{1/2}-x^{1/2}$ was chosen in equation (2) because in the division

$$y \risingdotseq \frac{(x^{1/2}-x^{1/2})B}{-p+2xp^2+x^3p^2-6x}$$

the first term is $(Bx^{1/2}/(-p))=-2B\,x^{3/2}/3)$ which contains x with the exponent $3/2$.

Another example is

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0.$$

The index equation is $c(c-1)+c=0$; $c=0$, 0.

The first term in one of the series is a constant, and in the other $B\log x$, where B is a constant.

For the series beginning with a constant put

$$y \risingdotseq \frac{Ap}{p+xp^2+1}, \tag{1}$$

$$y=A\left[1-\frac{x^1}{1^2}+\frac{x^2}{1^2.2^2}-\frac{x^3}{1^2.2^2.3^2}+\cdots\right]. \tag{2}$$

For the other series put

$$y \risingdotseq \frac{B(x^{-1}-x^{-1})}{p+xp^2+1}, \tag{3}$$

$$\begin{aligned} y = B\log x &\left[1-x+\frac{x^2}{2^2}-\frac{x^3}{2^2.3^2}+\frac{x^4}{2^2.3^2.4^2}-\cdots\right] \\ &+B\left[2x-\frac{6x^2}{2^3}+\frac{22x^3}{2^3.3^3}-\frac{100}{2^3.3^3.4^3}x^4+\cdots\right]. \end{aligned} \tag{4}$$

The division of equation (3) is interesting and illustrates a condition met with in certain equations. When the operational division of $x^{-1}-x^{-1}$ by $p+xp^2+1$ is performed the quotient is

$$\begin{aligned} &B \log x \left[1 - x + \frac{x^2}{2^2} - \frac{x^3}{2^2.3^2} + \frac{x^4}{2^2.3^2.4^2} - \cdots\right] \\ &\quad + B\left[x - \frac{x^2}{2^3} + \frac{x^3}{2^2.3^3} - \frac{x^4}{2^2.3^2.4^3} + \cdots\right] \end{aligned} \tag{5}$$

with a remainder which forms the following infinite series

$$B\left[1 - \frac{3x}{2} + \frac{5x^2}{2^3.3} - \frac{7x^4}{2^2.3^3.4} + \cdots\right]. \tag{6}$$

Next the remainder (6) is divided by $p+xp^2+1$ and the quotient is

$$B\left[x - \frac{5x^2}{2^3} + \frac{20x^3}{2^3.3^3} - \frac{94x^4}{2^3.3^3.4^3} + \cdots\right] \tag{7}$$

By adding (7) and (5) the result (4) is obtained. The complete primitive is (2)+(4) or

$$\begin{aligned} y &= (A+B\log x)\left[1 - \frac{x^2}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2} + \cdots\right] \\ &\quad + B\left[2x - \frac{6x^2}{2^3} + \frac{22x^3}{2^3.3^3} - \frac{100x^4}{2^3.3^3.4^3} + \cdots\right], \end{aligned} \tag{8}$$

4. Equations with exponential or other functions. The solution of certain equations containing exponential (including trigonometric) functions or other functions can sometimes be obtained very readily by operational division.

For example $p^n y + e^x y = e^{2x}$,
$$y \approx \frac{e^{2x}}{p^n + e^x}. \tag{1}$$

$$\begin{array}{l|l} \underline{p^n+e^x|}\ e^{2x} & \dfrac{e^{2x}}{p^n} - \dfrac{e^{3x}}{2^n p^n} + \dfrac{e^{4x}}{2^n.3^n\, p^n} - \cdots \\ \qquad\quad \underline{e^{2x} + \dfrac{e^{3x}}{2^n}} & \\ \qquad\qquad -\dfrac{e^{3x}}{2^n} & \dfrac{e^{2x}}{2^n} - \dfrac{e^{3x}}{2^n.3^n} + \dfrac{e^{4x}}{2^n.3^n.4^n} - \cdots \\ \qquad\qquad \underline{-\dfrac{e^{3x}}{2^n} - \dfrac{e^{4x}}{2^n.3^n}} & \\ \qquad\qquad\qquad \dfrac{e^{4x}}{2^n.3^n} & \end{array}$$

So we get
$$y = \frac{e^{2x}}{2^n} - \frac{e^{3x}}{2^n.3^n} + \frac{e^{4x}}{2^n.3^n.4^n} - \cdots \tag{2}$$

The complementary functions are obtained by equating the left side of (1) to k_1p, k_2p^2, k_3p^3, $\cdots$, k_np^n respectively and dividing.

5. Application to linear equations with constant coefficients. All the examples given have been of linear equations with variable coefficients. The methods outlined will apply to equations with constant coefficients, in fact, the process of division is in general much simpler when the coefficients are constant. The complementary functions are obtained by equating the left hand side of the equation to k_1p, k_2p^2, etc. and dividing. If the equation

contains $p^n y$, the complementary functions are obtained by using $k_1 p$, $k_2 p^2$, $\cdots$, $k_n p^n$ successively as the dividends.

For example, $p^n y + y = 1$.
The complementary functions are found by evaluating

$$\frac{k_1 p}{p^n+1}, \quad \frac{k_2 p^2}{p^n+1}, \cdots, \frac{k_n p^n}{p^n+1}.$$

6. Theory: The reader has probably noted the great similarity between the operational division and ordinary long division of algebra. In algebra if we divide with the divisor and dividend arranged in terms of descending powers of x we get one answer and if the division is with terms arranged in ascending powers of x we get another answer, generally one series is divergent and the other convergent for certain values of x, provided that the quotient is not finite. The same was true of the operational division.

We saw in the operational division, especially the compound division, that we could substitute the generating function for a series in the quotient and the result would satisfy the equation even though the series for which the generating function was substituted was divergent.

The same is true of ordinary division in algebra.

Take the equation $xy - y = x$ or $y = x/(x-1)$. By division we get

$$= 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$$

or

$$y = - x - x^2 - x^3 - x^4 - \cdots$$

according to the arrangement of terms in the divisor. Each series is divergent for certain values of x, but the generating function of each series is $x/(x-1)$ and it, of course, satisfies the equation $xy - y = x$.

These remarks are to pave the way for the theory which indicates that the operational division merely follows the rules of an algebra whose multiplication table is closely allied to the integration and differentiation of calculus.

Suppose we have an algebra whose multiplication table is expressed by

$$x^a \odot x^b = \frac{x^{a+b}(a)!\,(b)!}{(a+b)!} \tag{1}$$

where the symbol $\odot$ is used to indicate this particular type of multiplication as distinguished from the multiplication of ordinary algebra.

In (1), the symbol $(a)!$ is intended to mean the gamma product of a. If a is an integer it is the same as factorial a.

The operation in (1) is evidently commutative, for

$$x^b \odot x^a = \frac{x^{b+a}(a)!\,(b)!}{(a+b)!} = x^a \odot x^b = \frac{x^{a+b}(a)!\,(b)!}{(a+b)!}\,.$$

It will be shown later that the operation is also distributive. That is $x^a \odot (x^b + x^c) = x^a \odot x^b + x^a \odot x^c$.

The operation is also associative for

$$(x^a \odot x^b) \odot x^c = \frac{x^{a+b}\,(a)!\,(b)!}{(a+b)!} \odot x^c = \frac{x^{a+b+c}(a)!\,(b)!\,(c)!\,(a+b)!}{(a+b)!\,(a+b+c)!} = \frac{x^{a+b+c}(a)!\,(b)!\,(c)!}{(a+b+c)!},$$

and

$$x^a \odot (x^b \odot x^c) = x^a \odot \frac{x^{b+c}\,(b)!\,(c)!}{(b+c)!} = \frac{x^{a+b+c}\,(b)!\,(c)!\,(a)!\,(b+c)!}{(b+c)!\,(a+b+c)!} = \frac{x^{a+b+c}(a)!\,(b)!\,(c)!}{(a+b+c)!}.$$

The proof can easily be extended to products of more than three terms.

Division is defined, as in ordinary algebra, as the reverse of multiplication. As a consequence of the definition of multiplication it follows that we may write

$$\frac{(x^{a+b})!\,(a)!\,(b)!}{(a+b)!\;x^a\odot} = x^b$$

or, replacing $a+b$ by m and a by n,

$$\frac{x^m}{x^n\odot} = \frac{(m)!\;\;x^{m-n}}{(n)!\,(m-n)!}. \tag{2}$$

In other words, x^m in this algebra divided by x^n equals

$$\frac{(m)!\;x^{m-n}}{(n)!(m-n)!}.$$

Constants are merely scalar multipliers as in ordinary algebra.

Addition and subtraction are the same as in ordinary algebra.

Now if a is a positive integer

$$\frac{x^a}{(a)!}\odot x^n = \frac{x^{a+n}(a)!\,(n)!}{(a)!\,(a+n)!} = \frac{x^{a+n}}{(n+1)(n+2)\cdots(n+a)}.$$

But

$$\frac{1}{p^a}\,x^n = \frac{x^{a+n}}{(n+1)(n+2)\cdots(n+a)} \tag{3}$$

for all fractional values of n, and for all positive integral values of n, and for all negative integral values of n, provided $|\,a\,| < |\,n\,|$.

Again if a is a positive integer

$$\frac{x^{-a}}{(-a)!}\odot x^n = \frac{x^{-a+n}(-a)!\,(n)!}{(-a)!\;(-a+n)!} = n(n-1)\cdots(n-a+1)\,x^{n-a}.$$

But

$$p^a x^n = n(n-1)\;(n-2)\cdots(n-a+1)\,x^{n-a}. \tag{4}$$

If a is fractional, then multiplication by $(x^a/a!)\odot$ defines fractional integration and multiplication by $(x^{-a}/(-a)!)\odot$ defines fractional differentiation.

We see therefore that with the limitation given above multiplying by $(x^a/(a)!)\odot$ is equivalent to operating by $1/p^a$ and multiplying by $(x^{-a}/(-a)!)\odot$ is equivalent to operating by p^a.

The exception as outlined above where this multiplication does not parallel the operation of calculus may be explained more clearly by an example

$$\int x^{-1}dx = \log x$$

but

$$x\odot x^{-1} = (-1)! = \infty = \int_0^x x^{-1}dx.$$

In other words, where the integration of dx/x is involved, the multiplication $x\odot x^{-1}$ gives not the indefinite integral, but the definite integral $\int_0^x (dx/x)$.

The introduction of the constant of integration is also interesting. We have

$$\int x dx = \frac{x^2}{2} + c.$$

The parallel operation is $x\odot x$ but this can be written $x\odot\left[x + \frac{cx^{-1}}{(-1)!}\right]$ since $\frac{cx^{-1}}{(-1)!} = 0$.

$$x \odot x = \frac{x^2}{(2)!} + \frac{c(-1)!}{(-1)!} = \frac{x^2}{(2)!} + c.$$

Since the multiplication of this algebra parallels integration and differentiation and since the two latter processes are distributive it follows that the multiplication of this algebra follows the distributive law.

Now I will give some examples of algebraic division and operational division.

Pure algebraic division

$$\begin{array}{r|l}
x-x^2 & x^2 \\
 & \underline{x^2 - x^3} \\
 & \quad x^3 \\
 & \quad \underline{x^3 - x^4} \\
 & \qquad x^4 \\
 & \qquad \underline{x^4 - x^5} \\
 & \qquad\quad x^5
\end{array}
\qquad
\begin{array}{|l}
\dfrac{x^2}{x}+\dfrac{x^3}{x}+\dfrac{x^4}{x}+\dfrac{x^5}{x}+\cdots \\
x + x^2 + x^3 + x^4 + \cdots \\ \hline
\end{array}$$

Therefore $y=x+x^2+x^3+x^4+\cdots$ or its generating function $y=x/(1-x)$ is a solution of $xy-x^2y=x^2$.

In the above division two lines are shown in the quotient, the indicated division and the actual result. This is done in order to bring out the correspondence between the algebraic and operational division.

If the terms in the divisor are arranged in descending powers of x the result is

$$y = \frac{x^2}{-x^2+x} = -1 - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \cdots$$

This series also formally satisfies the equation $xy-x^2y=x^2$.

Pure operational division:

$$
\begin{array}{l|l|l}
x\odot - x^2\odot & x^2 & \dfrac{x^2}{x\odot}+\dfrac{2x^3}{3x\odot}+\dfrac{2x^4}{6x\odot}+\dfrac{2x^5}{3.5x\odot}+\cdots \\
 & x^2-\dfrac{2x^3}{3} & 2x+2x^2+\dfrac{4x^3}{3}+\dfrac{2x^4}{3}+\cdots \\
\cline{2-2}
 & \dfrac{2x^3}{3} & \\
 & \dfrac{2x^3}{3}-\dfrac{2x^4}{6} & \\
\cline{2-2}
 & \dfrac{2x^4}{6} & \\
 & \dfrac{2x^4}{6}-\dfrac{2x^5}{3.5} & \\
\cline{2-2}
\end{array}
$$

Therefore $y=2x+2x^2+(4x^3/3)+(2x^4/3)+\cdots$ is a solution of $x\odot y-x^2\odot y=x^2$ or

$$\int ydx-2\int\int ydxdx=x^2.$$

Another example:

$$
\begin{array}{l|l|l}
\dfrac{x^{-1}}{(-1)!}\odot+1 & 1 & \dfrac{1}{\dfrac{x^{-1}}{(-1)!}\odot}-\dfrac{x}{\dfrac{x^{-1}}{(-1)!}\odot}+\dfrac{x^2}{\dfrac{2x^{-1}}{(-1)!}\odot}-\cdots \\
 & 1+x & x-\dfrac{x^2}{(2)!}+\dfrac{x^3}{(3)!}-\cdots \\
\cline{2-2}
 & -x & \\
 & -x-\dfrac{x^2}{2} & \\
\cline{2-2}
 & \dfrac{x^2}{2} & \\
\end{array}
$$

Therefore $y=x-(x^2/(2)!)+(x^3/3!)-\cdots$ is a solution of $(x^{-1}/(-1)!)\odot y+y=1$ or $(dy/dx)+y=1$.

Mixed algebraic and operational division:

$$
\begin{array}{l|l|l}
x\cdot - x^2\odot & x^2 & x+\dfrac{x^2}{3}+\dfrac{x^3}{3^2.2}+\dfrac{x^4}{5.4.3^2}+\cdots \\
 & x^2-\dfrac{x^3}{3} & \\
\cline{2-2}
 & \dfrac{x^3}{3} & \\
 & \dfrac{x^3}{3}-\dfrac{x^4}{3^2.2} & \\
\cline{2-2}
 & \dfrac{x^4}{3^2.2} & \\
 & \dfrac{x^4}{3^2.2}-\dfrac{x^5}{5.4.3^2} & \\
\cline{2-2}
 & \dfrac{x^5}{5.4.3^2} & \\
\end{array}
$$

Therefore

$$=x+\frac{x^2}{3}+\frac{x^3}{3^2.2}+\frac{x^4}{5.4.3^2}+\cdots$$

is a solution of

$$xy-2\int\int ydxdx=x^2.$$

The operational algebra outlined differs from the calculus under certain conditions. If, however, we had defined the multiplication table as

$$x^a \odot x^b = \frac{x^{a+b}(a)!\,(b)!}{(a+b)!} \tag{1}$$

with the exception that, when a and b are positive integers and b is not greater than a,

$$x^a \odot x^{-b} = a!\ \underbrace{\int\!\!\int \cdots \int\!\!\int}_{a \text{ times}} x^{-b}dx \cdots dx,$$

$x^{-b} \odot x^a$ being defined as in (1), then the algebra and the calculus will be exactly parallel. With this definition of multiplication the algebra is of course not always commutative as, for example, $x \odot x^{-1} = \log x$ but $x^{-1} \odot x = (-1)!$.

When this is done it is easier to think of the rules of integration and differentiation when performing the operational division, than to remember the rule of multiplication in (1), and the rule of division outlined elsewhere.

7. Conclusion. Let me conclude with a general proof of the

THEOREM: *If the result of operational division is a polynomial or a convergent infinite series, then it is a solution of the differential equation.*

PROOF: Let

$$[p^n + \psi(x)p^{n-1} + \phi(x)p^{n-2} + \cdots + \theta(x)]y \equiv (p^n + p_s)y = \Delta \tag{1}$$

be any linear differential equation, where Δ represents any function of x, and where, for convenience,

$$p_s = \psi(x)p^{n-1} + \phi(x)p^{n-2} + \cdots \theta(x)\ \cdot$$

Then

$$y \Bumpeq \frac{\Delta}{p^n + p_s} \tag{2}$$

and can be found by operational division as follows:

$$\begin{array}{r|l} p^n+p_s \,\big|\, \Delta & \frac{\Delta}{p^n} - \frac{1}{p^n}p_s\frac{\Delta}{p^n} + \left(\frac{1}{p^n}p_s\right)^2\frac{\Delta}{p^n} + \cdots \\ \Delta + p_s\frac{\Delta}{p^n} & ----+(-1)^{r-1}\left(\frac{1}{p^n}p_s\right)^{r-1}\frac{\Delta}{p^n} + ---- \\ \hline -p_s\frac{\Delta}{p^n} & \\ -p_s\frac{\Delta}{p^n} - p_s\frac{1}{p^n}p_s\frac{\Delta}{p^n} & \\ \hline +p_s\frac{1}{p^n}p_s\frac{\Delta}{p^n} & \end{array}$$

In the quotient $(-1)^{r-1}((1/p^n)p_s)^{r-1} \cdot \Delta/p^n$ represents the rth term and the expression $((1/p^n)p_s)^{r-1}$ indicates that the operator $(1/p^n)p_s$ is repeated $r-1$ times.

Now we know that the formal process represented by this operational division can be performed. We know that in many cases the result is a convergent series and in cases we know the operational division gives a divergent series. Let us consider the case where the result is a convergent series.

Substitute the first r terms of the quotient in the left side of (1) and denote the result by S_r. Then

$$S_r=\Delta - p_s\frac{\Delta}{p^n}+p_s\left(\frac{1}{p^n}p_s\right)\frac{\Delta}{p^n}- \cdots +(-1)^{r-1}p_s\left(\frac{1}{p^n}p_s\right)^{r-2}\frac{\Delta}{p^n}$$

$$+ p_s\frac{\Delta}{p^n}-p_s\left(\frac{1}{p^n}p_s\right)\frac{\Delta}{p^n}+ \cdots + (-1)^r p_s\left(\frac{1}{p^n}p_s\right)^{r-2}\frac{\Delta}{p^n}$$

$$+ (-1)^{r-1} p_s\left(\frac{1}{p^n}p_s\right)^{r-1}\frac{\Delta}{p^n}$$

. . . .

$$S_r=\Delta+(-1)^{r-1} p_s\left(\frac{1}{p^n}p_s\right)^{r-1}\frac{\Delta}{p^n}\cdot$$

Since by hypothesis we are considering only the case when the quotient is a convergent series, it follows that as r is indefinitely increased that $((1/p^n)p_s)^{r-1}\cdot\Delta/p^n$ approaches zero as its limit and $(-1)^{r-1}p_s((1/p^n)p_s)^{r-1}\cdot\Delta/p^n$ will also approach zero as a limit provided p_s is not an operator which will change an infinitesimal into a finite quantity.

We have then, in case the series is convergent and we substitute an infinite number of its terms in the equation, $\lim_{r\to\infty} S_r=\Delta$, and the equation is satisfied.

Since p_s is $\psi(x)p^{n-1}+\phi(x)p^{n-2}+\cdots+\theta(x)$, the operator p_s in general will be a finite operator, unless one or more of the functions $\psi(x)$, $\phi(x)$, etc., are infinite for certain values of x. For such values of x the remainder may not approach zero as the number of terms is indefinitely increased and the operational division then will not satisfy the equation.

The proof for the complementary functions is similar except that for Δ are substituted respectively A_1p, A_2p^2, $\cdots$, A_np^n, giving n complementary functions.

A NEW TYPE OF FRESHMAN COURSE IN MATHEMATICS

By N. J. LENNES, University of Montana

In view of the present somewhat unsettled condition of courses in mathematics for college freshmen, a brief account of an experiment now under way in the University of Montana may be of sufficient interest to warrant publicity. The idea which led to the new course to be described in this paper arose

from experience with a definite concrete case, which is described in more detail in the May number, 1914, of the *Educational Review*, under the caption "Mathematics for Culture." In that paper there is given also a synopsis of a possible course to meet the demand presented by this particular case. In the main the course now being given at the University of Montana follows that synopsis.

In brief, the case was that of a young man of fine intelligence and high educational purposes whose ultimate aim was the law, the profession of his father, but who wanted to spend four years in college as an intellectual "sporting proposition." He came to the department of mathematics in one of the great American universities with the request that he be given an opportunity to learn something about the nature of mathematics, what it has done in developing our present civilization both on its material and intellectual (also spiritual) sides, and what it is continuing to do in man's present inroads into the realm of the unknown. He was not afraid of hard work or difficult subjects, and was possessed of abundant intellectual curiosity. The response to this request was an offer of two years of college mathematics ending with the calculus. "Anything less would be far too fragmentary and entirely inadequate for your purpose. More would be better." But this was much more time than this young man could judiciously give to the subject. It would consume at least one-eighth of his college course—possibly one-sixth. There were many other departments with equal claims on his attention and his curiosity was at least as keen about the mysteries of modern physics and chemistry as about mathematics. The atom had been broken up! What are electrons and protons and "quanta," which he had seen mentioned in semi-popular scientific writings? What is the present status of the doctrine of evolution? What about heredity and bacteria and the geological history of the earth and the recent investigations into the extent of the sidereal universe? Certainly he could not devote two years, even a few hours a week, to each of the departments in which he might hope for light on these varied subjects. Besides, it was necessary that a large share of the four year course should be devoted to the subjects nearer his proposed life work. History, economics, politics in the more general sense, were entitled to more time than the sciences. Besides all these he should study at least one foreign language and possibly some of the world's great literary productions.

The upshot of it all was that the university had nothing to offer in mathematics which this young man could afford to take. A course in trigonometry or in college algebra would be entirely too fragmentary and too barren of results for his purpose.

The question then presented itself as to whether this state of affairs is, in the nature of the case, a necessary evil or whether it might be remedied.

Certainly this young man represents a class of the highest type, to whose wants the college should seek to minister if that is at all possible. It may be granted for the sake of the argument that our traditional sequence of courses is the best that can be arranged for those who are to devote a large share of their college work to mathematics, but may it not be possible to organize a course which should meet the requirements of our somewhat precocious freshman? Is it possible that in our interest in our subject and in our more or less conscious zeal to "build up our department" in the sense of attracting students to the more advanced courses, we have neglected entirely legitimate demands on the part of those whose center of interest lies quite remote from our subject?

With this situation in mind and with a feeling that the correct answers to the questions just stated may very likely be in the affirmative, it was decided some years ago by the members of the department of mathematics in the State University of Montana to offer a course entitled "A Survey of College Mathematics" to be open to all freshmen who enter with at least two years of preparatory mathematics—one year of algebra and one year of plane geometry. The course was first given in the autumn quarter of 1924 and has been repeated each quarter since then, including the summer quarter of 1925, so that now it is in its seventh quarter.

The course is based upon a manuscript text which has been mimeographed for student use. This text consists of five chapters whose titles will give a first indication of the content of the course. I. Trigonometry: indirect measurement of distances; II. Functions and their graphs; III. Derivatives and their uses; IV. Integrals and their uses; V. An historical sketch.

The first chapter confines itself to the definition of trigonometric functions and the proof of the sine- and cosine-formulas. (Logarithms are not introduced.) It is shown that by means of these the triangle may be solved completely and thus distances measured indirectly. It is shown how, by means of these formulas and the necessary tables as a mathematical nucleus, a vast body of interesting information has been obtained and tasks have been performed which without them would be very difficult or altogether impossible. The dimensions of the solar and sidereal systems, the heights and exact locations of mountains that have never been scaled, would be forever beyond us without these formulas or their equivalents. The correct mapping of a great cordilleran system such as that of western North America, or of a long and intricate coastal region with thousands of indentations and islands would be impossible if "triangulating" could not be used, and this requires the use of our two innocent-looking formulas.

In the chapter on functions and their graphs, there is a natural continuation of the subject of graphs as begun in elementary algebra. The Cartesian system is described fully; and then follow in succession brief considerations

of the analytic geometry of the straight line, parabola, circle, ellipse, hyperbola, the cubic, the special exponential 2^x, and the "curve of chance" or "normal distribution." Some of the simple connections of these with phenomena of general interest are pointed out. The planets and comets travel in orbits which are conic sections (probably all ellipses). The relative rapidity with which the functions x^2 and x^3 increase as x increases represents a fundamental fact which limits the size of both animate and inanimate structures, except masses of matter that assume spherical forms under the pressure of gravitational attraction. Biological forms provided with unlimited food and not in any way checked in their increase in numbers, will multiply according to a law represented by a curve of the type a^x (2^x is the one actually studied); and money placed at compound interest and subject to no untoward vicissitudes will increase according to the same law. Examples are given to show the statistical distribution of "traits" in plants and animals and of results which are said to depend upon chance. The equation involved here is of course too difficult for study at this stage and this is stated freely. The geometric series is studied and its use in financial problems is pointed out. The student is informed that the "Mathematical Theory of Investments," which is becoming of increasing importance, depends very largely on the geometric series and its sum; that critical statistical studies in economics, psychology, education, and other subjects are rapidly becoming indispensable, and that in all these the curve of "normal distribution" and its equation are fundamental. In short, the subject of which we are making a first study is in use everywhere about us. It is the heart of the equipment which is of constant use in man's present rapid conquest of the vast unknown.

In chapter III, the derivative is defined, beginning with a critical examination of the meaning of speed at a given instant (point) of time, followed by a similar consideration of acceleration, and then of the concept of tangent to a curve. Derivatives of simple algebraic polynomials and of simple cases involving square roots are found and these are used in writing the equations of tangents to curves and in finding maxima and minima. Derivatives of areas and volumes are also found. It is pointed out that there are many other types of functions whose derivatives are not found, and that there are many other uses to which the derivative may be put besides finding tangents and maxima and minima.

In Chapter IV the problem is to find an expression whose derivative is given. The meaning of constants of integration is studied and problems in finding areas and volumes are solved. The ease with which a volume such as that of the sphere may be found is truly impressive, especially to those who have studied solid geometry. The superiority of the methods of Newton and Leibnitz over those of the Greeks is stupendous. Is there anything in history

which shows more clearly the value of an accumulating body of knowledge and how each step in advance becomes henceforth the heritage of the race?

Certain very simple differential equations are solved such as $(d^2s/dt^2) = -g$. It is pointed out that frequently the world which we are seeking to investigate presents her secrets to us in, what is in effect, the form of differential equations and that these are often so complicated that as yet only approximate solutions are possible, the "problem of three bodies" for instance, being such a one. There is therefore original work still to be done by the keenest among those who are occupying themselves with research in mathematics.

It is pointed out that with the information which the class has about the calculus the area of a circle cannot be found, nor can the length of a single curve be found, nor the area of a surface unless it is flat. Indeed all along the course the limitations of what we are learning are pointed out and problems suggested which could be solved if we were to take time to extend our study, using however essentialy the methods with which we are becoming familiar.

The experience of the last seven quarters in this course, with college freshmen presumably of normal intelligence and preparation, has proved beyond a doubt that the subject matter is easily within their reach and that it can be covered in one quarter meeting five hours a week, and of course equally easily in a semester meeting three times a week. This experience has proved also that the course is of real interest to such students. It is the unanimous testimony of those who have taught it that the first year students show greater interest in this course than in other freshman courses in mathematics. This is also shown by the fact that it is being elected by a considerable number of grade A students who are not expecting to specialize in mathematics and who presumably select what they think is worth-while. Our purpose to build a course that should meet the needs of the type of student described at the beginning of this paper seems to have been realized to a reasonable degree. Certain it is that the time required is not beyond what such a student can afford to give to the subject. Not a few have expressed the desire that similar survey courses might be offered in a number of other departments so that those whose first interests lie at the other end of the campus might have an opportunity to acquire a wide range of authentic information about the subject matter, method, and general significance of the various parts of the field of science.

While the satisfaction of the needs of the non-mathematical student was the sole early aim in organizing this course, we did not go far with that work before the question arose as to whether the course might not be the very best possible introduction to college mathematics even for those who are to specialize in this subject or to do enough work in it to satisfy the requirements for physics, chemistry, or general engineering, and we came to the conclusion that an

affirmative answer is the right one. It was fully recognized that there is justification for the feeling of those who espouse the cause of "unified mathematics" that the classical separate courses fail to show the interdependence of these courses, an understanding of which is necessary for adequate comprehension of each. But at the same time, it was believed that the newer "unified" courses have a tendency to be superficial and that they fail to give that continued attention to each of the well defined separate elements of the subject, which is also necessary for adequate comprehension. The question arises as to whether this new arrangement may possibly possess the merits of both older plans and at the same time avoid the pitfalls of each.

Employing an extended figure of speech, let us consider the exploration and study of the geography and geology of an unknown region. The work may conceivably be done in several ways. The geographical features may be studied entirely without reference to the geological structure and a complete topographical map prepared, and then as a "separate course" the geological formations investigated. This would correspond to the old type separate courses. Again each part of the region may be studied completely as the work proceeds, the topographical map prepared and the geological structure deciphered for each part the first time the region is traversed. This would typify the unified course. A third possible mode of procedure would be to make a general preliminary reconnaissance giving fairly accurate ideas of the lay of the land and of the main geological conformations and the agencies which have been at work to make the topographical features what they now are. This would be followed by intensive and fairly separate study of the topography, and the geological formation and history. This corresponds to the plan we are now describing. The arguments for this plan appealed so strongly to us that the general survey course has been made a prerequisite for all other courses in mathematics except intermediate algebra.

The students who have had this course come to the more specialized courses in trigonometry, analytic geometry, and calculus with considerable insight into their purpose and method. They know in a general way the character of the whole region and each separate part has a significance which is conducive to a higher type of work. The methods of these subsequent courses are known sufficiently so they can be used to some extent where they are of service in any of these courses. Thus, for instance, derivatives can be used in analytic geometry.

Finally, I should like to register the personal opinion that a course closely similar to this survey course would be the best final course in the high school for those who elect more than two-and-a-half of three years of mathematics and for students in the normal schools. In European schools the ideas of the calculus are brought in much earlier than we have done heretofore. This

course, if given in the high school, would do just that for us, and would give those who do not go to college a much broader outlook on mathematics than they get now even if a half year is devoted to trigonometry, solid geometry, or higher algebra. The analytical part of trigonometry, for instance, has little significance for those who take that subject (and to take it is the present custom), while this survey course is full of meaning and interest throughout.

In this course there is much more attention given to the general human setting of the subject than is customary in our early college courses, at least so far as the character of these is revealed by the texts in use. What was the origin of these methods and formulas—what was their historical setting? What beautiful illustrations we have of how a scientific theory is built! At the time of its formulation as a general theory—the building of an edifice of thought—parts which enter into the structure are taken from many points in the long story whose culmination it is. Some parts may come from India, others from Egypt, Babylonia, Greece, northern Europe. These parts do not constitute the general theory any more than a pile of bricks constitutes a building. But the bricks are necessary. Thus by slow and laborious steps the necessary material is accumulated and in its time placed at the disposal of a genius who perceives "the golden threads that run through all and do all unite" and who thus furnishes the human race with another instrument for investigation. In what ways have the methods and theories which we are studying been of service to the race? What elements in our present technical information and practice are dependent upon them? What do we know about the world in which we live which they have in part been instrumental in discovering? How is the method of thought used in mathematics related to the thinking about other subjects? Such questions are constantly kept to the fore. To the dullest they mean little, but to the more intelligent they are fraught with meaning and may be the starting point of the formation of sane ways of thinking.

Supplementary Note—Certain questions have been raised about this paper which I believe can be considered more appropriately in a supplementary note than by rewriting the paper. "What is the unifying element of the course?" "Is it a lecture course and what is the nature and extent of the students' participation?" "Does not a course of this kind skim the cream off this subject matter, leaving the subsequent more formal courses uninteresting?" "Should not such a course be given to upper class students rather than to freshmen?" "How does this course differ from the 'unified' courses with which we are familiar?"

To reply to these queries in order: The unifying element is the general human significance and interest of the subject matter and the mathematical

method of dealing with it. This has determined the inclusion and exclusion of topics as well as the mode of treatment.

The course is no more of a lecture course than any other course in elementary mathematics. Theorems are developed and exercises and problems solved precisely as in any other such course. If this statement needs any qualification whatever, it may be this, that in making proofs of theorems there is a flavor of mathematical intuition and insight as over against logical rigor. The purely logical element finds full use in applying the theorems in the solution of problems. There is just as much logic necessary in making certain that a fundamental theorem applies in a particular way in dealing with given situations as there is in showing that this fundamental theorem follows from still more fundamental propositions.

If this kind of course renders subsequent courses less interesting by "skimming the cream" off them, the same indictment lies against a great many courses whose propriety we do not question. Limits are considered in geometry, to some extent in algebra (the sum of a series such as $1+\frac{1}{2}+\frac{1}{4}+\cdots$), and in analytic geometry in defining a tangent, while the main treatment of this topic of course comes in the calculus. Some trigonometry is now brought into elementary algebra and geometry; a considerable part of formal analytic geometry is given in a strong course in high school algebra where graphs are stressed; many are urging that derivatives be used in college algebra and also in analytic geometry. My own impression is that interest attaches to that with which we are partly familiar and that this course se ves to make the subsequent courses sufficiently familiar to make them more interesting. If this "skimming the cream" objection is valid how *very far* from the right track must be our European colleagues who introduce derivatives comparatively early!

The time at which such a course should be given was considered with some care, having in mind the legitimate demands of other subjects and the departments in which lie the student's main interests. There is a pretty general demand in various departments that certain of their courses be open only to upperclassmen. If this demand is acceded to impartially for all such departments it follows as a necessary consequence that these courses will be available only to a few. As the student advances in his work his interests tend (and I believe properly) to concentrate about his major work and in courses which have a more immediate bearing on the profession for which he is making special preparation. It was this more general point of view, having in mind the college course as a whole, and the many different departments which later come to claim the students main effort which decided us to arrange this course for freshmen. This course was devised with the idea that it should fit in naturally in a scheme in which similar courses are given in a number of departments.

This course differs from the ordinary "unified" course in that it is a "preliminary survey," followed by the standard separate courses in trigonometry, analytic geometry, and calculus. In these later courses there is no unnecessary repetition. There is a brief review and extension of the concepts already studied and then a further study of the main topic. The situation is very much analogous to what we now have in college algebra—a review of the advanced part of high school algebra, the quadratics for example.

The historical material has been placed in a separate chapter to avoid the scrappy notes we find in many texts and to make possible a connected story which shows something of real historical development. This material is used as each topic is considered, and there is a final reading of the whole chapter.

The following questions are from the final examination given at the end of this winter quarter:

In the triangle ABC, $\angle C=62°40'$, $\angle A=49°50'$, $c=1263$ feet. Find the remaining parts of the triangle.

Write the geometric series whose sum represents the accumulated value of an annuity of p dollars per year if the rate of interest is 4.5% and the number of years is 15.

A point moves along the y-axis so that $y=4t^3-6t^2+t-6$. Find a function of t representing its velocity and also a function representing its acceleration. For what values of t will the point be at rest? For what value of t will the acceleration be zero? Find the position of the particle for each of these values of t.

Examine the curve $y=6x^2+3x+8-5x^3$ for maxima and minima. How do you decide which value of x represents a maximum and which represents a minimum?

The curve $y=2x^4-7x^3+11x^2-14x+8$ meets the x-axis in the points $x=1$ and $x=2$. Find the area inclosed between these points by the x-axis and this curve.

Find the values of x for which the tangents to the curves $y=x^2$ and $y=x^3$ are parallel.

Using the method of the calculus, derive the formula giving the volume of a sphere.

ON DIOPHANTINE RELATIONS

By W. H. RITTENHOUSE, Philadelphia, Pa.

1. Theorem. If we have a rational, integral, homogeneous function, $F(x,y)$, and if we put $(\partial F/\partial y)=h$, $(\partial F/\partial x)=k$,

$$F\{(xd-hn)\ ,\quad (yd+kn)\}=G(x,y,n,d)\ ,$$
$$F\{(xd+hn)\ ,\quad (yd-kn)\}=H(x,y,n,d)\ ;$$

then $G(x,y,n,d)$ *will be integrally divisible by* $F(x,y)$, in the same sense that $x^m - y^m$ is in all cases divisible by $(x - y)$. A similar remark applies to $H(x,y,n,d)$.

PROOF:[1] Expand $F(xd-hn,\ yd+kn)$, regarded as a function of six independent variables, in powers of h and k, by Taylor's theorem for functions of two independent variables. The expansion will contain all the various partial derivatives up to those of total order, m, if m is the order of the homo-

[1] This proof was supplied by A. A. Bennett.

geneous function, $F(x,y)$ in x and y. Collecting the terms of total order, r, in h and k, we have as a typical set of terms

$$\sum_i \frac{\partial^r F(xd,yd)}{\partial x^i\,\partial y^{r-i}}\binom{r}{i}(-1)^i(hn)^i(kn)^{r-i}.$$

This set of terms is homogeneous and integral in n and d, individually. Removing the common factor $d^{m-r}n^r$, we have left for examination the sum,

$$S=\sum_i \frac{\partial^r F(x,y)}{\partial x^i\,\partial y^{r-i}}\binom{r}{i}(-1)^i h^i k^{r-i}.$$

We shall show that this sum is always integrally divisible by $F(x,y)$ if h and k are regarded as functions of x and y, by the previous relation, $h=F_y(x,y)$, $k=F_x(x,y)$. By Euler's formula for homogeneous functions, $mF(x,y)=xF_x(x,y)+yF_y(x,y)$. By repetition, we have

$$\binom{m}{r}F(x,y)=\sum_i\binom{r}{i}x^iy^{r-i}\frac{\partial^r F(x,y)}{\partial x^i\,\partial y^{r-i}}.$$

As suggested by Sylvester's dialytic method, we may write out the following sequence of equations:

$$\begin{aligned}
x^rF_{x(r)}+rx^{r-1}yF_{x(r-1)y}+\binom{r}{2}x^{r-2}y^2F_{x(r-2)y(2)}+\cdots+rxy^{r-1}F_{xy(r-1)}+y^rF_{y(r)}&=\binom{m}{r}F\,,\\
x^rF_x+x^{r-1}yF_y\qquad\qquad&=mx^{r-1}F\,,\\
x^{r-1}yF_x+x^{r-2}y^2F_y\qquad&=mx^{r-2}yF,\\
\cdots\cdots\cdots\cdots\cdots&\\
xy^{r-1}F_x+y^rF_y&=my^{r-1}F\,.
\end{aligned}$$

Now each pair of values of x,y, not both zero, which makes F vanish, reduces all the right hand members to zero, and leaves a set of $r+1$ homogeneous linear equations in the $r+1$ quantities, x^r, $x^{r-1}y$, $\cdots$, xy^{r-1}, y^r. The determinant of the coefficients in this set of equations must therefore vanish for such pairs of values. This determinant is exactly the expression, S. Since S vanishes with F, S must contain F as a factor.

2. The theorem above written gives, then, four expressions. G, G/F, H, H/F, which we shall find useful in studying diophantine properties of the function F. The arguments involved are x, y, n, and d, and one may study the variation of these functions as any one or all of the arguments are regarded as varying.

Consider the diophantine relation imposed upon x and y by the equation: $x^3+y^3\equiv f$, (1), where f is a given integer. In accordance with the theorem we have, if we put $F(x,y)\equiv x^3+y^3$,

$$(-y^2n+xd)^3+(x^2n+yd)^3=\{d^3+3xydn^2+(x^3-y^3)n^3\}\cdot f \tag{I}$$

$$(y^2n+xd)^3+(-x^2n+yd)^3=\{d^3+3xydn^2-(x^3-y^3)n^3\}\cdot f\,\cdot \tag{II}$$

If $h(x,y)$ and $k(x,y)$ have common numerical factors it is sometimes possible to suppress these if they do not appear as algebraic factors of the function, F, mentioned in the theorem. This remark is made here to prepare the reader should he find such factors omitted. We study then from the diophantine standpoint the relations obtained from the four expressions:
$X_1=-y^2n+xd,\ Y_1=x^2n+yd,\ X_2=y^2n+xd,\ Y_2=-x^2n+yd.$ (2)

The numerical tabulation of these four expressions, (2), selecting any fixed x and y, and allowing n and d to vary over integral values from zero to as high as one cares to go, will give interesting sets of four numbers, and their relations may be studied from the second members of (I) and (II).

3. Application of X^3+Y^3 to problems in fifth powers. Though we usually think of the cubic relation of X_1, and Y_1 in connection with (I) and (II), there exists the relation in *squares* and *biquadrates*: $X_1^4+Y_1^4-X_2^4-Y_2^4$ is algebraically divisible by $X_2^2+Y_2^2-X_1^2-Y_1^2$ for any X_1, Y_1, X_2, Y_2, satisfying (2). Suppose more generally that we have any four quantities A_1, B_1, A_2, B_2, for which

$$A_1^4+B_1^4-A_2^4-B_2^4=2(A_2^2+B_2^2-A_1^2-B_1^2)q^2\,\cdot \tag{3}$$

In (3) whether we use X_1, Y_1, X_2, Y_2, or any four integers of which the condition is true, as A, etc., we will always have:

$$(A_1+q)^5+(B_1+q)^5+(A_2-q)^5+(B_2-q)^5=$$
$$(A_1-q)^5+(B_1-q)^5+(A_2+q)^5+(B_2+q)^5. \tag{4}$$

This may be seen from the expansion of (4). In the numbers of the left-hand members of (2) to make q^2 a rational square in (3) we must solve the relation

$$xyd^2-q^2=n^2(x^5-y^5)/(x-y)\,\cdot \tag{5}$$

Given an x and y this is not always solvable, but solvable cases exist. For example: if $x=2$ and $y=1$, this becomes $2d^2-q^2=31n^2$, which is solvable, and solvable conditions may be obtained at least frequently if we make xy a square, by selecting x and y as squares, 1, 4, 9, etc. We then solve by finding X_1, Y_1, X_2, Y_2, of (2), writing finally the eight numbers of (4).

In one's search for such numbers, A_1, B_1, A_2, B_2, it is merely noted that the four expressions of (2) fulfill the condition. It is doubtful whether any of the eight numbers (4) obtained by using (2) will become zero, or whether two may be equated, in opposite members, but the solution of the reduced problems in which the number of terms in one or both members is reduced may be easily obtained by using expressions slightly different from (2). Take instead: $A_1=(-rn+xd)$, $B_1=(tn+yd)$, $A_2=(rn+xd)$, $B_2=(-tn+yd)$. (6)

If these values be substituted in (3), it reduces to

$$\frac{ty^3-rx^3}{rx-ty}d^2-q^2=\frac{r^3x-t^3y}{rx-ty}n^2\,\cdot \tag{7}$$

The two principal reduced problems will now be treated in detail.

First. To find the sum of three fifth powers equal to four fifth powers, with sums of corresponding first powers also equal, let $A_1+q=0$. Then

$$q=rn-xd\cdot \tag{8}$$

Second. To find the sum of three fifth powers equal to three fifth powers with sums of corresponding first powers also equal, let $A_1+q=B_1-q$. Then

$$q=\tfrac{1}{2}(y-x)d+\tfrac{1}{2}(r+t)n\cdot \tag{9}$$

Select any x and y in integers. Substitute in (7). It is noted that now q^2 in (7) contains a term in d^2 with a square as coefficient, with either (8) or (9). Make the coefficient of d^2 in (7) equal to this square. From (9) we write:

$$\frac{(ty^3-rx^3)d^2}{rx-ty}-\tfrac{1}{4}(y-x)^2d^2-\tfrac{1}{2}(y-x)(r+t)dn-\tfrac{1}{4}(r+t)^2n^2=\frac{(r^3x-t^3y)n^2}{rx-ty} \tag{10}$$

Make the *entire* coefficient of d^2 in (10) zero by writing:

$$t=x^3+\tfrac{1}{4}x(y-x)^2, \tag{11}$$

$$r=y^3+\tfrac{1}{4}y(y-x)^2. \tag{12}$$

As d^2 has vanished, the factor n will divide out of (10), and

$$d=\tfrac{1}{4}(r+t)^2(rx-ty)+(r^3x-t^3y), \tag{13}$$

$$n=-\tfrac{1}{2}(y-x)(r+t)(rx-ty). \tag{14}$$

In (13) and (14) r and t must be written in values of x and y, from (11) and (12), and finally we write q in (9).

Illustration. If $x=3$ and $y=5$, we find, (dropping the common factor 10 from r and t, and -24 from d and n): $r=13$; $t=3$; $d=-333$; $n=16$; $q=-205$. These are the final ratios which may be used, and we obtain:

$$1412+1822+586+1508=1002+1412+996+1918.$$

The fifth powers are also equal, and finally:

$$911^5+293^5+754^5=501^5+498^5+959^5.$$

NOTE: We have followed out in detail and illustration only (9). Equation (8) may be similarly followed out and illustrated, but the results are apparently the same as Haldeman's solution (*1918*, *399–402*) of three fifth powers equal to four fifth powers.

MATHEMATICIANS IN THE AMERICAN DICTIONARY OF NATIONAL BIOGRAPHY

A fund of half a million dollars has been established to make possible the publication of an *American Dictionary of National Biography*, comparable with the English work of similar title. The editor is Professor Allen Johnson of Yale University. In order to make the proper selection of names, the collaboration of scientific and historical societies is earnestly desired. A tentative list of American mathematicians (no longer living) to be included in the dictionary is appended. It is understood that mathematicians born abroad are to be classed as American if their principal activities have been with American institutions. Division A includes the names of those supposedly worthy of longer notices; division B includes somewhat doubtful names, or those to have briefer notices.

The Editor of the MONTHLY would appreciate suggestions on this list both as to other names to be included and as to names whose inclusion is

QUESTIONS AND DISCUSSIONS

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate methematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

REPLIES TO QUESTIONS

51 [*1924*, 85; *1925*, 506]. Can any reader supply approximate formulas for the problem of a cable suspended from two points at different levels?

REPLY BY MICHAEL GOLDBERG, Philadelphia, Pa.

This problem is met in the stringing of electric wires and cables in the transmission of electrical energy. The following papers on it have appeared in the *Proceedings of the American Institute of Electrical Engineers*:

"Solutions to Problems in Sags and Spans," Wm. Le Roy Robertson (1911) vol. 30, pages 1111–1129,

"Calculations for Suspended Wires," Percy H. Thomas (1911) vol. 30, pages 1131–1141,

"Mechanical Characteristics of Transmission Lines," H. Pender and N. de K. Thompson, vol. 30, pages 1379–1400.

If the span length is X, the difference of elevation of the supports is h, the maximum tension is T, and the load per foot is w, then x_1, which is the horizontal distance from the upper support to the lowest point in the wire is given by Thomas

$$\left.\begin{aligned} x_1 &= \frac{X}{2} + \frac{h}{K}\left(K - \frac{h}{2}\right), \\ x_1 &= \frac{\sqrt{d}}{\sqrt{d} \pm \sqrt{d-h}}, \end{aligned}\right\} \quad \begin{aligned} K &= T/w \\ d &= \text{distance of lowest point in wire below lower support.} \end{aligned}$$

$$x_1 = \frac{X}{2} + \frac{h}{X}K, \qquad \text{Molesworth's } \textit{Pocketbook}.$$

If S is the sag or dip for supports at the same elevation, the distance d of the lowest point below the lower support for different elevations is given by

$$d = s\left(1 - \frac{h}{4s}\right) \cdot \qquad \text{Pender.}$$

A still more accurate formula, which I have used, is given below:

$$x_1 = \sqrt{\left(\frac{3a\,X}{h}\right)^2 + \frac{3a}{h}(X^2+2ah)} - \frac{3a\,X}{h}$$

where $a = K - h/2$.

DISCUSSIONS

I. On the Vanishing of the Remainder of the Binomial Series

By C. F. Gummer, Queen's University

Let $S_{n,r}(x)$ represent the sum of the finite series $\binom{n}{0}+\binom{n}{1}(x-1)+\binom{n}{2}(x-1)^2+\cdots+\binom{n}{r}(x-1)^r$, where n is a real number. If x lies between 0 and 2, so that $x-1$ is numerically less than 1, the same series continued to infinity converges to the value $\{1+(x-1)\}^n$, or x^n. Whether x lies between these limits or not, let us denote the remainder $x^n-S_{n,r}(x)$ by $R_{n,r}(x)$.

Within the region of convergence, $R_{n,r}(x)$ is the sum of the infinite series $\binom{n}{r+1}(x-1)^{r+1}+\cdots$; and, if x is near enough to 1, the sign of this series is that of its leading term $\binom{n}{r+1}(x-1)^{r+1}$, the first term of the binomial series lacking from $S_{n,r}(x)$. If, for more remote values of x, these signs cease to agree, the change must occur where $R_{n,r}(x)$ becomes either infinite or zero. The infinite cases are found only at $x=0$, for values of n (such as -3) which make x^n infinite there with change of sign; and these need not detain us. The location of the values of x for which $R_{n,r}(x)$ is *zero* forms however a more interesting problem.

We may omit throughout the cases where $R_{n,r}(x)$ vanishes identically in x; these being the cases $n=0, 1, 2, \cdots, r$. In all other cases $R_{n,r}(x)=0$ is an equation having a root of multiplicity $r+1$ at $x=1$. Hence, if $S_{n,r}(x)$ is arranged in powers of x as $c_0+c_1x+c_2x^2+\cdots+c_rx^r$, $R_{n,r}(x)$ must, by Descartes' rule, exhibit at least $r+1$ variations of sign in its coefficients when the powers of x are in numerical order, x^n being inserted in its proper position. But the expression for $R_{n,r}(x)$ contains only $r+2$ terms. Therefore the number of variations must be exactly $r+1$, the coefficients having alternate signs.[1] It follows that $R_{n,r}(x)$ does not vanish for any positive value of x other than 1.[2]

The only real values of n for which the question of *negative* roots arises are of the form p/q, where p is an integer and q a positive odd integer. When x is replaced by $-x$, each variation of sign between consecutive coefficients becomes a permanence, except in some cases where one of the terms is x^n. Hence there cannot be more than one negative root in any case, and the following results are easily verified.

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[1] This fact may be proved directly by finding the values of the c's the result being $c_i=(-1)^{r-i}\binom{n}{i}\binom{n-i-1}{r-i}$.

[2] It is of course easy to see from the finite Taylor development that $R_{n,r}(x)$ has the same sign as $\binom{n}{r+1}(x-1)^{r+1}$ for every *positive* x.

For $n=p/q$, q odd;

(1) If $n>r$, and $p-r$ is even; there is one negative root to $R_{n,r}(x)=0$.

(2) If $n>r$, and : $p-r$ is odd; no negative root.

(3) If $0<n<r$: one negative root.

(4) If $n<0$, p even; one negative root.

(5) If $n<0$, p odd; no negative root.

It is easy to illustrate these statements by drawing the curve $y=R_{n,r}(x)$ in the various cases. To do so it is necessary to note the character of n, the parity of r , which decides whether the curve crosses the x-axis at $x=1$, and, if $n<r$, the sign of $\binom{n}{r}$. It will also be observed that, when p is an even positive integer and q an odd integer greater than p, there is a cusp with vertical tangent where the y-axis is crossed.

When $n=p/q$, p an odd integer and q an even positive integer, the complete algebraic curve has, besides $y=R_{n,r}(x)$, a lower branch $y=T_{n,r}(x)$, where $T_{n,r}(x)=-x^n-S_{n,r}(x)$. This curve has no points to the left of the y-axis. When p is greater than q it has a cusp on the y-axis, with non-vertical tangent.

The question now arises whether the lower branch may have points in common with the x-axis; that is, whether, for certain values of x, $S_{n,r}(x)$ may be equal to the *negative* qth root of x^p. To settle this we may note the signs of $T_{n,r}(x)$ when $x=1$ and as x becomes zero and infinite. From what we already know of $R_{n,r}(x)$, these are readily found to be as in the following table.

					No. of roots in	
		$T_{n,r}(0)$	$T_{n,r}(1)$	$T_{n,r}(\infty)$	$(0,1)$	$(1,\infty)$
(6)	$n>r$, r odd.	$+$	$-$	$-$	1	0
(7)	$n>r$, r even.	$-$	$-$	$-$	0	0
(8)	$0<n<r$, r odd, $\binom{n}{r}<0$.	$+$	$-$	$+$	1	1
(9)	$0<n<r$, r odd, $\binom{n}{r}>0$.	$-$	$-$	$-$	0	0
(10)	$0<n<r$, r even, $\binom{n}{r}<0$.	$-$	$-$	$+$	0	1
(11)	$0<n<r$, r even, $\binom{n}{r}>0$.	$+$	$-$	$-$	1	0
(12)	$n<0$, r odd.	$-$	$-$	$+$	0	1
(13)	$n<0$, r even.	$-$	$-$	$-$	0	0

The numbers of roots noted in the two right-hand columns are evidently correct *modulo* 2. Moreover, if $r=0$, which may occur in cases (7) and (13), the figures are exactly true. To show that they are exact in all cases, we may use an inductive argument, using the fact that the derivative of $T_{n,r}(x)$ is $n\cdot T_{n-1,r-1}(x)$. For instance, if there were more than one root in the interval $(0,1)$ under case (6), there would be at least two values in the interval at which the derivative was zero, so that case (7) would be incorrect for a smaller value of r. This argument will be found sufficient for each case, except where

the zeros are given in italics; and these zeros are evidently correct, since here $R_{n,r}(x)$ (and therefore also $T_{n,r}(x)$) is negative throughout.

Some numerical instances follow of the roots of $R_{n,r}(x)=0$ and $T_{n,r}(x)=0$ under the various cases.

(1) $R_{3,1}(-2)=0$; *i. e.* $1+3(-3)=(-2)^3$.

More generally, if n is a positive integer, and $x=-n+1$, then $x^n=S_{n,n-2}$.

(3) $R_{2/3,1}(-1/8)=0$; *i. e.* $1+\frac{2}{3}(-1\frac{1}{8})=(-1/8)^{2/3}$.

(4) $R_{-2,r}(x)=0$ if $x=-1/(r+1)$.

(6) $T_{3/2,1}(1/4)=0$; *i. e.* $1+\frac{3}{2}(\frac{1}{4}-1)=-(1/4)^{3/2}$.

(8) $T_{3/2,3}(7\pm4\sqrt{3})=0$.

(10) $T_{1/2,2}(9)=0$; *i. e.* $1+\frac{1}{2}\times8-\frac{1}{8}\times8^2=-9^{1/2}$.

(11) $T_{3/2,2}(1/9)=0$.

(12) $1-\frac{1}{2}(4-1)=-4^{-1/2}$.

II. Al-Bîrûnî's Computation of the Value of π

By Carl Schoy, Frankfurt am Main

Al-Bîrûnî,[1] who wrote at Ghazna *c.* 1000, was perhaps the first of the Arab writers to find the value of π to any very high degree of accuracy; or rather, in his case, the reciprocal of this value. In this he anticipated the work of al-Kashî[2] by about four centuries, although the value found by each was the reciprocal of that found by the other.

Al-Bîrûnî's treatment of the subject is found in the work *al-Qânûn al-Mas'ûdî*, Book III, Chapter V, and is as follows:[3]

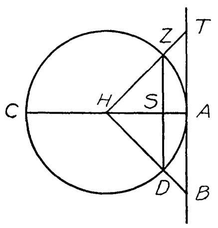
It is a simple matter to express by a single number the ratio of the diameter of a circle to the circumference. Indeed, in making the attempt, altering the problem somewhat, the only major change refers to the procedure and improvement in the graduation of the circle, the extent depending upon the amount by which scientists differ.[4] The circumference consists of 360 parts, each part being divided sexagesimally. The origin (ašl) of this usage is to be found in the fact that the number 360 is the mean between the number of days of a sun year and a moon year, although there is no confirmation of this

[1] Mohammed ibn Aḥmed, Abû'l Rîḥan (or Raiḥân), al-Bîrûnî (or al-Bêrûnî), a native of Khwarezm, *c.* 973–1038. See Smith, *History of Mathematics*, I, 285.

[2] Jemshîd ibn Mes'ûd ibn Maḥmud, Ġiyâṯ ed-dîn al-Kashî. He died *c.* 1436. See Smith, *loc. cit.*, I, 289, 290.

[3] Translated by David Eugene Smith from Dr. Schoy's manuscript.

[4] Cf. *Isis*, XIV, 371.



hypothesis.[1] The circumference of a circle bears a fixed ratio to its diameter, and if the number of the circumference be placed in proportion to that of the diameter there results the approximate number of this ratio.

On the figure there should be placed the necessary symbols. Through A, the extremity of the diameter AHC, we draw the perpendicular AT and connect (the center) H and Z,[2] producing it to cut the perpendicular at T. Because SZ is half the side of a 180-gon, which subtends two parts (degrees) of the 360 of the circumference, 2. SZ is the chord of 2°, the value of which is $2^P\ 5^I\ 39^{II}\ 43^{III}\ 36^{IV}$, the circumference being (approximately) made up of 180 such chords. The sum of all these chords is $6^P\ 16^I\ 59^{II}\ 10^{III}\ 48^{IV}$. Taking the diameter as 2, we have

$$\frac{\text{Diameter}}{6^P16^I59^{II}10^{III}48^{IV}} = \frac{1}{3^P8^I29^{II}35^{III}24^{IV}}. \tag{1}$$

But the circumference of the circle exceeds the sum of all these sides of the (regular) inscribed 180-gon.

Furthermore, since we have the proportion $TA:AH = ZS:SH$ we have $TA = 0^P\ 1^I\ 2^{II}\ 50^{III}\ 19^{IV}\ 43^V$; and since $TB = 2TA$, we have $TB = 0^P\ 2^I\ 5^{II}\ 40^{III}\ 39^{IV}\ 26^V$, and the number representing the perimeter of the circumscribed polygon is $6^P\ 17^I\ 1^{II}\ 58^{III}\ 19^{IV}$.

Hence we have

$$\frac{\text{Diameter}}{6^P17^I1^{II}58^{III}19^{IV}} = \frac{1}{3^P8^I30^{II}59^{III}10^{IV}}. \tag{2}$$

But the circumference of the circle is less than the sum of the sides of the [regular] circumscribed 180-gon, whence

$$\frac{1}{3^P8^I29^{II}35^{III}24^{IV}} > \frac{\text{Diameter}}{\text{Circumference}} > \frac{1}{3^P8^I30^{II}59^{III}10^{IV}}.$$

Therefore the circumference lies between the two values $3^P\ 8^I\ 29^{II}\ 35^{III}\ 24^{IV}$ and $3^P\ 8^I\ 30^{II}\ 59^{III}\ 10^{IV}$. We now make the same assumption that Ptolemy makes in Book VI of his Almagest, taking the arithmetic mean between the two, whence we have

$$\frac{\text{Diameter}}{\text{Circumference}} = \frac{1}{3^P8^I30^{II}17^{III}16^{IV}46^V30^{IV}}. \tag{3}$$

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[1] A very original idea of al-Bîrûnî's, but purely conjectural.

[2] As in the original. He seems to have taken AHZ as 1° and then to have constructed the chord of 2° the value of which he knew. See this MONTHLY, (*1926*, 95–96).

Since the value of that part of the denominator which follows 3^p is approximately 1/7, equation (3) may be written[1]

$$\frac{\text{Diameter}}{\text{Circumference}} = \frac{5184000}{1628681471}. \tag{4}$$

If we take the circumference as 360^p, the diameter is 114^p, and the remaining fraction is $954312306/1628681471 \cdot 360$, so that

$$\frac{\text{Diameter}}{\text{Circumference}} = \frac{114}{360} + \frac{954312306}{1628681471 \cdot 360}.^{2} \tag{5}$$

III. New Light on Babylonian Mathematics

By L. C. Karpinski, University of Michigan

In his recent *Babylonia und Assyrien*,[3] Bruno Meissner, includes a summary of the natural and exact sciences as known to the Babylonians. This presents considerable material in the mathematical sciences not yet found in works on the history of mathematics although the material has been extant for some years in monographs and journals devoted to the history of the Orient.

The greatest interest attaches to two methods of approximation to find the diagonal of a rectangle in terms of the sides. The one method corresponds directly to the formula $\sqrt{x^2+a^2}=x+(a^2/2x)$ while the other method gives for $\sqrt{x^2+a^2}$ the value $x+2a^2x/3600$. Both methods are found on tablets

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[1] The reciprocal value of the right side of (3) is 3.141742, and that of the right side of (4) is 3.141746, so that one agrees very well with the other. The approximation for π thus found by al-Bîrûnî is therefore somewhat larger than the value 3.14166, already known to the Greeks and the Hindus. At any rate the work in this chapter shows that the Moslem scientists had their own peculiar method of approach tothe calculation of this ratio. The above plan shows clearly the line of approach and is much better than the one which Ibn al-Haitam (965–1039) had given in his work on the quadrature of the circle. (Compare H. Suter, "Die Kreisquadratur des Ibn al-Haiṯam," *Zeitschrift für Mathematik und Physik*, histor.-literar. Abteilung, 1899, pp. 33–47.)

Of the values for chord 2° and chord 1° calculated by al-Bîrûnî, he selected the worst (it being too large) for the determination of π. Similarly in his preceding computation, where he gives $AT=\tan 1°$,although it is greater than the values of tan 1° as given in his "table of shadows" (tangent table). (*Orient. MS.*, Okt. 275 Berlin, p 78, where $\tan 1° = 0^P\ 1^I\ 2^{II}\ 50^{III}\ 17^{IV}$.) For a later computation, in al-Bîrûnî's style, of a very exact value of chord 2°, compare the value of Jamshîd, al-Kashî, already mentioned, namely, $0'\ 2^I\ 5^{II}\ 39^{III}\ 26^{IV}\ 22^V\ 29^{VI}$ (See *Isis*, No. 14,p. 389; Smith, *History*, I, 290), and for tan 1° the value given in the "table of shadows" of Ulugh Beg (*Berliner Pers. MS.* 280, fol. 36, *v.*), namely, $0^P\ 1^I\ 2^{II}\ 50^{III}\ 17^{IV}\ 38^V$, whence $=3.141671$. On Ulugh Beg (1393–1449), see Smith, *History*, I, 289.

[2] It is easy to derive equation (5) if we increase (4) by 360, multiply 5184000 by 360, and divide, the result by 1628681471, after the quotient 114 discontinues, etc.

[3] Heidelberg, 1925, 2 vols. In *Kulturgeschichtliche Bibliothek*. The references are to chapter 21, in vol. II, pp. 380–418.

in the British Museum and the methods were published in 1916 by E. F. Weidner;[1] both are applied to the rectangle of sides 40 and 10, giving 41:15, *i. e.*, $41\frac{15}{60}$, 42: 13: 20, or $42\frac{13}{60}\ \frac{20}{60^2}$, respectively. The material dates from *c.* 2000 B.C. The Pythagorean theorem is not mentioned, yet it is implicitly involved in the first method mentioned, if not in both.

Another interesting series of problems concerns the measurement of the area of irregular fields by subdividing the fields into a number of rectangles and triangles. The method is not materially different from modern methods with such areas.

Recent discoveries in the history of astronomy make it quite certain that Hipparchus drew considerable material from the Babylonian scientists. The above material, and equally the recently discovered Egyptian papyrus[2], show the dependence of early Greek mathematics (notably work on rectangles) upon Egyptian and Babylonian mathematics.

IV. Napier's Rods in China

By Père Louis Vanhée, S. J. Brussels

Professor Smith's statement in his *History of Mathematics* (vol. II, p. 203), "The Napier Rods found their way unto China at least as early as the beginning of the 18th century," and Mr. David Chin-te Cheng's recent article in the Monthly (*1925*, 492), both open up a very interesting question as to the precise date of their introduction. Fortunately we have sources at hand which give us this information to a high degree of accuracy, but they seem not to have found their way heretofore into western literature relating to the history of mathematics.

The Jesuit fathers, Johann Adam Schall von Bell (1591–1666), a native of Cologne, and Giacomo Rho (1593–1638), a native of Milan,[3] held office in the Peking Astronomical Board near the close of the Ming Dynasty (1368–1644) and were the first to introduce this device into China. Since their time it has been known under the name *ch'ow suan* (*Computing by Rods*) and has been highly esteemed by Chinese astronomers and mathematicians. It is discussed by Mei Wen-ting (1633–1721),[4] the ablest Chinese mathematician of his time and the leading writer upon the history of the mathematical sciences of his people. His memoirs on the subject appeared in his collected works in sixty sections (the *Li-suan-ch'uan-shu*, published by Wei Li-t'ung

[1] "Die Berechnung rechtwinkliger Dreiecke bei den Akkadern um 2000 v. Chr.", *Orientalische Literaturzeitung*, vol. 19, columns 257–263.

[2] First published by B. Touraeff in *Ancient Egypt*, 1917, and later by the writer in *Science*, vol. 57, pp. 528-529.

[3] Smith, *History of Mathematics*, vol. I, p. 436.

[4] *Ibid.*, vol. I, p. 436; vol. II, p. 170.

in 1725). Unfortunately the term *ch'ow*, which is used to mean the Napier Rods, is the same one that had long been used to mean "counters," "computing sticks," and similar devices, and hence great care is necessary in order to avoid confusion as to its precise meaning. In Mr. Cheng's article, for example, the term "computing rods" is used in such a way as to add to the confusion and to obscure the real meaning of the expression.

In order to clarify the situation it should be observed that, according to the works of Rho, Schall von Bell, and Mei Wen-ting, who really set the standard of usage, the Napier Rods should be called *ch'ow*, and all the other similar instruments should be spoken of as *ts'e* (*computing sticks*, or *counters*). Mr. Cheng states (p. 494) that the use of "computing rods" began about the thirteenth century, and he gives a number of illustrations showing rods with numerals carved upon them. Now according to Mei Wen-ting[1] there were no numerals on the ancient Chinese counters, and he distinctly states that those having numerals were purely a European invention. It follows that whatever these old counters may have been, they were not of the type described by Mr. Cheng and certain other modern Chinese scholars. The rods with numerals upon them are, according to all reliable historical evidence, of European origin. Indeed, I have personally made a careful and extended analysis of Schall von Bell's astronomical works as published in Peking, this analysis appearing in the well-known Sinological monthly, *T'oung-pao* (Paris). In this it is clearly shown that the introduction of the rods is due to him and Rho as already stated.

Chinese mathematicians always make use of a loose terminology. For example, in 1744 Tai Chen published a work entitled *Ts'e-suan*, *ts'e* being merely another name for "counter" or "rod," and the work refers to Napier's rods, although this would hardly be inferred from the title. On the other hand, Ch'in K'iё, in his *Suan-fa-ta-ch'eng* (*Mathematical Compendium*) of 1843, a work in twenty-one sections, mentions the *suan pan* as the most convenient device for calculating, and the *ch'ow suan* (meaning the Napier Rods) as the next in order of merit. In this usage he followed an earlier compiler, *Fang Chung-t'ung*, who described the rods in his *Shu-tu-yen* (*Mathematical Summary*), a work written in 1687 but, owing to political reasons, not printed until 1721. In this he includes material from Matteo Ricci's translation of Euclid[2] and from an earlier treatise on the abacus, a section on arithmetic, one on the sector from Ricci's *T'ung-wen-suan-che*, and other material from the Jesuit Collection known as the *Sin-fa suan shu*, and other early treatises.

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[1] *Ku-suan-k'i-k'ao*. See Smith, vol. II, p. 170.
[2] Smith, *History*, vol. I, p. 303.

Mr. Cheng's article is valuable as showing the Chinese modification of the Napier Rods, and is very fair in its statement that the latter may have been due to the Jesuit fathers. What we would here emphasize is that this is the case, and we would call attention to the fact that the booklet by Fathers Schall von Bell and Rho is available in the large collection of works edited at Peking under the direction of Minister Siu.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

An Introduction to Mathematical Probability. By J. L. COOLIDGE. New York, Oxford University Press, American Branch, 1925. viii+216 pp. Price $5.00.
Calcul des Probabilités. By PAUL LEVY. Paris, Gauthier-Villars et Cie, 1925. viii+350 pages. Price 40 fr.

The publication by a firm with English and American connections of a text in English on mathematical probability written by an able and clear mathematician is a noteworthy and most welcome occurrence. Regardless of the requirement, laid down by most graduate schools, of a reading knowledge of French or German, it is nevertheless true that many American teachers and students are not willing and ready to delve in the existing treatises in those languages, particularly in a subject so full of pitfalls and often so difficult of exposition as the theory of probability. The book by Keynes has been sufficiently well criticized[1] to make it evident that it did not fill the need for such a text. It may be merely pointed out here that it was written, to quote Keynes' own words, "as a Fellowship Dissertation"; it can scarcely be styled a mathematical treatise, even though it may continue to be praised as a so-called philosophical treatment. One may well contrast the latter book with the very excellent but brief article "On Probability" wherein Charles Jordan[2] of the University of Budapest distinguishes sharply and accurately mathematical, empirical and philosophical probability, sketches concisely the leading theorems (Bayes, Bernoulli, Inversion, Rule of Succession), and discusses the mutual bearings of these without indulging in a wealth of "philosophical" discussion which brings about only a befogging of the ideas.

Coolidge's book may be divided roughly into the theory of probability, contained in Chapters I–VI, and its applications in Chapters VII–XI. After an introductory chapter in which he discusses the scope and meaning of mathematical probability and adopts three definitions suited to three chief

[1] Review by E. B. Wilson, *Bulletin A. M. S.*, vol. 29, 1923, p. 319.
[2] *Proc. Physico-Math. Soc. of Japan*, 3rd ser., vol. 7, no. 5, 1925, pp. 96–109.

conditions, he treats the elementary principles (total and compound probability, expectation, risk), Bernoulli's theorem, mean value and dispersion, geometric probability, and probability of causes. Only a thoughtful and repeated reading of the book will show how clearly a topic such as Bernoulli's theorem is presented and is related in its logical bearing to the first definitions and is safeguarded in its practical application to problems in chance. Equally important is it that our students have in Chapter VI a discriminating statement and discussion of Bayes' principle, both for a finite number of causes and for continuous probability, both for the probability of an operative cause and as applied to future events.

The later chapters treat errors of observation, errors in many variables, indirect observations, the statistical theory of gases, and life insurance so far as this involves considerations of probability. In determining "the best value" among a set of discordant measures, Coolidge lists carefully the postulates leading to the weighted arithmetic mean, preferring Schiaparelli's postulate of a singly differentiable continuous function of the measures to the Schimmack postulate sometimes used. Further assumptions, advanced and supported as being natural and plausible, lead to a modification of Gauss' first proof of his law of error; the author wisely prefers to adopt a single perspicuous set of assumptions rather than to adopt a less plausible proof which might be more elegant from the axiomatic standpoint or to make an extended critique of various methods of historical importance. Coolidge incorporates here his development of the Gaussian law of error for any number of variables as given in the *Transactions of the American Mathematical Society* for 1923, with a reference to the error ellipse applied in artillery practice and the "Big Bertha" shots of 1918. A fuller account of the use of the correlation coefficient in statistics with some illustrative examples might well have been given in the treatment of that subject. The method of least squares and the theory of curve fitting are treated satisfactorily but concisely, a particularly clear explanation of the meaning of the "best representation" of a given function by a curve being given. Depending on Castelnuovo as the moving spirit and assembling as before a list of clean-cut assumptions, the author develops Maxwell's law of the distribution of velocities in a gas and the supplementary work concerning the probability that a gas shall be in a nearly normal state. The book closes with the placing of the principles of life insurance on their basis in the theory of probability, a description of commutation tables, and, at the end, tables containing the common logarithms of e^x and e^{-x}, and the probability integral.

The book is made more valuable for class use through well selected problems throughout, especially in the first two-thirds of the book.

Levy consciously makes the law of large numbers (Bernoulli's) the central feature of his treatise and the foundation of the calculus of probability, problems in games of chance being regarded merely as simple problems which enable one to grasp the real meaning of the later principles. The second outstanding feature of the book is that Levy bases his rigorous development of Gauss' law for accidental errors on the notion of characteristic function, a method which at least in considerable part goes back to Cauchy and 1853. Levy justifies this, as against the direct derivation of Gauss' law from the binomial formula, on the ground that when once the fundamental properties of the characteristic function are found, a large number of important consequences are obtained, adopting thus a unifying principle similar, say, to that which Hilbert uses in his organization of integral equations.

The book is divided into two main parts. The first part on the principles of the calculus of probability contains chapters on subjective probability, principles derived experimentally, objective value of probability, laws of probability (the law of Gauss and its relation to the law of large numbers), probability derived from experience (statistics), and a critique of the theory of expectation or risk. The second part on the mathematical theory of probability treats in successive chapters general notions about the laws of probability and the theory of sets, probable values, characteristic coefficients and functions (his basic chapter, including his proof of Gauss' law of error), the composition or combination of laws of probability, laws of variable probabilities, a more extended study of the law of large numbers, exceptional laws the theory of errors, and the kinetic theory of gases.

W. D. CAIRNS

Romance in Science. By BESSIE I. MILLER. Boston, The Stratford Co., 1924. 87 pages. Price $1.00

This very interesting little book contains a series of lectures delivered by the author in an elective course offered by the mathematical department of Rockford College. It first appeared in the college catalogue under the double heading, "The Romance of Science," or "The Grammar of Science," according to which you wish to get out of it. Officially it is known as "Some Literature of Science." But in the registrar's office it has always been known by the significant title *Browse*, and that is evidently the name favored by the author. We read:

"It is Browse which best describes the method of study advised in the course. Have you ever seen a sleek, biscuit-colored Jersey cow, hock deep in the lush grass of a meadow where it slopes down toward a transparent stream tinkling over brown stones in the shadow of large weeping-willow trees? It is a Jersey cow which browses. A Jersey crops only now and then, and between times gazes meditatively over the landscape."

That it is a very rich meadow growing an extraordinary number of different varieties of grasses in which this browsing is done is evident from the following

chapter headings. The scientific method, Law, The fourth dimension, The fourth dimension and non-Euclidean geometry, Einstein's theory of gravitation, Transformations, The human significance of mathematics.

The "browsing" is done not only in the field of mathematics, but also in astronomy, architecture, physics, biology, literature, philosophy, psychology, geology, chemistry, medicine, law, art, music, and religion. Like *The Education of Henry Adams* this book is delightfully informal, not to say unmethodical and contains many interesting and instructive bits of philosophy.

To an old college teacher of mathematics the most astonishing thing about this "*Browse*" course is that the prerequisite for it is only "Freshman mathematics or enrollment in that course with a minimum grade of B for the first semester. This secures some knowledge of analytical geometry, limits, and differentiation on the part of all, although it is a very elementary knowledge." That the lectures are delivered to college girls, some of whom do not even elect mathematics after the freshman year, also seems rather extraordinary.

One meaning of browse, as given in the dictionary, is: "To eat or nibble off, as the tender branches of trees, shrubs, etc." This is the kind of browsing preferred by most of the undergraduates I have met; they are very partial to those tender shoots from the tree of knowledge that are easy to masticate and digest. Like children they like to have the meal all dessert. But that is bad for the digestion. A little browsing in every lesson, in any subject, is a good wholesome dessert; that is what makes a subject interesting to the student. But it should be given in homeopathic doses and confined to topics within the range of the student's knowledge and mental capacity.

In the appendix is found a Reading List for "Browsing," comprising 63 books ranging all the way from *Jungle Peace* by Beebe, through the whole gamut of scientific knowledge, to *The Einstein Theory of Relativity*, by Lorentz.

W. A. GRANVILLE

A Brief Course in Analytic Geometry and the Elements of Curve Tracing. By W. B. FORD. New York, Henry Holt and Co., 1924. 288 pages. Price $2.40.

This text follows pretty closely the usual material presented in such a book. Starting with the definition and fundamental ideas, the author works through chapters on the line, circle, and the conics in the usual order, the transformation of coordinates, and polar coordinates. Then comes a chapter on other well known curves and one on curve fitting, a subject which seldom is found in a book of this kind. There is also a very short study of solid geometry, limited as usual to the line, plane and quadric. The book closes with a few trigonometric formulas and tables, answers to the problems, and an index.

Only a reference is made to oblique coordinates and in many places the author hesitates to give much of what one might call theory. If he showed that the graph of an equation is unaltered by such operations as the transposition

of a term, or multiplication or division by a constant different from zero, he would not have to give such definite instructions about the reduction of the general linear equation to any of the standard forms. It seems unfortunate to give the student the impression that he is not qualified to study a proof of the theorem that a first degree equation in two variables must graph into a straight line. The theorem on the classification of the graphs of quadratic equations is stated without reference to the exceptional cases. The degenerate cases are picked up in the next article, but no reference is made to the fact that all the solutions of an equation may be imaginary.

The author's choice of material in the chapters on the conics is very good, though some will wish that he had included a study of the diameter as a locus of middle points. The figures are very good and the general form is very pleasing.

C. R. MacInnes

Analytic Geometry. By R. W. Brink. New York, The Century Co., 1924. 274 pages. Price $2.25.

This is intended to be a text book for the usual course in the subject. It aims to be a book on the general subject of analytic geometry rather than one on the conic sections, as so many have been. The author opens with a short chapter devoted to things involving the coordinates of isolated points, such as the distance between two points and the point dividing a line segment in a given ratio. One wonders why it did not include also the slope of the line joining two points and the area of a triangle, derived by the trapezoid method, a method which seems to most beginners less cumbersome than the one involving the base and altitude. This is followed by a chapter on the straight line and by one giving a general discussion of the graphs of equations, including the ideas of symmetry, values of one coordinate for which the other becomes imaginary or infinite, and the degenerate cases. Then follow chapters on locus problems, transformation of coordinates, polar coordinates, and tangents. After these general ideas are disposed of, the author brings in two substantial chapters, the first on the circle and the next on the other conics. It seems strange that the existence of a second focus and directrix in the ellipse and hyperbola should not be mentioned. A chapter on some well known curves finishes plane geometry and a short study of solid geometry closes the book. This includes the line, plane and quadrics, except the cone.

An appendix gives a collection of formulas of trigonometry and algebra. It also gives the fundamentals of two and three row determinants, which are used throughout the book. The author has gathered a great many problems of fairly varied type, some with answers and many more without. Oblique coordinates are not even mentioned. The figures are very good and the detail work has been well done. Problem 2, page 156, might have been more carefully stated.

C. R. MacInnes

Statistical Analysis. By E. E. DAY. New York, MacMillan Co., 1925. xxvii+459 pages. Price $4.00

While this book does not "enter upon the more refined mathematical phases of statistical method" and therefore calls for little comment here, it is of importance that no serious objection can be found to the statement that "nothing in the book should prove inconsistent with the findings of the most advanced mathematical statistics." Writers of such books and teachers of corresponding courses are not always as sympathetic toward the demands of the mathematician for rigor and it is a distinct pleasure to recommend this book for its sound doctrine and logic. A more extended title of "The Logic of Statistical Analysis" is suggested by the author.

In addition to the traditional treatment of averages, correlation, etc., there is, at the beginning, a thorough discussion and treatment of the various terms used in the tabulation, classification and analysis of observed data. The latter part is devoted to the treatment of time series and of index numbers. The treatment throughout is careful and thorough. C. H. FORSYTH

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

American Journal of Mathematics, volume 48, no. 1, January, 1926: "Maps of Twelve Countries with Five Sides with a Group of Order 120 Containing an Ikosahedral Subgroup" by H. R. Brahana and A. B. Coble, 1–20; "On Surfaces and Curves which are Invariant under Involutory Cremona Transformations" by A. Emch, 21–44; "The Invariant System of Two Associated Bilinear Connexes" by O. E. Glenn, 45–56; "Arithmetics of Generalized Quaternion Algebras" by C. G. Latimer, 57–66; "Conditions under which One of Two given Closed Linear Point Sets may be Thrown into the Other One by a Continuous Transformation of a Plane into Itself" by R. L. Moore, 67–72.

Annals of Mathematics, second series, volume 27, no. 2, December, 1925: "Notes on the Ampère-Cauchy derived function" by H. L. Smith, 69–72; "Note on the existence of analytic solutions of non-homogeneous linear q-difference equations, ordinary and partial" by C. R. Adams, 73–83; "Incidence and parallelism in biaffine geometry" by A. A. Bennett, 84–86; "New proofs of the simplicity of every alternating group whose degree is not four" by G. A. Miller, 87–90; "Symmetric tensors of the second order whose covariant derivatives vanish" by H. Levy, 91–98; "Generalizations of the eight square and similar identities" by E. T. Bell, 99–104; "On a geometrical theory of continuous groups. I. Families of path-curves of continuous one-parameter groups of the plane" by B. de Kerékjártó, 105–117; "Group velocity and the propagation of disturbances in dispersive media" by F. Wood, 118–128; "On an expression of a minor of order two of the Mth compound of a determinant A in terms of minors of A of order higher than M" by W. H. Metzler, 129–132; "The nullity of a matrix relative to a field" by C. C. MacDuffee, 133–139; "Determination of the ternary collineation groups whose coefficients lie in the $GF(2)$" by R. W. Hartley, 140–158; "On the oscillation of harmonic functions" by F. W. Perkins, 159–170.

Journal of Mathematics and Physics, Massachusetts Institute of Technology, volume 5, no. 2, February, 1926; New Formulation of the Laws of Quantization of Periodic and Aperiodic Phenomena"

by Max Born and Norbert Wiener, 84–98; "The Harmonic Analysis of Irregular Motion" by Norbert Wiener, 99–121; "A Method of Solving Power Networks by Means of Conjugate Vectors" by W. L Lyon and F. L. Hitchcock, 122–125; "Note on a Method of Evaluating the Complex Roots of a Quartic Equation" by Y. H. Ku, 126–128.

Mathematische Zeitschrift, volume 24, no. 4, February, 1926: "Note on the location of the roots of a polynomial" by J. L. Walsh, 733–742.

UNDERGRADUATE MATHEMATICS CLUBS

EDITED BY H. J. ETTLINGER, 3110 Harris Park Ave., Austin, Texas

CLUB ACTIVITIES

PI MU EPSILON, Hunter College.

The following play was put on by the Hunter College chapter of Pi Mu Epsilon at the American Mathematical Society dinner, Jan. 1, 1926. The author requests that his name be withheld.

A RELATIVITY DRAMA

CAST OF CHARACTERS: HUMANITY, LIGHT, MICHELSON, MORLEY, MILLER, EINSTEIN, FOUR ASTRONOMERS, FIVE TOOLS.

Light, a young woman, is beloved of all on account of her many virtues and extreme usefulness. She has been accused by the astronomers of an affair with the Sun and, at their behest, has been summoned into court to be tried before Humanity. The scene opens with Humanity sitting as judge, the astronomers as plaintiffs on the right. To the left stand certain persons who describe themselves in the action. To the left forward, stands Light herself in the prisoner's box.

HUMANITY: You astronomers, here you are with a charge against fair Light. Let me warn you that it must be a clear case and well proved, otherwise, you will be eternally discredited in our eyes. Stand out one at a time! Who are you?

FIRST ASTRONOMER: I, noble Humanity, am a noted professor from ....

SECOND ASTRONOMER: I, your Honor, am that most learned and far-seeing astronomer from ....

THIRD ASTRONOMER: I, your Honor, am that great astronomer who writes popular astronomy for the ....

FOURTH ASTRONOMER: I am from the famous observatory ....

HUMANITY: State your case, first Astronomer.

FIRST ASTRONOMER: Your Honor, we astronomers are great detectives. None can see as far as we. We look, no detail escapes us.

ALL (*slowly*): Look, look, look.

FIRST ASTRONOMER: We measure as no other can measure.

ALL: Measure, measure, measure.

FIRST ASTRONOMER: We have turned our attention on Light. She is young and good to look upon and as such, as your Honor well knows, should constantly go in a straight line. Such a one had we thought her. But (ALL: Sh! sh! sh!) we were mistaken. We have seen her pass by the Sun. We have focused our instruments upon her. Verily, to see her at eclipses, we have journeyed to the South Seas, to barbarous Mexico, to desert sands and to arctic New Haven. Each time she passes the Sun, she departs from the straight line of rectitude.

ALL: Here are the instruments, the charts, the pictures.

FIRST ASTRONOMER: Behold the telescope!

SECOND ASTRONOMER: Behold the blue prints!

THIRD ASTRONOMER: Behold the pictures!

FOURTH ASTRONOMER: Behold the bent line!

HUMANITY: Ah! Ah, a grave charge indeed.

ALL: Grave, very grave.

HUMANITY: Light, what have you to say for yourself?

LIGHT (*timidly*): Your Honor, if I did turn from the straight line a teeny, weeny bit, it was all due to Gravity, the all powerful. What could a poor girl do? It was not my fault, not a bit.

FIRST ASTRONOMER: You see she admits it. A clear case.

SECOND ASTRONOMER: And Gravity, how ridiculous! I know all about Gravity.

ASTRONOMERS: And I, and I.

HUMANITY: Stand out and tell about Gravity, you from . . . .

ASTRONOMER: Gravity, your Honor, is a very strong demon who lies in wait to grab all kinds of men. It grabs and pulls and pulls but,—Gravity never possibly touched a woman. Light herself knows that Gravity has nothing whatever to do with her nor she with Gravity.

FIRST ASTRONOMER: Right, right, Gravity never influenced Light.

ALL: Impossible, impossible.

HUMAINTY: Astronomers, where did you learn about Gravity?

FIRST ASTRONOMER: Newton told us.

SECOND ASTRONOMER: Yes, the great Newton.

ALL: Newton, Newton.

HUMANITY: But Newton has been dead over two hundred years.

FIRST ASTRONOMER: No matter.

SECOND ASTRONOMER: Newton knew.

HUMANITY (*dubiously*): Where did Newton learn?

ASTRONOMER: He saw an apple.

ALL (*slowly*): Apple, apple, apple.

HUMANITY: I do not understand but we shall proceed. Are there witnesses for Light? (*Michelson steps out*).

HUMANITY: Your name, please.

MICHELSON: I am Michelson of Chicago.

HUMANITY: Do you know Light?

MICHELSON: All my life I have loved her. All my life I have studied her.

HUMANITY: Tell us of Light.

MICHELSON: Light, oh learned Judge, is a young woman of perfect honor. I tell what she does by the fringes of her hair.

HUMANITY: How so? But never mind, proceed.

MICHELSON: She goes at the same speed going toward the Sun or coming from it, whether she goes with you or across your path. It matters not whence she comes nor whither she is going.

HUMANITY: You have measured her speed?

MICHELSON: Yea, verily. (*Morley steps out*)

MORLEY: Your Honor, I am Morley, I agree with Michelson. (*Miller jumps up*)

MILLER (*loudly*): Your Honor, I disagree. The ether . . .

HUMANITY: Who are you?

MILLER: I am Miller of the Case School.

HUMANITY: Sit down. Michelson, what of Gravity?

MICHELSON: I have not studied him.

HUMANITY: Are there other witnesses for Light? (*Einstein steps out and remains standing without speaking*)

HUMANITY: Your name?

EINSTEIN: I am Einstein.

HUMANITY: What know you of this case?

EINSTEIN: I am a mathematician.

HUMANITY: Ah, a worthless calling.

EINSTEIN: I think not, your Honor, and I hope by this case to show you the contrary.

HUMANITY: The truth is, no one knows you. Mathematicians do not often put themselves before Humanity. How am I to know it? Proceed.

EINSTEIN: I have here my tools. Stand forth and announce yourselves.

FIRST TOOL: I am Noneuclidean Geometry. My parents Riemann and Lobatchewsky were the greatest of men.

SECOND TOOL I am Space-time. My father, Minkowski stands second to none.

THIRD TOOL: I am her sister, Higher Dimensional Geometry. No one boasts more honorable descent than I.

FOURTH TOOL: I am Tensors, young but as good a woman as any.

EINSTEIN: I have other tools that are not here but the greatest is here. Now, you the chief of all my tools, my right hand, step out.

FIFTH TOOL: I am Mathematical Analysis.

HUMANITY: And now what of the case?

EINSTEIN: We have worked with unerring accuracy. These my tools never fail me. I, too, love Light. I have striven and have vindicated her name. There is no Demon Gravity. Poor Light did not understand. She could not help her actions which were due to the position she was in, solely. The course of correct behavior was to bend the path as she bent hers; to go not in a straight line, but in a hyperbola, and this is not what astronomers think it is, for in my opinion space is non-Euclidean. So would the astronomers, perforce. They, poor old fogies, wished the modern Woman, Light, to go according to their old-fashioned notions, the same as with Newton over two hundred years ago. But Light, the girl personified, goes according to where she is and swiftly.

HUMANITY: The case is dismissed. Light, I congratulate you. Einstein, you and Michelson escort Light from the court.

All go out. Michelson, Einstein, Light, arm in arm.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF NEBRASKA, Lincoln, Nebraska [*1924*, 146]

The program for the year 1924–1925 was the following:

November 6, 1924. "Magic squares" by Dr. A. L. Candy.
December 3. Social meeting.
February 25, 1925. "Construction of triangles" by Mr. Ogden.
March 25. "Lenses and cameras" by Mr. Russell.
April 20. "Relation of mathematics to astronomy" by Professor Sweezy.
May 20. Annual picnic.

(Report by Irma E. Wiedeman, secretary)

THE EUCLIDEAN CIRCLE OF THE UNIVERSITY OF INDIANA, Bloomington, Indiana [*1924*, 495]

The program for the year 1924–1925 was the following:

November 10, 1924. "The continuity of the number system and the creation of rational numbers based on the Dedekind postulate" by Myrtle M. Comer. "Archimedes" by T. H. Rawles.
November 17. "The development and application of mathematics" by Professor S. C. Davisson.
January 19, 1925. "Ciphers" by Professor J. S. Galland.
March 30. "Relativity and the relation of philosophy to science" by Chas. Kasper.
April 27. "Asteroids" by N. H. Long.
May 18. "The cosine theorem as it is developed from the analytical standpoint" by Professor D. A. Rothrock.

(Report by Professor H. T. Davis)

PI MU EPSILON, UNIVERSITY OF PENNSYLVANIA, Philadelphia, Pa. [*1922*, 148]

For the first time since the University of Pennsylvania chapter was organized, the undergraduate members of the fraternity have been taking an active part in the lectures and mathematical discussions offered by members and friends. It is hoped that this active participation of the younger element in

the affairs of the fraternity will serve as an impetus towards increased attendance at the meetings of the fraternity and towards increased interest being shown by the student body of the school.

The folowing papers were read during the year 1924–1925:

1. "Mathematical fallacies" by Miss S. M. Wolfe.
2. "Theory of numbers" by Dr. E. E. Witmer.
3. "Introduction to the calculus of variations" by Dr. J. D. Eshleman.
4. "The duplication of the cube" by Miss R. H. Stauffer.
5. "The Einstein theory" by Mr. L. Zippin.
6. "Stereographic projection" by Mr. Chadwick.
7. "Some regular polyhedra" by Miss I. Shields.
8. "English mathematics and mathematicians" by Dr. Wilson of Haverford College.

(Report by Miss Ruth H. Stauffer, secretary)

The Mathematics Club of Albion College, Albion, Mich.

The program of the Mathematics Club of Albion College for the year 1924–25 was the following:

October 14, 1924. Election of officers for the first semester.

President, William Wylie, '25;
Vice-president, Irene Bauer, '25;
Secretary-treasurer, Esther Richard, '26;
Member of the Program Committee, Omar Bartow, '25.

October 21. "Unrealized possibilities in teaching mathematics" by William Wylie, '25.

November 18. Roll call responded to by giving a trigonometric formula. "Early mathematics in the United States" by Harmon Camburn, '25. "Early mathematics in the United States by Arthur Dewey, '26. Critic's report by Omar Bartow, '25.

December 2. Roll call responded to by giving a geometrical construction. "Ruler and compasses construction" by Omar Bartow, '25. Critic's report by William Wylie, '25.

January 6, 1925. "How to make geometry interesting" by Esther Richard, '26. "Some problems to be met by a mathematics teacher" by Keith Friend, '25. Critic's report by Irene Bauer, '25.

Election of officers for the second semester.

President, George Price, '26;
Vice-president, Omar Bartow, '25;
Secretary-treasurer, Rupert Cortright, '26;
Member of the Program Committee, Keith Friend, '25.

February 3. Roll call responded to by facts from the lives from English mathematicians. "The planimeter" by Lester Smith, '26. "The slide rule" by Mr. Sears. Critic's report by Omar Bartow, '25

March 3. Roll call responded to by facts from the lives of German mathematicians. "Magic squares" by Irene Bauer, '25. "Algebraic magic squares" by George Price, '26. Critic's report by Lester Smith, '26.

April 21. "Method of extracting the square root" by Arthur Dewey, '26. "Method of extracting any root" by Harmon Camburn, '25. "The teaching of logarithms" by Frank Sanders, '26. Critic's report by Irene Bauer, '25.

May 5. Debate: Resolved, that two years of mathematics should be required in the high school,—that two years to consist of one year of algebra and one year of geometry.

Affirmative:	Negative:
Esther Richard, '26	Keith Friend, '25
William Wylie, '25	Rupert Cortright, '26

May 19. Social meeting.

(Report by Rupert Cortright, secretary)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3187. Proposed by Frank Morley, Johns Hopkins University.

Taking six points, 1, 2, 3, 4, 5, 6, on a conic, let the areas of the triangles 612, 123, $\cdots$ be denoted by $a_1, a_2, \cdots, a_6$. Prove that the area, A, of the hexagon, formed by the points as ordered, is given by

$$(A-a_1-a_3-a_5)a_2a_4a_6=(A-a_2-a_4-a_6)a_1a_3a_5.$$

3188. Proposed by B. F. Finkel, Drury College.

Find the equation of the curve whose radius of curvature at any point of the curve is n times the radius vector to the same point.

3189. Proposed by C. G. Latimer, Tulane University.

Find integers, h, k_i, c_i, n, where $|k_i|<h$, such that

$$\log(x+h)=\sum_{i=1}^{n}c_i.\log(x+k_i)+\sum_{i=1}^{\infty}a_iz_i, \qquad (i=1,2,\cdots),$$

where the a_i are constants and $z=b/f(x)$ where b is a constant and $f(x)$ is a polynomial of degree $\geqq 6$.

Borda's and Haro's series are similar to this except that they involve polynomials of the third and fourth degrees respectively.

3190. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Two given lines x, y are met in the points P, Q by a circle passing through their common point and through another fixed point in the plane. Find the locus of the point of intersection of the lines projecting P, Q upon y, x, respectively.

3191. Proposed by A. A. Bennett, Lehigh University.

Prove that each positive integer less than 2×10^9, has the property of containing an odd prime factor or else of being expressible as a product (with distinct factors) of the form $\Pi_i(p_i-1)$ where each p_i is a prime.

3192. Proposed by J. L. Riley, Ouachita College.

Find all pairs of numbers x and y such that $rx\equiv s(\text{mod } y)$ and $r'y\equiv s'\ (\text{mod } x)$; r, s, r', s', being given integers.

3193. Proposed by W. H. Rasche, Virginia Polytechnic Institute.

Prove that two coplanar, copolar triangles are homological; and, conversely, two coplanar, homological triangles are copolar.

NOTE.

(1) In the proof of the foregoing theorem, use only Euclidean geometry and the positive and negative signs to distinguish the internal division of a linear segment from its external division.

(2) It is of interest to note, too, that a similar theorem holds for spherical geometry, namely: Two spherical, Coplanar triangles are homological; and, conversely, two spherical homological triangles are coplanar, two spherical triangles ABC and A'B'C' being coplanar when the great circle arcs AA', BB', CC' are concurrent and two spherical triangle ABC and A'B'C' being homological when the three pairs of corresponding sides meet in three points which lie on the same great circle.

SOLUTIONS

Problem 2676 [1918, 75]. **Proposed by E. R. Smith, Iowa State College.**

Find the greatest term of the series

$$\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}F(-r,\ -sq,\ sp-r+1,\ 1)$$

where s, r, sq and sp are positive integers, $r<s$, p and q are proper fractions such that $p+q=1$, and $F(-r,$ $-qs,\ sp-r+1,\ 1)$ is a hypergeometric series. If $s,r,\ s-r$ are large, show that the greatest term is approximately equal to

$$\sqrt{\frac{s}{2\pi pqr(s-r)}}\,.$$

Solution by F. M. Weida, Lehigh University.

Let us consider the function

$$\phi=\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}F(a,b,c,z) \tag{1}$$

$$\text{where } F(a,b,c,z)\ =\ 1+\frac{a\cdot b}{1!c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+\ \cdots.$$

If in (1) we put $a=-r$, $b=q$, $c=sp-r+1$, $z=1$, it is readily found that

$$\phi\ =\ \frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}\left\{1+\frac{rqs}{ps-r+1}+\frac{r(r-1)}{2!}\cdot\frac{qs(qs-1)}{(ps-r+1)(ps-r+2)}+\cdots\right\}. \tag{2}$$

At this point, it may be of interest to note[1] that the successive terms of (2) give the chances, respectively, of getting r, $r-1,\cdots,0$, black balls from a bag containing ps black and qs white balls when r balls are drawn from the bag.

Let us now write the general term of (2), namely,

$$\phi_x=\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}\cdot\frac{r(r-1)\cdots(r-x+2)}{(x-1)!}\cdot\frac{qs(qs-1)\cdots(qs-x+2)}{(ps-r+1)\cdots(ps-r+x-1)}. \tag{3}$$

It is now our problem to find the value of x in (3) for which ϕ_x as given by (3) is a maximum. Now

$$\Delta y_x=y_{\ +1}-y_x=y_x\left\{\frac{r-x+1}{x}\cdot\frac{qs-x+1}{ps-r+x}-1\right\}=y_x\left\{\frac{(r+1)(qs+1)-x(s+2)}{x(ps-r+x)}\right\} \tag{4}$$

for $p+q=1$.

Imposing the condition for a maximum, we find from (4) that

$$(r+1)(qs+1)-x(s+2)=0\ \cdot$$

If $x<(r+1)(qs+1)/(s+2)$, then $\Delta y>0$; while if $x>(r+1)(qs+1)/(s+2)$, then $\Delta y<0$. Since

$$\lim_{s\to\infty}\left[\frac{(r+1)(qs+1)}{s+2}\right]=rq+q,$$

---

[1] See Elderton, *Frequency Curves and Correlation*, p. 37.

we find that, if r and s are sufficiently large, ϕ_x is a maximum when

$$x=rq. \tag{6}$$

Substituting the value of $x=rq$ in (3) and combining terms, we find that

$$\phi_{rq}=\frac{ps(ps-1)\cdots(ps-pr)}{s(s-1)\cdots(s-r+1)}\cdot\frac{r(r-1)\cdots(rp+2)qs(qs-1)\cdots(qs-qr+2)}{(rq-1)!}. \tag{7}$$

From (7) we see that we may also write that

$$\phi_{rq}=\frac{(ps)!\times(s-r)!\times(r)!\times(qs)!}{(ps-pr-1)!\times(s)!\times(r-rq+1)!\times(qs-rq+1)!\times(rq-1)!} \tag{8}$$

Let us now replace each factorial in (8) by Stirling's[1] approximation to the factorial, namely,

$$n!=n^n\times e^{-n}\times\sqrt{2\pi n},\ \text{and we have}$$

$$\phi_{rq}=\frac{(ps)^{ps+1/2}\times(s-r)^{s-r+1/2}\times r^{r+1/2}(qs)^{qs+1/2}}{\sqrt{2\pi}(ps-pr-1)^{ps-pr-1/2}\times s^{s+1/2}\times(rp+1)^{rp+3/2}\times(qs-qr+1)^{qs-qr+3/2}\times(rq-1)^{rq-1/2}} \tag{9}$$

If in (9), we divide $(ps-pr-1)$ into two factors of which $(s-r)^{ps-pr-1/2}$ is one of the two; and if we divide $(rp+1)$ into two factors of which $r^{rp+3/2}$ is one of the two; and if we divide $(qs-qr+1)$ into two factors of which $(s-r)^{qs-qr+3/2}$ is one of the two; and if we divide $(rq-1)$ into two factors of which $r^{rq-1/2}$ is one of the two; and if we then impose the condition that r,s, $s-r$ are large, it is easily seen after combining terms and simplifying that, approximately,

$$\phi_{rq}=\sqrt{\frac{s}{2\pi rpq(s-r)}} \tag{10}$$

and this is the result desired.

3123 [1925, 138]. **Proposed by B. F. Finkel, Drury College.**

A circular hole, radius r, in the bottom of a flat-bottomed water-tank is covered with a weightless spherical rubber shell, radius R. Water is then poured into the tank to the depth h. What is the ratio of R to r when the shell is just on the point of rising?

II. Solution by the Editors.

Let R be the radius of the spherical shell and r_2, the radius of the circular hole in the bottom of the tank. If water is poured into the tank, the only portion of the ball's surface upon which there is an upward pressure lies between the level of depth $\sqrt{R^2-r_2^2}$ and the bottom, and the portion upon which there is a downward pressure lies above the first level. To each zone of the sphere between this level and the bottom there is an equal zone above this level symmetrical to the first zone. Since the pressure on the lower zone is greater than that on the upper, the ball will rise if $h\leqq 2\sqrt{R^2-r_2^2}$. We have now two cases to consider: Case I. $2\sqrt{R^2-r_2^2}<h\leqq R+\sqrt{R^2-r_2^2}$; Case II. $h>R+\sqrt{R^2-r_2^2}$.

Case I. Let r_1 be the radius of the section of the surface of the water with the sphere, and x, the radius of a horizontal section between the two sections of radii r_1 and r_2. Then $h=\sqrt{R^2-r_2^2}+\sqrt{R^2-r_1^2}$. The force acting vertically downward upon the wetted part of the segment of radius r_2 is

$$F_1=2\pi wg\int_{r_1}^{r_2}\left[x\sqrt{R^2-r_1^2}-x\sqrt{R^2-x^2}\right]dx$$

$$=2\pi wq\left[\frac{1}{2}(r_2^2-r_1^2)\sqrt{R^2-r_1^2})+\frac{1}{3}(R^2-r_2^2)^{3/2}-\frac{1}{3}(R^2-r_1^2)^{3/2}\right].$$

[1] Whittaker and Robinson, *Calculus of Observations*, p. 140.

The buoyant force is the weight of the water which could fill the space between sphere and a cylinder of radius r_2,

$$F_2 = -\frac{4\pi wg}{3}(R^2-r_2^2)^{3/2}.$$

If $F_1+F_2=0$, we have after certain cancellations $h=3\sqrt{R^2-r_2^2}$.

If $r_1=0$, $h=R+\sqrt{R^2-r_2^2}=3\sqrt{R^2-r_2^2}$. Hence $r_2/R=\sqrt{3}/2$, $h=3R/2$.

Case II. $h>R+\sqrt{R^2-r_2^2}$, Here

$$F_1 = 2\pi wg\int_0^{r_2}[h-\sqrt{R^2-r_2^2}-\sqrt{R^2-x^2}\,]x\,dx,$$

$$=2\pi wg\left[\frac{r_2^2}{2}(h-\sqrt{R^2-r_2^2})+\frac{1}{3}(R^2-r_2^2)^{3/2}-\frac{R^3}{3}\right].$$

Hence if $F_1+F_2=0$, we have $(2R^2+r_2^2)\sqrt{R^2-r_2^2}=3hr_2^2-2R^3$; and setting $r_2/R=k$, we have

$$(2+k^2)\sqrt{1-k^2}=\frac{3hk^2}{R}-2,$$

an equation of the sixth degree in k when freed of radicals.

Also solved by H. S. UHLER whose solution was similar to that of E. M. BERRY (*1926*, 50).

3137 [1925, 261]. **Proposed by Harry Langman, New York City.**

Show how to draw, using straight-edge only, a tangent to the circumference (or an arc) of a circle at a given point, without making use of Pascal's hexagon theorem.

SOLUTION BY NINA M. ALDERTON, Mills College.

Through P, the given point on the arc C of the circle, draw a straight line cutting the arc again at A. Take any point P_1 on PA and draw through it the chord BD. Draw the lines AB,DP meeting at E, PB, DA meeting at F. Then the line EF is the polar of P_1. Now take a second point P_2 on PA and repeat this construction finding the polar MN of P_2. Let MN and EF meet in L, which must be the pole of PA. Then the line drawn through L and P is the tangent at P.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Associate Professor A. F. KOVARIK has been promoted to a full professorship of physics at Yale University.

Associate Professor LOUIS LINDSEY, of Syracuse University, has been promoted to a full professorship of applied mathematics.

Assistant Professor J. C. MOREHEAD has been promoted to an associate professorship at the Carnegie Institute of Technology.

At the University of Michigan the following appointments to assistant professorship are announced: G. Y. RAINICH, formerly lecturer and research fellow

at Johns Hopkins; R. L. WILDER, formerly assistant professor at Ohio State University; J. A. NYSWANDER, formerly assistant professor at Swarthmore College.

Dr. F. H. MURRAY, of Dalhousie University, has been promoted to an assistant professorship of mathematics.

Dr. J. R. MUSSELMAN, of John Hopkins University, has been promoted to an associate professorship of mathematics.

Miss MARGARET C. PACKER, of Hood College, has been promoted to an associate professorship of mathematics.

Assistant Professor L. P. SICELOFF, of Columbia University, has been promoted to an associate professorship of mathematics.

Professor J. D. TAMARKIN has been appointed assistant professor of mathematics at Dartmouth College.

Professor J. H. TANNER, head of the department of mathematics at Cornell University, has retired.

Assistant Professor J. S. TAYLOR, of the University of Pittsburgh, has been promoted to an associate professorship of mathematics.

Assistant Professor J. T. TRACEY, of Yale University, has been promoted to an associate professorship of mathematics.

Dr. M. S. VALLARTA has been promoted to an assistant professorship of physics at the Massachusetts Institute of Technology.

Associate Professor R. M. WINGER, of the University of Washington, has been promoted to a full professorship of mathematics.

Dr. EUPHEMIA R. WORTHINGTON, of the Southern Branch of the University of California, has been promoted to an assistant professorship of mathematics.

The following appointments to instructorships are announced:
Bryn Mawr College, Dr. ECHO D. PEPPER;
University of Michigan, Dr. F. W. KOKOMOOR; I. M. SHEFFER;
University of Rochester, Miss ROSE L. ANDERSON;
Rutgers University, Dr. C. M. HUBER and Mr. A. E. MEDER.

Professor R. D. BOHANNAN, who had just completed thirty-nine years as head of the department of mathematics at the Ohio State University, died suddenly June 20, 1926, at the age of seventy-one. His death was due to apoplexy.

Professor S. M. BARTON, of the University of the South, died January 5, 1926, at the age of sixty-six.

Missing Numbers of the Monthly

Cash will be paid for certain single numbers as follows, up to a limited number of copies:

February, March, May or September, 1913; September, 1914; February, March, April or June, 1915; February or September, 1918—fifty cents; September, 1915—seventy-five cents; May, 1915—one dollar (See MONTHLY, March, 1921, p. 152).

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio

CONTENTS

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.
Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.
The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7-8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Annapolis, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

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THE MATHEMATICAL ASSOCIATION

The following forty-five persons and five institutions have been elected to membership in the Association, their applications having been duly certified.

To Individual Membership

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TROFIMOV, L. A., (Imperial Naval Acad., Petrograd). Development engr., Electric Controller and Mfg. Co., Cleveland, Ohio.

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W. D. CAIRNS, *Secretary-Treasurer.*

A NEW PUBLICATION OF THE ASSOCIATION

By H. E. SLAUGHT, University of Chicago

We take great pleasure in announcing that about December, 1926, the Mathematical Association of America expects to publish a notable new edition of the *Rhind Mathematical Papyrus*, often known as the *Ahmes Papyrus*.

This publication is the result of fifteen years of study and thought on the part of Chancellor A. B. Chace of Brown University, to whom the Association is already much indebted. In preparing it for the press he has been assisted by Professors H. P. Manning and R. C. Archibald of Brown University; and by Doctor L. S. Bull, of the Metropolitan Museum of Fine Arts, New York The work will be issued in two volumes, printed on specially manufactured paper, bound in a dignified style appropriate to such an important publication, and will represent the highest grade of typographical art.

Eisenlohr's German edition of the *Rhind Papyrus*, which appeared in 1877, is well known. The scholarly English edition of Peet, in 1923, reviewed in this MONTHLY (*1924*, 246–351), contained a hieroglyphic transcription of the original hieratic papyrus and a translation of the entire text. In 1898 the British Museum published what purported to be a facsimile of the papyrus, but which was merely a hand copy on a background prepared to imitate papyrus. The errors in transcription were so numerous, however, as to make the work unreliable for the use of scholars. For this reason Chancellor Chace was anxious

not only to publish a new translation but to place before his readers a real facsimile of this notable relic of ancient Egyptian mathematics. Through the courtesy of the Trustees of the British Museum photographs of the whole papyrus were made for this work. These are reproduced in twenty-eight plates in volume II, the scale being very little less than the original size. This volume is to contain 111 other plates in two colors, the red and black of the original. Underneath the copied hieratic original is the hieroglyphic transcription, under which is the (right-to-left) transliteration. On the opposite page is a left-to-right arrangement of the transliteration, under which is the literal translation. In this way any one, even though unfamiliar with the Egyptian language or with the scripts used by copyists, may obtain some idea of the meaning of the symbols used and of the language itself.

The first volume contains a free translation of the papyrus with copious notes, an extensive introduction, and an elaborate and critical bibliography, 1706–1926, occupying with indexes more than eighty printed pages. Professor Archibald is alone responsible for the bibliography.

Chancellor Chace is meeting all the expenses of publication, and is presenting the entire edition, with the exception of certain personal gift copies, to the Association. The money received through its sale is to constitute an endowment fund of the Association, and is to be known as the ARNOLD BUFFUM CHACE FUND. The Trustees of the Association desired if possible to set a price for the sale of the work so that the five hundred copies available for this purpose might be rapidly distributed. *A member of the Association may procure the work for personal use* at the special price of fifteen dollars per set, post paid, on ordering it from the Secretary and making a declaration regarding its purchase for personal use. An institutional member may also purchase the work for its library at fifteen dollars per set by ordering it through the Secretary. Other institutions and the public in general may procure it only through the Open Court Publishing Company of Chicago at the regular price of twenty dollars per set. Further announcement will be made and subscription blanks supplied shortly before the date of publication.

An examination of the work will show that the regular price of twenty dollars is much below the actual cost of publication. Members will greatly assist the Association by suggesting that their college and university libraries subscribe to the work and thus aid in the establishment of a fund which will materially assist the Association in its activities.

AFFILIATION OF THE ASSOCIATION OF TEACHERS OF MATHEMATICS IN NEW ENGLAND WITH THE MATHEMATICAL ASSOCIATION OF AMERICA

The council of the Association of Teachers of Mathematics in New England have voted to accept a plan of affiliation originating in 1925 in conferences between President J. L. Coolidge and the officers of that Association. As modified by the Trustees of the Mathematical Association and put into effect by this recent action, it is agreed

1. That the members of the Association of Teachers of Mathematics in New England may become members of the Mathematical Association of America without the payment of the customary initiation fee, the A.T.M.N.E. to supply annually to the Mathematical Association a list of new members whom the Association may invite to their membership under this special condition.

2. That it is understood that whenever the M.A.A. holds meetings in New England, the A.T.M.N.E. engages to aid in every way, and that meetings of the A.T.M.N.E. in connection with those of the M.A.A. as an affiliated organization will be welcomed.

3. That the present agreement may be terminated by either party on six months' notice.

4. That the vote by this council here recorded makes this agreement effective.

W. D. CAIRNS, *Secretary-Treasurer.*

ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The tenth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado Agricultural College at Fort Collins, Colorado, on April 16, 17. There were thirty-six present including the following seventeen members of the association: A. G. Clark, I. M. DeLong, G. W. Finley, Philip Fitch, J. C. Fitterer, H. C. Gossard, A. J. Kempner, Claribel Kendall, G. H. Light, W. V. Lovitt, S. L. Macdonald, J. Q. McNatt, L. R. Odell, O. H. Rechard, W. J. Risley, H. E. Russell, and C. H. Sisam.

The section voted to hold the next meeting at Colorado College, Colorado Springs. The secretary was instructed to invite the Association to hold a summer meeting at the University of Colorado sometime in the near future.

A committee was appointed consisting of the secretary, Miss Odell, Professor Russell, and Professor Risley to compile material on the significance and value of the study of mathematics. This material is to be published by the section.

The following officers were elected: W. V. LOVITT, chairman; W. J. RISLEY, vice-chairman; PHILIP FITCH, secretary and G. H. LIGHT, treasurer.

The section was favored Friday evening with an address "The New Heavens" by Dr. D. W. Morehouse, president of Drake University.

A complimentary dinner was served on Friday to all present, at which time President C. A. Lory of the Colorado Agricultural College delivered an address of welcome to which Professor I. M. DeLong made an appropriate response.

The following sixteen papers were read:

(1) "Report on the experiment with a standardized test in college algebra" by Professor H. C. GOSSARD.

(2) "The interpretation of errors" by Professor R. L. PARSHALL (by invitation).

(3) "Graphic solutions" by Mr. J. Q. MCNATT.

(4) "Mathematical logic" by Professor H. V. CRAIG (by invitation).

(5) "Concerning d'Alembert's principle" by Professor J. C. FITTERER.

(6) "Finite trigonometric series" by Professor A. G. CLARK.

(7) "Mathematics for freshman women" by Mrs. NELLIE LANDBLOM (by invitation).

(8) "On the reliability of the composite score of a battery test" by Mr. PHILIP FITCH.

(9) "Concerning rigorous proofs" by Professor S. L. MACDONALD.

(10) "Note on the limit functions of sequences of functions of certain types" by Professor O. H. RECHARD.

(11) "The use of the discriminant in differential equations" by Professor G. H. LIGHT.

(12) "Index number bias" by Professor W. V. LOVITT.

(13) "A system of vector coordinates" by Professor H. C. GOSSARD.

(14) "On a property of the Hessians of cubic forms" by Professor C. H. SISAM.

(15) "Complex roots of equations" by Professor A. J. KEMPNER.

(16) "Root extraction with the adding machine" by Professor F. H. LOUD. In the absence of the author, the abstract of Professor Loud's paper was read by the secretary.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. A report of the experiment by the colleges and universities of the Rocky Mountain Section with a standard college algebra speed-accuracy test. This experiment under the direction of the University of Wyoming was voted to be continued through the coming year.

2. This paper dealt with the interpretation of the errors arising in results obtained from experiments on the flow of water over weirs.

3. Mr. McNatt gave some examples of the use of graphic solutions of problems in surveying.

4. The paper discussed the primitive ideas and primitive propositions of the system of Whitehead and Russell; analogies among the calculus of propositions, calculus of classes, and calculus of relations; and the definition of the cardinal numbers 1 and 2. It was shown that a class cannot be used as an argument of anyone of its determining functions and that a hierarchy of functions is necessary.

5. A résumé of its statement and presentation in various texts on mechanics and kinetics was given and it was pointed out that its function, particularly for the teacher of dynamical subjects—especially the practical applications in engineering and allied sciences, consists primarily in simplifying and reducing forces involved in accelerated systems to the field of statics. This service is comparable with the idea Monge had in descriptive geometry, of reducing solid problems to the realm of the plane; and also, as another example, with the conception involved in influence lines in the theory of structures whereby the effect of live loads is simplified to the status of dead loads.

6. Professor Clark indicated that the summation of

$$\sum_{i=1}^{n} F(i) \begin{matrix}\sin\\ \\ \cos\end{matrix} \phi(i)\theta$$

may be effected when $F(i)$ is a rational, integral polynomial by successive application of the parts formula for finite integration,

$$\Sigma u_x \Delta v_x = u_x v_x - \Sigma v_{x+1} \Delta u_x + C_1$$

provided the summation of

$$\sum_{i=1}^{n} \begin{matrix}\sin\\ \\ \cos\end{matrix} \phi(i)\theta$$

is possible.

When $F(i)$ is the quotient of two rational integral polynomials, the summation may be effected by setting up and solving a system of two linear differential equations, the order of the system being the same as the degree of the polynomial forming the denominator.

7. The question as presented by Mrs. Landblom was "What should be the content of a course in mathematics for freshman women and why?" Arguments set forth dealt primarily with problems arising in home life, teaching, special subjects required by curricula, as physics and chemistry, extension work, child welfare, and social settlement. The solution was given in an outline of topics to be covered in a fifty hour course.

8. In this paper it was shown that the composite score of a battery test could be reliable only if the respective scores of its elements were weighted

according to their reliability, validity and independence after having been reduced to equal spread.

9. In his paper Professor Macdonald pointed out the necessity of making conclusions definite in rigorous proofs.

10. An example was given to show that quasi-uniform convergence is not necessary in order that the existence of a limit of a sequence of functions which are pointwise discontinuous shall itself be proved. The following theorem was then proved: A sufficient condition that the limit of a sequence of functions with upper continuity shall have upper continuity is that the sequence is quasi-uniformly convergent.

11. This paper was intended to show how the discriminant can be applied to the solution of differential equations of the type $f(x,y,p)=0$ and showed how the extraneous factor, if any exists, can be detected before solving the equation.

12. Four primary systems of weighting have been devised. In the customary notation these are p_0q_0, p_0q_1, p_1q_0, p_1q_1. The speaker noted the absence of strict mathematical proofs of the assertions made as to the upward or downward bias of an index number weighted with the weights specified above. In this paper some proofs were given of bias when such exists. Professor Irving Fisher has given a proof that the unweighted arithmetic average of relative prices has an upward bias. This paper gave a new proof of this fact.

13. Professor Gossard presented a system of coordinates based upon naming the points (x) of the Gaussian plane by rotations (t) on a base circle plus such translations as called for by the given equation $x=f(t)$. The expressions arising are symmetric functions of (t) and for many types of theorems in geometry the analytic work is exceedingly simple.

14. This paper dealt with the determination of a simplified form for the equation of the Hessian of a cubic in any number of variables.

15. The roots (real and complex) of an equation $w=a_0z^n+a_1z^{n-1}+\cdots$ $a_n\neq 0$, $w=u+iv$, $z=x+iy=re^{i\varphi}$, a_i real or complex, may be isolated and determined to any desired degree of accuracy in the following manner: In a rectangular system of coordinates with a φ-axis and an axis which serves at the same time for $u-$axis and $v-$axis plot, for appropriately chosen values r_i of r, the two curves $u=u(r_i,\varphi)$, $v=v(r_i,\varphi)$. From the order in which the points of intersection of the u-curve with the φ-axis and the points of intersection of the v-curve with the φ-axis follow each other, the number of roots of $w=0$ of absolute value $<r_i$ is read off by a very simple rule. The absolute values of the roots are then determined to any desired degree of accuracy.

Professor Kempner next showed how limits for the arguments of the roots may be determined. The value of the method lies in the fact that instruments

are in existence (harmonic analyzers) which will trace mechanically curves of type $u=u(r_i,\varphi)=\sum c_k \cos k(\varphi)$, $v=v(r_i,\varphi)=\sum d_k \sin k(\varphi)$.

16. This paper dealt with the employment of Newton's binomial formula as a method, and of the adding machine as an instrument, in the extraction of roots. Several cube roots were extracted, as illustrations of certain elementary and rather obvious devices for securing rapid convergence in the series, and otherwise minimizing the labor of computation. In the latter part of the paper, some of the actual records taken from the adding machine were inserted, as fuller demonstration of the method employed.

PHILIP FITCH, *Secretary.*

ELEVENTH ANNUAL MEETING OF THE OHIO SECTION

The eleventh annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, April 2, 1926, in connection with the meetings of the Ohio College Association.

Thirty-four persons registered attendance, among whom were the following twenty-five members of the Association: R. B. Allen, C. L. Arnold, Grace M. Bareis, I. A. Barnett, H. Blumberg, R. D. Bohannan, W. D. Cairns, V. B. Caris, R. Crane, W. Dancer, O. L. Dustheimer, T. M. Focke, B. C. Glover, H. Hancock, H. W. Kuhn, S. E. Rasor, P. L. Rea, Hortense Rickard, S. A. Singer, C. E. Stout, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, J. B. Winslow, B. F. Yanney.

Officers elected for the coming year were: Chairman, H. W. KUHN; Secretary-Treasurer, RUFUS CRANE; Member of Executive Committee, H. M. BEATTY; Member of Program Committee, T. M. FOCKE. A resolution was adopted expressing the appreciation of the members of the section for the life and services of the late Professor G. N. Armstrong, and their regret at his death. A committee was appointed to study the advisability of attempting to organize a new section of the American Mathematical Society, centering in Ohio. A committee was appointed to study ways and means of improving the teaching situation in the secondary schools. It is expected that the next meeting will be held on April 8, 1927.

The following papers were read:

(1) "Alphabetic symbolism applied to some operations on power series" by the Chairman, Professor R. D. BOHANNAN, Ohio State University.

(2) "Euclidean invariants of plane second degree curves" by Professor C. C. MACDUFFEE, Ohio State University.

(3) "Controversial mathematics" by Professor H. BLUMBERG, Ohio State University.

(4) "A major in mathematics" by Professor O. L. DUSTHEIMER, Baldwin-Wallace College.

(5) "The advisability of organizing a section of the American Mathematical Society in this part of the country" by Professor HARRIS HANCOCK, University of Cincinnati.

(6) "Qualifications of teachers in the subjects which they teach" by Professor HANCOCK.

In the absence of Professor MacDuffee, the second paper was read by Mr. L. J. Paradiso, student at Ohio State University. Abstracts of some of these papers follow:

2. This paper has already appeared in full in the MONTHLY (*1926*, 243–252).

3. Professor Blumberg discussed the difficulties that have arisen, especially since the appearance of Cantor's work on the transfinite, in connection with the notions of "set," "infinity," and "existence," and sketched the theories of Zermelo, Brouwer, Weyl, and Hilbert, that were projected to meet such difficulties.

4. Professor Dustheimer presented a summary of replies from 136 institutions to a questionnaire, which showed that about 50 per cent of the colleges require 2 units of high school mathematics for entrance, while about 30 per cent require $2\frac{1}{2}$ units; 50 per cent require mathematics for graduation; only 34 per cent require solid geometry of students majoring in mathematics; 53 per cent give college credit for solid geometry; 25 per cent give college credit for intermediate algebra; an average of 7 per cent of seniors specialize in mathematics; this percentage is increasing in 55 per cent of the colleges. According to this survey, the following would be a good major in mathematics: college algebra 3 hours, trigonometry 3 hours, analytic geometry 3 hours, calculus 6 hours, differential equations 3 hours, history of mathematics 2 hours, theory of equations or some applied mathematics 3 hours, and teaching of mathematics 2 hours.

5. Professor Hancock paid high tribute to what has been accomplished by the American Mathematical Society, stating that thereby the mathematical standard of excellence in every university of the country has been greatly elevated. Professor Hancock said that there are at the present time more good mathematicians within the area that would be included by the section in question than there were thirty years ago in the whole country. He declared that such a section would give a greater impetus to mathematical research and would thereby create a greater interest in mathematics in the Ohio Valley Section, and that such increased interest would eventually strengthen the society as a whole.

6. Professor Hancock called attention to the exceedingly low marks that were made by freshmen in mathematics in examinations that were held last year in three of the larger universities of the state. He ascribed this in great measure

to the modern conditions that have existed for the last twenty years. To remedy such conditions he proposed:

That teachers of mathematics in the secondary schools who have majored in mathematics, and who hold an A.B. or some similar degree, be put on a more advanced salary scale than those teachers of mathematics who are not so qualified.

That those teachers of mathematics who have attained the A.M. degree in mathematics be put on a still higher scale, and that the highest salary scale should be reserved for those teachers who have majored in mathematics with the degree of Doctor of Philosophy.

That those teachers who have no such degrees be put on similar scales of salary and promotions, if they have already shown themselves able teachers and if they pass examinations in mathematics equivalent to the examinations leading to the A.M. and Ph.D. degrees.

RUFUS CRANE, *Secretary-Treasurer.*

THIRD ANNUAL MEETING OF THE MICHIGAN SECTION

The third annual meeting of the Michigan section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, on April 1, 1926. Chairman E. R. Sleight called the meeting to order at 9 A.M.

The attendance was fifty-six including the following thirty members of the Association: N. H. Anning, P. N. Blessing, J. W. Bradshaw, H. C. Carver, R. W. Clack, C. C. Craig, W. W. Denton, B. F. Dostal, P. Field, S. E. Field, W. B. Ford, J. W. Glover, L. A. Hopkins, M. F. Johnson, L. C. Karpinski, D. K. Kazarinoff, H. B. Lemon, T. Lindquist, A. L. Nelson, H. L. Olson, J. R. Overman, O. J. Peterson, V. C. Poor, T. E. Raiford, C. Reid, T. R. Running, J. Shohat, E. W. Sleight, C. C. Spooner.

At a short business session, reports of the retiring officers were received and the following new officers were elected: Chairman, A. L. NELSON, College of the City of Detroit; Secretary-Treasurer, C. REID, University of Michigan; Member of Executive Committee, H. L. OLSON, Michigan State College.

The following papers were presented, those numbered 6 and 8 being read by title. Correspondingly numbered abstracts of the others follow.

(1) "Some elementary applications of the theory of correlation" by Mr. CLAIR REID, University of Michigan.

(2) "Stepping up to get curl and divergence" by Professor W. W. DENTON, University of Michigan.

(3) "On Legendre polynomials" by Professor J. SHOHAT, University of Michigan.

(4) "Conditions for a fixed point in projective differential geometry" by Professor A. L. NELSON, College of the City of Detroit.

(5) "On Pappus's problem" by Mr. D. K. KAZARINOFF, University of Michigan.

(6) "A focal point construction for a certain type of problem in the calculus of variations" by Professor V. G. GROVE, Michigan State College.

(7) "Parallel surfaces with straight line orthogonal trajectories and orthogonal curvilinear coordinates" by Professor W. S. KIMBALL, Michigan State College (by invitation).

(8) "A consideration of the relativity of magnitudes" by Mr. C. E. SMITH, Northwestern High School, Detroit (by invitation).

1. In this paper, Mr. Reid gave four applications of the theory of correlation. The first two were examples of the correct use of correlation coefficients. The third was a case in which there was perfect correlation although the linear coefficient was zero. The fourth illustration showed a high linear correlation between two sets of variates not directly correlated but correlated with a hidden third variate.

2. In elementary textbooks on mathematical physics, the proofs of Stokes' formula, Gauss' formula, the relation of curl to angular velocity, the equation of continuity, and similar theorems are, in general, rather poorly presented. The proofs in question involve the expansion of functions by Taylor's formula and integration of results over an infinitesimal curve or surface. The object of Professor Denton's paper was to explain why these particular proofs are weak and to point out how they may be strengthened. The improvements suggested depend upon the following device. With each point of the given vector field there is associated an auxiliary system of coordinates having that point as origin, and the integration over the infinitesimal curve or surface is actually carried out.

3. Professor Shohat showed that the asymptotic expression (for $n \to \infty$) of the Legendre polynomial $X_n(x)$ can be derived directly from the difference equation

$$(n+2)X_{n+2}(x) - (2n+3)xX_{n+1}(x) + (n+1)X_n(x) = 0,$$

by a method similar to that given by Liouville (*Journal des Mathêmatiques*, vol. II (1837), pp. 16–35), and by W. B. Ford (*Transactions, American Mathematical Society*, vol. XV (1909), pp. 319–336). An important point is the proof of the formula

$$\frac{2 \cdot 4 \cdot \cdot \cdot \quad 2n}{1 \cdot 3 \cdot \cdot \cdot (2n-1)} = \sqrt{\pi\,(n + \tfrac{1}{4} + O\,(n^{-1}))}.$$

4. In two special theories of Wilczynski's projective differential geometry, it has been noted that one of the coordinates of a fixed point with reference to a local tetrahedron must be a solution of differential equations geometrically adjoint to the basic system. Professor Nelson explained and generalized this fact. The paper will appear in the *Bulletin of the American Mathematical Society.*

5. To give a direct proof of the theorem: "If a triangle have two equal angle-bisectors then it will have two equal sides" is a problem which has recently attracted the interest of readers of the MONTHLY (*1917*, 344; *1918*, 182). The history of the problem was traced and it was pointed out that a neat solution follows from use of a construction given by Pappus for placing between the arms of a given angle a line-segment which shall be bisected at a given point.

7. As a result of an investigation of the potential theory of thermionic currents where the electrons pursue straight line paths, Professor Kimball was led to the three following theorems:

I. A necessary and sufficient condition for $F(x,y,z)=a$ to be a family of surfaces having straight lines for its orthogonal trajectories is that h (Lamé's first differential parameter), be a function of a only: $h=\sqrt{F_x^2+F_y^2+F_z^2}=f(a)$.

II. These surfaces are identified with families of parallel surfaces.

III. These surfaces are shown to be "Families of Lamé": they may be one of three orthogonal curvilinear coordinates, the other two of which turn out to be families of developable surfaces.

NORMAN ANNING, *Secretary-Treasurer.*

ON THE ASYMPTOTIC EXPRESSIONS FOR JACOBI AND LEGENDRE POLYNOMIALS DERIVED FROM FINITE-DIFFERENCE EQUATIONS[1]

By J. A. SHOHAT, University of Michigan

1. Introduction. We denote by

$$\varphi_n(x) \equiv \varphi_n(x\,;\ \alpha,\beta) = a_n(\alpha,\beta)x^n + \cdots\ (n=0,1,2\cdots\ ;\ a_n>0)$$
$$\left[\varphi_n' \equiv \varphi_n'(x) \equiv \frac{\varphi_n(x;\alpha,\beta)}{a_n(\alpha,\beta)} = x^n + \cdots\ \right] \tag{1}$$

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[1] Presented to the American Mathematical Society, Kansas City, December, 1925, and (on Legendre polynomials) to the Michigan Section of the Mathematical Association of America, Ann Arbor, April, 1926.

the system of orthogonal and normal polynomials of Jacobi corresponding to the interval $(-1, 1)$, with the characteristic function $(1+x)^{\alpha-1}(1-x)^{\beta-1}$, $(\alpha, \beta>0)$. We have then,[1]

$$\int_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{\beta-1}\varphi_n(x)\varphi_m(x)dx=\begin{cases}0, & m\neq n,\\ 1, & m=n;\end{cases} \tag{2}$$

$$a_n(\alpha,\beta) = \sqrt{\frac{(\alpha+\beta+n-1)\cdots(\alpha+\beta+2n-2)\Gamma(\alpha+\beta+2n)}{\Gamma(\alpha+n)\Gamma(\beta+n)\Gamma(n+1)}}\cdot 2^{-n-\frac{\alpha+\beta-1}{2}}$$
$$= 2^{n+\frac{\alpha+\beta-1}{2}}\sqrt{\frac{1}{\pi}}\,(1+o(1)). \tag{3}$$

[We use here the formula:

$$p(p+1)\cdots(p+n)=\frac{n^p\Gamma(n+1)(1+o(1))}{\Gamma(p)},\ p>0,\ n>0 - \text{integer.}]$$

$$\varphi_n'(1\,;\,\alpha,\beta) = \frac{2^n\beta(\beta+1)\cdots(\beta+n-1)}{(\alpha+\beta+n-1)\cdots(\alpha+\beta+2n-2)},$$
$$\varphi_n'(-1\,;\alpha,\beta) = \frac{(-1)^n2^n\alpha(\alpha+1)\cdots(\alpha+n-1)}{(\alpha+\beta+n-1)\cdots(\alpha+\beta+2n-2)}. \tag{4}$$

Special case:

$$\alpha=\beta=1\,;\ \varphi_n(x\,;1,1)=\sqrt{\frac{2n+1}{2}}\ X_n(x)\text{—Legendre polynomial.} \tag{5}$$

For any function $f(x)$ we have the formal development,

$$f(x)\sim\sum_0^\infty{}_iA_i\varphi_i(x),\quad A_i=\int_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{\beta-1}f(x)\varphi_i(x)dx, \tag{6}$$

provided, of course, the integrals A_i exist. The convergence of the development (6) depends essentially upon the behavior of $\varphi_n(x)$ for n very large. This has been investigated by Darboux,[2] Stekloff,[3] W. B. Ford[4] and others.

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[1] W. Stekloff, Sur l'approximation des fonctions, *Bulletin of the Russian Academy of Sciences* (1917), 187–218; p. 211.

[2] Darboux, Mémoire sur l'approximation des fonctions de très grandes nombres, *Journal des Mathématiques*, t. IV, sér. III (1878), 5–57, 377–416; pp. 23–24.

[3] W. Stekloff, Sur les expressions asymptotiques, *Transactions of the Kharkoff Mathematical Society* (1907), 1–103; pp. 21–27.

[4] W. B. Ford, On the integration of the homogeneous linear difference equation of second order, *Transactions of the American Mathematical Society*, 2nd ser., v. XV (1909), 319–336; pp. 335–6.

In this paper *we derive the asymptotic expression (for* $n \to \infty$*) of* $\varphi_n(x;\alpha,\beta)$ *in an elementary way, directly from the fundamental difference-equation satisfied by* $\varphi_n(x;\alpha,\beta)$. The method used is of the same character as that developed by Liouville[1] and Ford (*loc. cit.*).

2. Jacobi polynomials in general. We get readily from (2) a difference-equation for $\varphi'_n(x)$,

$$\varphi'_{n+2}(x) - (x - c_{n+2})\varphi'_{n+1}(x) + \lambda_{n+2}\varphi'_n(x) = 0 \quad (n \geqq 0), \tag{7}$$

where c_{n+2}, λ_{n+2} are constants, which do not depend on x. Making use of (4), we get

$$c_{n+2} = 1 - \frac{1}{2}\frac{(n+\alpha+\beta)\left(n+\frac{\alpha+\beta}{2}\right)(n+\beta+1) + \left(n+\frac{\alpha+\beta}{2}+1\right)(n+\alpha)(n+1)}{\left(n+\frac{\alpha+\beta}{2}\right)\left(n+\frac{\alpha+\beta}{2}+\frac{1}{2}\right)\left(n+\frac{\alpha+\beta}{2}+1\right)}$$

$$= \frac{(\alpha-\beta)(\alpha+\beta-2)}{4n^2} + \cdots,$$

$$\lambda_{n+2} = \frac{1}{4}\,\frac{(n+1)(n+\alpha)(n+\beta)(n+\alpha+\beta-1)}{\left(n+\frac{\alpha+\beta}{2}\right)^2\left(n+\frac{\alpha+\beta}{2}-\frac{1}{2}\right)\left(n+\frac{\alpha+\beta}{2}+\frac{1}{2}\right)} \tag{8}$$

$$= \frac{1}{4}\left(1 + \frac{-2\alpha^2-2\beta^2+4\alpha+4\beta-3}{4n^2} + \cdots\right),$$

$$c_{n+2} = \frac{-\delta'_n}{n^2}, \quad \lambda_{n+2} = \frac{1}{4} - \frac{\delta''_n}{n^2}, \quad \delta'_n, \delta''_n \text{ finite for } n \to \infty\,;$$

$$\varphi'_{n+2} - x\varphi'_{n+1} + \tfrac{1}{4}\varphi'_n = \frac{1}{n^2}\left[\delta'_n\varphi'_{n+1} + \delta''_n\varphi'_n\right]. \tag{9}$$

Consider the interval

$$-1+\epsilon \leqq x \leqq 1-\epsilon \quad (\epsilon > 0 \text{ arbitrarily small, but fixed}). \tag{10}$$

We get in (10) as solution of the homogeneous equation

$$z_{n+2} - xz_{n+1} + \tfrac{1}{4}z_n = 0,$$

$$z_n = \frac{1}{2^{n-1}}\left\{C'\cos n\varphi + C''\frac{\sin n\varphi}{\sin\varphi}\right\} \quad (x = \cos\varphi\,;\ 0 < \varphi < \pi). \tag{11}$$

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[1] Liouville, Second mémoire sur le developpement des fonctions, *Journal des Mathématiques*, t. II (1837), pp. 23–4.

Therefore, the solution of the non-homogeneous equation (9) is

$$\varphi_n' = \frac{1}{2^{n-1}}\left\{ C_n' \cos n\varphi + C_n''\frac{\sin n\varphi}{\sin \varphi} \right\}, \tag{12}$$

where C_n', C_n'' are functions of n such that

$$\Delta C_n' \cos (n+1)\varphi + \Delta C_n'' \frac{\sin (n+1)\varphi}{\sin \varphi} = 0, \quad (\Delta C_n = C_{n+1} - C_n),$$

$$\Delta C_n' \cos (n+2)\varphi + \Delta C_n'' \frac{\sin (n+2)\varphi}{\sin \varphi} = \frac{2^{n+1}}{n^2} (\delta_n' \varphi_{n+1}' + \delta_n'' \varphi_n'), \tag{13}$$

$$C_n' = C' - \sum_1^n \frac{2^{n+1}}{n^2} (\delta_n' \varphi_{n+1}' + \delta_n'' \varphi_n') \frac{\sin (n+1)\varphi}{\sin \varphi},$$

$$C_n'' = C'' + \sum_1^n \frac{2^{n+1}}{n^2} (\delta_n' \varphi_{n+1}' + \delta_n'' \varphi_n') \cos (n+1)\varphi, \tag{14}$$

C', C'' do not depend on n.

Taking in (6) $f(x) = \varphi_n(x; \alpha, \beta)$, $\varphi_i(x) = \varphi_i(x; 1, 1)$ and using the known property: $|X_n(x)| \leqq 1$ for $-1 \leqq x \leqq 1$, Stekloff shows[1] that

$$2^{n-1} \varphi_n'(x\,;\ \alpha, \beta) = O(n^{1/2}), \qquad (-1+\epsilon \leqq x \leqq 1-\epsilon). \tag{15}$$

(15) assures the uniform and absolute convergence of

$$\sum_1^\infty \frac{2^{n+1}}{n^2} (\delta_n' \varphi_{n+1}' + \delta_n'' \varphi_n') \begin{matrix}\cos\\ \sin\end{matrix} (n+1)\varphi,$$

and we get from (12, 14),

$$C_n' = C_1 + \frac{1}{\sin\varphi} \sum_n^\infty (a_n' \varphi_{n+1}' + a_n'' \varphi_n'),$$

$$C_n'' = C_2 + \sum_n^\infty (b_n' \varphi_{n+1}' + b_n'' \varphi_n')\,; \tag{16}$$

C_1, C_2 do not depend on n; a_n', a_n'', b_n', b_n'' are finite for $n \to \infty$.

$$\varphi_n'(\cos \varphi) = \frac{C_1 \cos n\varphi + C_2 \sin n\varphi / \sin \varphi}{2^{n-1}}$$
$$+ \frac{\cos n\varphi}{2^{n-1} \sin \varphi} \sum_n^\infty \frac{2^{n+1}}{n^2} (a_n' \varphi_{n+1}' + a_n'' \varphi_n')$$
$$+ \frac{\sin n\varphi}{2^{n-1}} \sum_n^\infty (b_n' \varphi_{n+1}' + b_n'' \varphi_n') \frac{2^{n+1}}{n^2}\,. \tag{17}$$

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[1] *Loc. cit.* (2), p. 205.

The two sums $\sum_n^\infty$ in (17) become infinitely small with $1/n$. Therefore, (17) shows, in the first place, that $2^{n-1}\varphi_n(x)$ *is finite for* $n\to\infty$ *in the interval* (10). But we can go further. We notice in general that

$$U_n = O\left(\frac{1}{n^{3/2+\alpha}}\right) \text{ with } \alpha \geqq 0 \text{ implies : } \sum_n^\infty U_n = O\left(\frac{1}{n^{1/2+\alpha}}\right). \tag{18}$$

This, applied to (17), gives (since in the sums $\sum_n^\infty$ under consideration the first terms are of order $1/n^2$) $1/n$ *as the order of each sum.* We get finally, using (3),

$$\left.\begin{aligned} \varphi_n'(x\,;\,\alpha,\beta) &= \frac{A_1\cos n\varphi + A_2\sin n\varphi + O\left(\frac{1}{n}\right)}{2^{n-1}} \\ \varphi_n(x\,;\,\alpha,\beta) &= B_1\cos n\varphi + B_2\sin n\varphi + O\left(\frac{1}{n}\right) \end{aligned}\right| \begin{aligned} &x=\cos\varphi, \\ &(-1+\epsilon\leqq x\leqq 1-\epsilon), \end{aligned} \tag{19}$$

A_1, A_2 ,B_1, B_2 do not depend on n.

This is the required asymptotic expression for Jacobi polynomials with arbitrary positive parameters α, β. We may add, that (4) gives

$$\varphi_n(1\,;\,\alpha,\beta) = \frac{2^{1-\frac{\alpha+\beta}{2}}n^{\beta-\frac{1}{2}}(1+o(1))}{\Gamma(\beta)}\,;\ \varphi_n(-1\,;\,\alpha,\beta) = (-1)^n\frac{2^{1-\frac{\alpha+\beta}{2}}n^{\alpha-\frac{1}{2}}(1+o(1))}{\Gamma(\alpha)}.$$

Note: (8) shows that $c_n\equiv 0$ for $\alpha=\beta$, and that $\lambda_n=\frac{1}{4}$, $c_n=0$ simultaneously, if and only if

$$-2\alpha^2 - 2\beta^2 + 4\alpha + 4\beta - 3 = 0, \quad (\alpha-\beta)(\alpha+\beta-2) = 0$$

i.e., $O(1/n)\equiv 0$ in (19) in the following cases[1] only:

$$\begin{aligned} &\alpha=\beta=\tfrac{1}{2}\,; \quad \varphi_n(\cos\varphi) = \sqrt{\frac{2}{\pi}}\cos n\varphi\left(n\geqq 1\,;\,\varphi_0 = \sqrt{\frac{1}{\pi}}\right) \\ &\alpha=\beta=\frac{3}{2}\,; \quad \varphi_n(\cos\varphi) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(n+1)\varphi}{\sin\varphi} \\ &\alpha=\frac{3}{2},\ \beta=\tfrac{1}{2}\,;\ \varphi_n(\cos\varphi) = \sqrt{\frac{1}{\pi}}\,\frac{\cos(2n+1)\varphi/2}{\cos\varphi/2} \\ &\alpha=\tfrac{1}{2},\ \beta=\frac{3}{2}\,;\ \varphi_n(\cos\varphi) = \sqrt{\frac{1}{\pi}}\,\frac{\sin(2n+1)\varphi/2}{\sin\varphi/2}. \end{aligned} \tag{20}$$

[1] To obtain (20), we use (2), substituting $x=\cos\phi$.

3. Polynomials of Legendre. (19) holds for Legendre polynomials, as a particular case ($\alpha=\beta=1$; see (5)). It seems to be of interest, however, to treat Legendre polynomials directly from the classical difference equation

$$X_{n+2}(x) - \frac{2n+3}{n+2} X_{n+1}(x) + \frac{n+1}{n+2} X_n(x) = 0, \ (n \geqq 0). \tag{21}$$

The method developed above obviously cannot be applied to (21). We apply, therefore, to (21) a certain transformation, which may be useful in similar problems. We introduce, instead of $X_n(x)$, a new function

$$U_n(x) \equiv \frac{X_n(x)}{\theta_n}, \tag{22}$$

where θ_n depends on n only and satisfies the relation

$$\theta_{n+2} = \frac{n+1}{n+2}\theta_n, \tag{23}$$

so that the transformed equation (21) for $U_n(x)$ is

$$U_{n+2} - \frac{2n+3}{n+1}\,\frac{\theta_{n+1}}{\theta_n}\,xU_{n+1} + U_n = 0. \tag{24}$$

We must find the asymptotic expression of θ_n for n very large. (23) gives, with two arbitrary constants θ_0, θ_1,

$$\theta_{2m} = \frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots\ \ 2m}\theta_0;\ \theta_{2m+1} = \frac{2\cdot 4\cdots\ \ 2m}{1\cdot 3\cdots(2m+1)}\theta_1. \tag{25}$$

We write now, with Tchebycheff, Wallis' formula as follows:

$$\frac{\pi}{2} = \left[\frac{2\cdot 4\cdots\ \ 2m}{1\cdot 3\cdots(2m-1)}\right]^2\cdot\frac{Y}{2m+1} = \left[\frac{2\cdot 4\cdots\ \ 2m}{1\cdot 3\cdots(2m-1)}\right]^2\cdot\frac{X}{2m}, \tag{26}$$

$$\begin{aligned} Y &= \frac{2m+2}{2m+1}\cdot\frac{2m+2}{2m+3}\cdots = \left(1+\frac{1}{(2m+1)(2m+3)}\right)\left(1+\frac{1}{(2m+3)(2m+5)}\right)\cdots, \\ X &= \frac{2m}{2m+1}\cdot\frac{2m+2}{2m+1}\cdots = \left(1-\frac{1}{(2m+1)^2}\right)\left(1-\frac{1}{(2m+3)^2}\right)\cdots. \end{aligned} \tag{27}$$

Tchebycheff[1] gets, using the fact, that $Y>1$, $X<1$,

$$\left[\frac{2\cdot 4\cdots\ \ 2m}{1\cdot 3\cdots(2m-1)}\right]^2 = \frac{\pi(m+\frac{1}{2})}{Y} = \frac{\pi m}{X} = \pi(m+\epsilon_m),\ \ (0<\epsilon_m<\tfrac{1}{2}). \tag{28}$$

[1] Tchebycheff, On functions deviating the least from zero, *Collected Papers* (in Russian) t. II, 189–215; p. 209.

The asymptotic expression (28) *is insufficient for our analysis*. We must estimate ϵ_m. We proceed to prove:

$$\lim_{m\to\infty} \epsilon_m = \tfrac{1}{4} \;;\; \frac{2\cdot 4\cdots 2m}{1\cdot 3\cdots(2m-1)} = \sqrt{\pi\left(m+\tfrac{1}{4}+O\left(\frac{1}{m}\right)\right)}. \tag{29}$$

In order to obtain (29), we investigate X or Y in (26, 27). We get from (27), using the inequalities

$$\log(1+x) = \frac{x}{1+\theta x}\,(0<\theta<1),\; x>\log(1+x)>\frac{x}{1+x}\,(x>0),$$

$$\begin{gathered}\frac{1}{(2m+1)(2m+3)}+\frac{1}{(2m+3)(2m+5)}+\cdots>\log Y\\ >\frac{1}{1+(2m+1)(2m+3)}+\frac{1}{1+(2m+3)(2m+5)}+\cdots.\end{gathered} \tag{30}$$

We get further, since

$$\frac{1}{(2m+1)(2m+3)}+\frac{1}{(2m+3)(2m+5)}+\cdots=\frac{1}{2(2m+1)}=\frac{1}{4m+2},$$

$$\begin{aligned}&\frac{1}{1+(2m+1)(2m+3)}+\frac{1}{1+(2m+3)(2m+5)}+\cdots\\ &\quad=\tfrac{1}{4}\sum_{1}^{\infty}{}_{i}\frac{1}{(m+i)^2}>\tfrac{1}{4}\int_{m+1}^{\infty}\frac{dx}{x^2}=\frac{1}{4m+4},\end{aligned}$$

$$\log Y=\frac{1}{4(m+\theta)}\;;\quad Y=e^{\frac{1}{4(m+\theta)}}\;(\tfrac{1}{2}<\theta<1). \tag{31}$$

(31), substituted into (28), gives (29), or

$$\frac{1\cdot 3\cdots(2m-1)}{1\cdot 2\cdots\quad m}=2^m\sqrt{\frac{1}{\pi\left(m+\frac{1}{4}+O\left(\frac{1}{m}\right)\right)}}. \tag{32}$$

(25) gives now

$$\theta_{2m}=\theta_0\sqrt{\frac{1}{\pi\left(m+\frac{1}{4}+O\left(\frac{1}{m}\right)\right)}},\quad \theta_{2m+1}=\theta_1\frac{\sqrt{\pi\left(m+\frac{1}{4}+O\left(\frac{1}{m}\right)\right)}}{2m+1},$$

and with $\theta_0=\pi$, $\theta_1=2$, we get a single formula

$$\theta_n=\sqrt{\frac{2\pi}{n}}\left(1+O\left(\frac{1}{n}\right)\right). \tag{33}$$

(24), combined with (33), gives:

$$U_{n+2}(x) - 2xU_{n+1}(x) + U_n(x) = \frac{x\delta_n'}{n^2} U_{n+1}(x), \tag{34}$$
$$\delta_n' \text{ finite for } n \to \infty,$$

an equation quite similar to (9), *which can be treated similarly*, since

$$U_n(x) = \frac{X_n(x)}{\theta_n} = \sqrt{\frac{n}{2\pi}}\, X_n(x)\left(1 + O\left(\frac{1}{n}\right)\right);$$
$$|U_n(x)| = O(n^{\frac{1}{2}}) \text{ for } -1 \leqq x \leqq 1. \tag{35}$$

We get then for any x in $(-1+\epsilon, 1-\epsilon)$ an asymptotic expression of $U_n(x)$ similar to (19), and then, by means of (35),

$$X_n(\cos\varphi) = \sqrt{\frac{2\pi}{n}}\left(C_1 \cos\varphi + C_2 \sin\varphi + O\left(\frac{1}{n}\right)\right), \tag{36}$$

where C_1, C_2 do not depend on n, $0 < \epsilon \leqq \varphi \leqq \pi - \epsilon$.

SOME APPLICATIONS OF MATHEMATICS TO ARCHITECTURE: GOTHIC TRACERY CURVES[1]

By E. C. PHILLIPS, S. J., Georgetown University

1. Some fifteen years ago Professor Frank Morley of Johns Hopkins University suggested as an interesting problem the study of rose-window curves. In this paper, which was inspired by that suggestion, an attempt is made to express by means of mathematical equations some of those beautiful forms of architectural adornment which give to the great cathedrals and other buildings of the middle ages much of the graceful detail which has won and held the admiration of succeeding centuries. As an introduction to the subject a brief historical account of the origin and development of Gothic architectural decoration will be given. Much has been written on this subject which has been treated at length in most of the histories of architecture, but a very brief summary is sufficient for our purpose and the following notes have been extracted chiefly from an excellent article on "The Gothic Window" by P. R. McCaffrey, O.C.C., published in *The Ecclesiastical Review* for March, 1924, (vol. LXX, no. 3, pp. 254–269).

2. In the earliest stone buildings of medieval times, when the houses of the great were fortresses as well as dwellings, the windows were small rectangular

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[1] Presented at the Kansas City meeting of the Association, December 31, 1925.

apertures which were splayed inward in order to help diffuse the light while the narrowness of the external opening served as a protection against attack from without. Next round windows, either entirely circular or with a semi-circular head on a rectangular body came into use; then two round windows were made adjacent as in Norman and Romanesque architecture, while a later modification placed a circular window above two adjacent ones, as is found in certain Cistercian churches.

A subsequent development of great importance was the *lancet*, a high narrow window with a pointed arch at the top, the two sides of the arch being arcs of circles of relatively long radius, and in this type of construction we find the origin of the later Gothic style which tends ever towards the heavens and invites us to raise our minds and hearts above the sordid things of earth. After the development of the single lancet, larger and more lightsome windows were constructed by bringing lancets together in twos, threes, fives or sevens and enclosing these multiple lancets under a single external arch. After the advent of the double lancet with its enclosing common arch or drip stone, *plate tracery* was introduced which, in its earliest form, consisted of a single circular hole pierced in the solid stone slab between the two lancets and the external arch. Later, this single circle was replaced by trefoils, quatrefoils and other forms which finally became very intricate. This style of construction or decoration is called plate tracery because the opening was cut in a single slab or plate of stone, or at least in a flat solid portion of the wall; but the architect soon rose above this limitation and began to build up his tracery by assembling into the structure of the window many more slender pieces of carved stone and thus originated what is called *bar tracery*; it is this kind of construction which is generally understood when we speak of Gothic tracery. The bars are the mullions, or solid portions, separating the lights, or clear portions, of the window and forming the framework which supports the glass. Gothic architecture dates from the twelfth century, its foundations being laid in 1144 when Abbot Suger completed the great Royal Abbey of St. Denis, near Paris, and mullions were first introduced towards the end of this same century, reaching their highest perfection in the middle of the thirteenth century. The designs of the tracery became more and more intricate and it is truly wonderful what lightness and grace were secured by the artists and artisans of those days who designed and chiseled and fitted together the hundreds of pieces of stone going to make up the framework of the great windows of the thirteenth and fourteenth centuries, as for example that of the marvelous rose windows in the transepts of the Cathedral of Notre Dame in Paris.

The designs of all the earlier Gothic windows consisted entirely of combinations of straight lines and circular arcs or complete circles, but other curves were introduced later and hence we have three main divisions of Gothic

tracery, known respectively as (1) geometrical tracery, (2) curvilinear or flowing tracery and (3) flamboyant tracery. The characteristics of these three styles may be summed up briefly as follows:

(1) In geometrical tracery all the lines are straight or circular, and hence, while the curves are mechanically continuous, they cannot be represented by any single continuous function of the coordinates, each part of the curve requiring a separate equation; there are no true cusps or flexes but we pass from straight line to circular arc or from one circular arc to another.

(2) In curvilinear or flowing tracery the architect made all the portions of the main design to flow on uninterruptedly, the apices formed in geometrical tracery by the juncture of two circular arcs being now replaced by true cusps and the juncture of a straight line and a circular arc or of two oppositely curved circular arcs becoming a true flex.

(3) In the flamboyant style the curves became more spiral in form and took on somewhat the aspect of tongues of flame. This style is considered as a decadent type of Gothic and was used chiefly on the continent while in England the flowing Gothic was being supplanted by the Tudor or rectangular style; there are however some beautiful examples of flamboyant tracery found in England, one of the best being that of the great west window of York Cathedral.

3. The discussion in this paper is restricted to flowing and flamboyant tracery since only they employ continuous curves having continuous first and second derivatives, though some examples of geometrical tracery contain designs which in appearance differ almost imperceptibly from such curves. In their general character most of these curves, in all three styles, resemble a flat flower with pointed petals arranged uniformly about a circular pistil, and our purpose is to find and discuss a system of equations which will represent the varied forms found in these circular designs.

The method of attack which suggested itself was first to find, in rectangular Cartesian coordinates, the equation of a periodic curve something like the cycloid but having cusps both above and below instead of concave arches bounded by cusps. When such a curve was obtained the next step was to bend this curve in its own plane into circular form so that each cuspidal arch should become a petal of our flower or rose-window. Several equations were tried but the simplest and most convenient was the following one given in parametric form:

$$x=(t-\tfrac{1}{2}\sin 2t)/p, \qquad y=1+k\cos t\,, \tag{1}$$

p being an arbitrary constant, and $k^2 \leqq 1$. On differentiating we find

$$p\,dx/dt=1-\cos 2t=2\sin^2 t, \qquad dy/dt=-k\sin t; \tag{2}$$

$$p\,d^2x/dt^2=2\sin 2t=4\sin t\cos t, \qquad d^2y/dt^2=-k\cos t. \tag{3}$$

Thus dx/dt and dy/dt are both zero whenever $\sin t$ is zero, that is for $t=0$, π, 2π, 3π, ... $n\pi$; whilst the second derivatives are simultaneously zero only when $\cos t$ is zero, that is for $t=\pi/2$, $3\pi/2$, ... $(2n-1)\pi/2$. Hence there is a cusp for $t=0$, π, 2π, ... $n\pi$; and a flex for $t=(2n-1)\pi/2$. The cusps for $t=0$, 2π, 4π, etc., all point upwards and lie on the line $y=1+k$; whilst those for $t=\pi$, 3π, 5π, etc., point downwards and lie on the line $y=1-k$. The flexes are midway between the cusps and lie on the line $y=1$. The form of the curve is shown in Fig. 1.

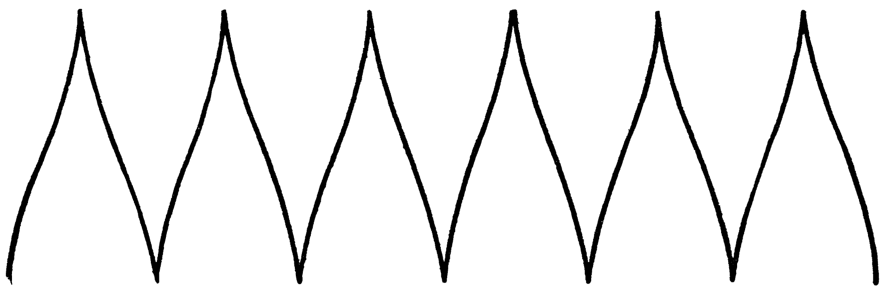


FIG. 1

If we now consider this strip bounded by the line $y=1\pm k$ to be flexible in the plane of the paper so that we can take any length of it and bend it into circular form, the cuspidal tangents $x=0$, π, 2π, etc., will go into concurrent straight lines, and the system of straight lines $y=$any constant, will go into concentric circles with the transforms of the x-lines as radii; each cuspidal arch of the original curve will then form a cuspidal petal and we have a window tracery of attractive appearance as shown in Fig. 2, which is the result of bending three or more arches into the circular form.

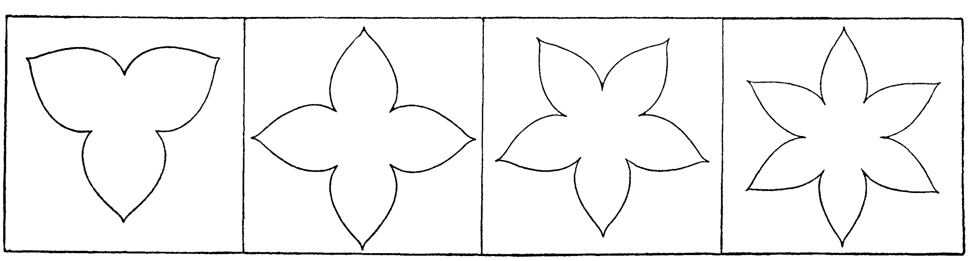


FIG. 2

In order to perform this wrapping process mathematically we merely substitute in equations (1) θ for x and ρ for y, where θ and ρ are polar coordinates; and in these new equations p will be the number of petals in our design. Hence the required equations are:

$$\theta = (1/p)(t - \tfrac{1}{2}\sin 2t), \qquad \rho = 1 + k\cos t. \tag{4}$$

For the present we will restrict the values of p to positive integers, so that we will have p entire petals in the circuit of 360°. Since the form of the differential equations (2) and (3) is unaltered by the above transformation, it follows that the cusps are transformed into cusps, while the flexes of the x-y arches do not go into flexes on the petals, and an examination of the figures given shows that the flexes on the petals are closer to the outer cusps than to the inner ones.

Since $\tan\psi = \rho d\theta/d\rho = -(2/pk)(1+k\cos t)\sin t$, and this is zero for $t=0$, π, 2π, etc., it follows that the curve is tangent to the radii vectores for these values which are the same as the values of t giving the cusps; hence the cusp tangents are radii of the concentric circles $\rho = 1+k$ and $\rho = 1-k$ on which the outer and inner rows of cusps lie.

Equations (4), even with the limitations on the values of the arbitrary constants p and k, allow us to vary the tracery very considerably, the number of petals being at our disposal as well as the depth of the circular ring which the petals occupy, and we give in Figs. 2 and 3 two series of curves in the first

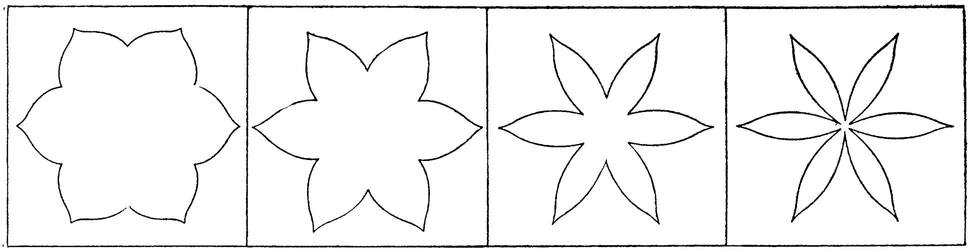


FIG. 3

of which the number of petals is varied while the depth of the ring is kept constant, and in the second of which the depth of the ring is varied whilst the number of petals is kept constant.

We may secure a further variation in form by replacing the integer p by a rational fraction, the equations now become

$$\theta = (c/p)(t - \tfrac{1}{2}\sin 2t), \qquad \rho = 1 + k\cos t, \tag{5}$$

where c and p are both integers and c is less than and prime to p. ρ now goes through its entire variation, and hence one petal is completed as t varies from 0 to 2π, or from 2π to 4π, etc., whilst for this same increase in the parameter t the angle θ increases by the quantity $(c/p)2\pi$ which is therefore the angular measure of the sector in which each petal is contained; hence in the entire circuit of 360° there will be p/c petals, but as p/c is not an integer the last petal will be incomplete and the curve will not close in at the first revolution of the radius vector but only when c complete revolutions have been made. Thus we will have a more complicated tracery consisting of p petals formed in

c complete circuits of the circle as shown in Figs. 4 and 5, for which c/p equals respectively 2/3 and 2/5. The greater the values of c and p the more complicated the figure becomes and if we were to allow, as a further extension, c/p to take on irrational values we would have an endless curve with an infinite number of petals filling the entire circular ring bounded by the two concentric circles of cusps. These forms are of course impossible in actual architectural designs; and in fact there are few if any examples of forms corresponding strictly to fractional values of p/c though we find pseudo examples of such forms constituted by two interlacing curves of the simpler form represented by equations (4).

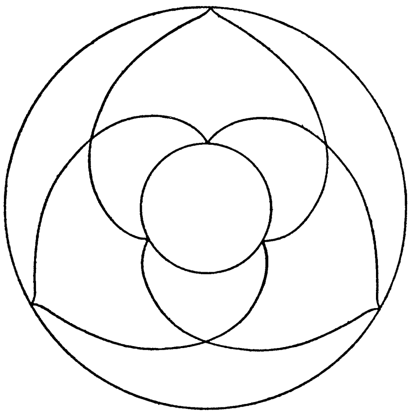


FIG. 4

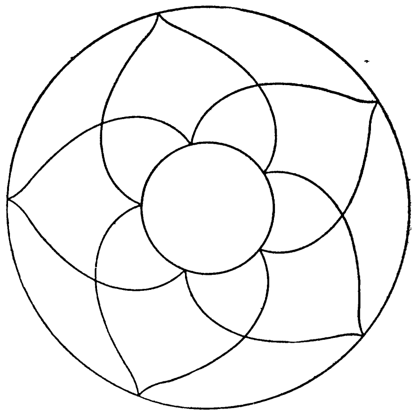


FIG. 5

Returning now to these simpler forms we find that the artistic character of the petals depends considerably on the position of the flex; if we compare the cuspidal arches of the x-y curve (Fig. 1) with the corresponding petals of the θ-ρ curve (Fig. 2) into which they are transformed we find that the former are more symmetrical (though not more graceful) than the latter inasmuch as the flex in the arches is midway between the cusps as measured either along the length of the curve or in the vertical direction between the cusps, while in the petals the flexes are much nearer the outer cusps than the inner ones. The position of the flex in flowing tracery differs considerably in the different examples which we have examined in Gothic structures, and hence it is well to have a means of modifying the shape of the petals more completely than the simple equations already discussed allow us to do. For this purpose we add a new term to the parametric value of ρ which will leave the position of the cusps unaltered but will bring the flexes nearer to the inner cusps. This added function of t should therefore become zero whenever the original petal has a cusp, that is for $t=0$, π, 2π, etc.; any function containing $\sin t$ as a factor will satisfy this condition but if we wish further to retain the symmetry of the petal

with respect to its central line or cusp tangent we must choose a function that will have the same value for $2n\pi+\varphi$ as for $2n\pi-\varphi$, where φ is any angle whatever; $\sin^2 t$ is a function of this character but we prefer to keep the central point of the portion of the petal joining the inner and outer cusps also unaltered and this leads us to $m \sin^2 t \cos t$ for the form of the added term and our modified equations are

$$\theta = (c/p)(t - \tfrac{1}{2}\sin 2t), \qquad \rho = 1 + k\cos t - m \sin^2 t \cos t, \tag{6}$$

where m is an arbitrary constant determining the direction and amount of the bending. The curves given in Fig. 6 show the effect of this new term on the six petal design.[1] The first curve (a) is the fundamental or unmodified curve

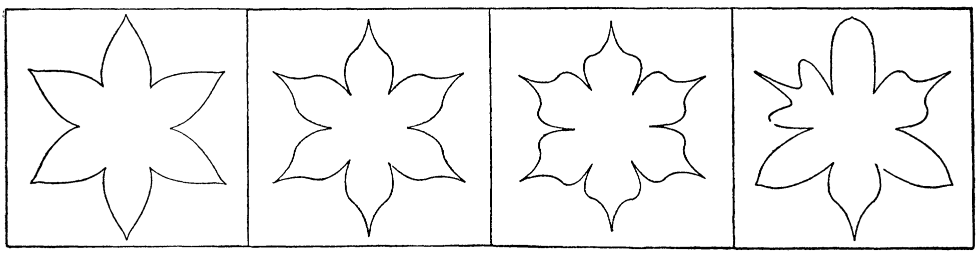


FIG. 6

given by equations (4); the second and third figures (b and c) are complete curves for which m equals respectively 1/4 and 1/2, and the fourth curve is a composite one in which the petals are taken from different curves with values of m varying from 1 to $-\frac{1}{2}$.

So far we have given the equations which represent the curves of pure flowing tracery; to complete the consideration of our subject we shall touch briefly on the flamboyant style. In this style the artist forsook the radial symmetry of the figure and began, apparently for the sake of securing novelty in his designs, to twist the figure spirally as can be seen in a number of the buildings erected during the period when Gothic architecture was declining from the purity and perfection it had attained in the thirteenth century. In order to introduce this twisting of the figure into our equations we merely make a transformation which will change radial straight lines into spirals through the origin. The equation of a radial straight line is $\theta=a$, where a is any constant, whilst the equation $\theta=a+b\rho$, where b is an arbitrary constant and ρ is the

[1] At the Kansas City meeting an informal vote was taken on the question "Which of the curves in Fig. 6 is the most graceful?" It would be interesting to know what is the opinion in this matter of a larger circle of individuals than were gathered at the meeting, and the author would be grateful to any reader of this article who would inform him by postal which one of these curves pleases him or her the most. (Address: Georgetown University, Washington, D. C.)

distance from the origin to the point (θ,ρ), gives us a spiral. Hence to give our curves the spiral twisting we use the equations

$$\theta = (c/p)(t - \tfrac{1}{2}\sin 2t) + b\rho, \quad \rho = 1 + k\cos t. \tag{7}$$

The value of b determines the amount of spiral twisting and Fig. 7 shows a couple of the flamboyant curves with the corresponding values of b indicated under each of them.

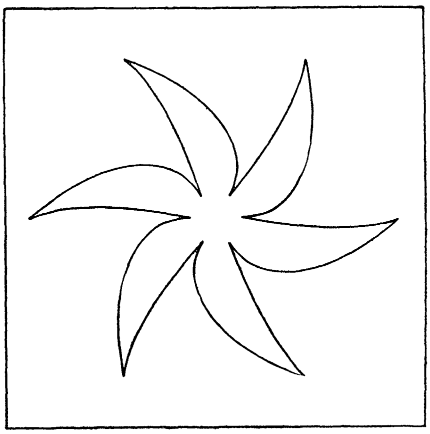


FIG. 7a. $b=0.087$

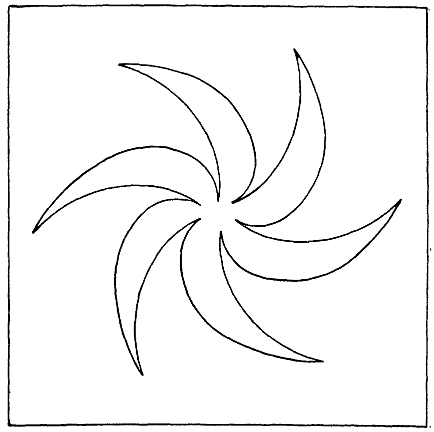


FIG. 7b. $b=0.174$

These curves could also be subjected to the various modifications introduced above in the case of the flowing traceries, but it seems hardly worth while pursuing the subject further in this direction as the flamboyant style is almost universally admitted to be a decadent form of Gothic; its fiery lines may indeed attract the attention of the onlooker but it lacks that perfection of proportion and harmony which is so characteristic of flowing Gothic and gives it its enduring charm.

ON THE DE LONGCHAMPS CIRCLE OF THE TRIANGLE

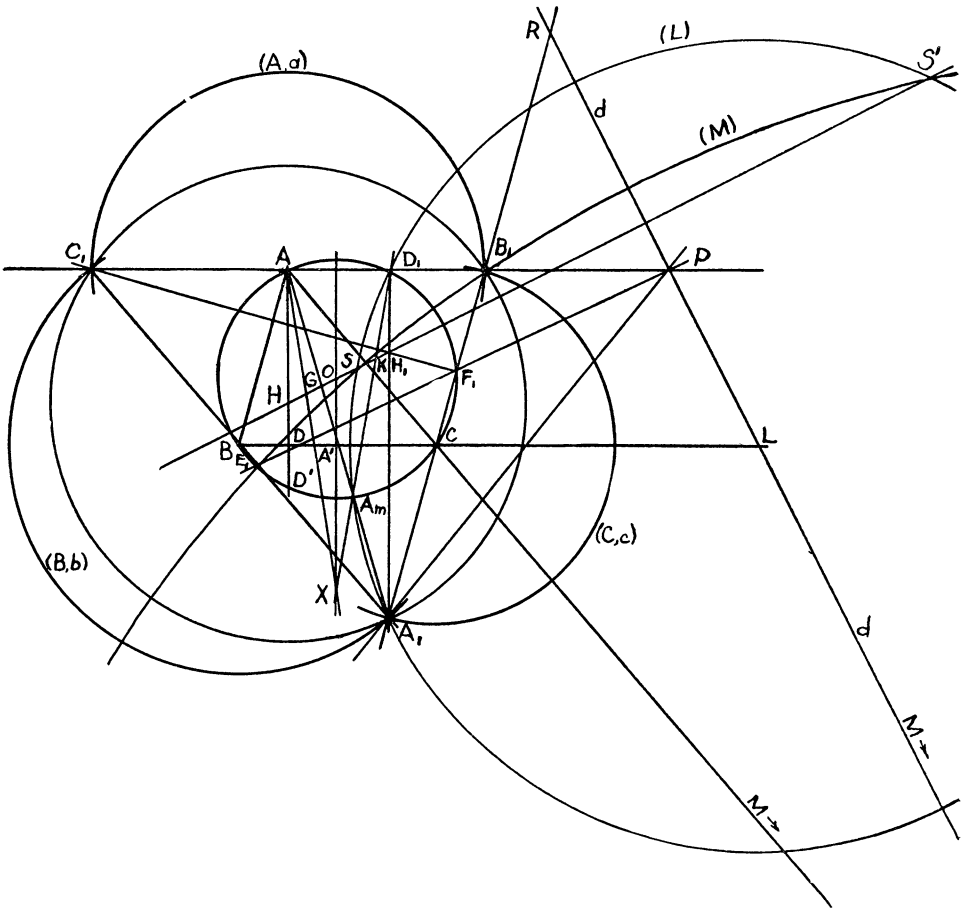
By NATHAN ALTSHILLER-COURT, University of Oklahoma

Introduction. The three circles (A,a), (B,b), (C,c) having for centers the vertices A, B, C of a given triangle ABC, and for radii the corresponding opposite sides were first considered by G. de Longchamps in an article "Sur un nouveau cercle remarquable du plan du triangle" published in the *Journal de Mathématiques Spéciales* in 1886, pp. 57 etc. In this analytical study the author was chiefly interested in the orthogonal circle of these three circles, and in the radical axis of this orthogonal circle with the circumcircle of the

basic triangle. This radical axis, the orthogonal circle, and the center of this circle are referred to respectively as the "de Longchamps axis, circle, and point" of the triangle.

In 1891, E. Vigarié gave some properties of the radical circles of the circles (A,a), (B,b), (C,c) in the *Journal de Mathématiques élémentaires*, pp. 63 etc.

E. Lemoine calculated the radii of the two circles which touch the circles (A,a), (B,b), (C,c) all internally and all externally (*Journal de Mathématiques élémentaires*, 1895, p. 139. See also *Mathesis*, 1906, pp. 59 etc.[1])



In the present article the bisecting circles and the circles of similitude of the circles (A,a), (B,b), (C,c) are considered. The results of the sections 1, 2, 3, 9b are known. For the convenience of the reader it was deemed advisable to reproduce them here.

[1] For bibliographical references to some related topics see M. Simon, *Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*, p. 104, Leipzig, Teubner, 1906.

1. The line of centers $BC=a$ of the two circles (B,b), (C,c) being smaller than the sum and greater than the difference of the radii b, c, the two circles intersect in two real points A_1, D_1; let A_1 denote the point lying on the opposite side of BC from the vertex A.

The two pairs of opposite sides of the quadrilateral ABA_1C are equal, hence A_1 is the point of intersection of the parallels to the sides AB, AC of ABC drawn through the opposite vertices C, B.

The triangles ABC, D_1BC are congruent as having their sides respectively equal, hence AD_1 is parallel to BC, and therefore perpendicular to D_1A_1, since the common chord is perpendicular to the line of centers BC. Similarly for the analogous points B_1, E_1; C_1, F_1.

The points A_1, B_1, C_1 are thus the vertices of the triangle formed by the parallels to the sides of ABC through the opposite vertices, *i. e.*, $A_1B_1C_1$ is the anticomplementary triangle of ABC, and $D_1E_1F_1$ is the orthic triangle of $A_1B_1C_1$. Thus: *The three circles (A,a), (B,b), (C,c) have for diameters the sides of the anticomplementary triangle of the basic triangle, and for common chords the altitudes of the anticomplementary triangle.*

The reader may notice that the points D_1, E_1, F_1, lie on the circumcircle (O) of the basic triangle ABC.

2. The altitudes A_1D_1, B_1E_1, C_1F_1 of $A_1B_1C_1$ being the radical axes of the circles (A,a), (B,b), (C,c) taken in pairs, the orthocenter H_1 of $A_1B_1C_1$ is therefore the radical center of these three circles. *i. e.*, the de Longchamps point of ABC.

The two triangles $A_1B_1C_1$ and ABC being homothetic with respect to their common centroid G (the homothetic ratio being -2), we have

$$GH_1 : GH = -2,$$

where H is the orthocenter of ABC. On the other hand O being the circumcenter of ABC we have

$$GH : GO = -1, \qquad \text{hence} \qquad OH_1 : OH = -1.$$

Consequently: *The de Longchamps point of a triangle*: (a) *lies on the Euler line of the triangle*; (b) *is the symmetric of the orthocenter of the triangle with respect to its circumcenter*; (c) *is the orthocenter of the anticomplementary triangle.*

3. The power of H_1 with respect to the circles (A,a), (B,b), (C,c) is respectively $H_1A_1 \cdot H_1D_1 = H_1B_1 \cdot H_1E_1 = H_1C_1 \cdot H_1F_1$, hence the center H_1 as well as the radius of the de Longchamps circle (H_1) of ABC is identical with the corresponding elements of the conjugate circle of $A_1B_1C_1$, hence the two circles are identical.

Now the conjugate circle of $A_1B_1C_1$ is coaxal with its circumcircle (O_1) and its nine point circle,[1] the latter being identical with the circumcircle (O)

of ABC. We have therefore: (a) *The de Longchamps circle of a triangle is the conjugate circle of the anticomplementary triangle.* (b) *The de Longchamps axis of a triangle is the radical axis of its circumcircle with the circumcircle of the anticomplementary triangle.*

4. For the foot D of the altitude AD of ABC we have

$$AB^2-BD^2=AC^2-CD^2$$

or

$$CD^2-BD^2=AC^2-AB^2=b^2-c^2.$$

Similarly for the feet E, F of the altitudes BE, CF. Hence:[2] *The altitudes of the basic triangle are the loci of the centers of the circles which bisect (i. e., cut in diametrically opposite points) the three circles (A,a), (B,b), (C,c) taken in pairs.*

5. The point H common to the three loci AD, BE, CF (4)[3] is the center of a circle bisecting the three circles (A,a), (B,b), (C,c). It is readily seen that this bisecting circle is identical with the circumcircle (O_1) of $A_1B_1C_1$.

The second point of intersection D' of AD with the circumcircle (O) of ABC is the symmetric of H with respect to the side BC, hence the circle (D') described with D' as center and bisecting the two circles (B,b), (C,c) is the symmetric of (O_1) with respect to BC. Similarly for the analogous circles (E'), (F').

It follows that the lines AB, AC are the radical axes of (O_1) with the circles (E'), (F') respectively, hence the radical axis of (E') and (F') is the perpendicular from A upon the line $E'F'$, *i. e.*, this radical axis is the radius OA of the circumcircle (O) of ABC.[4]

The line of centers OD' of the two circles (O), (D') is equal to the radius of (O), and since the radius of (D') is double the radius of (O), the two circles are tangent at the diametric opposite D_1 (1) of D' on (O). To sum up: *The circles (A,a), (B,b), (C,c) taken in pairs are bisected by the circles having for centers the second points of intersection of the altitudes of the basic triangle with its circumcircle, and for radii the diameter of this circumcircle.*

These three circles have for their radical center the center of the circumcircle and are tangent to this circle, the points of contact being the vertices of the orthic triangle of the anticomplementary triangle.

6. The line BC is the radical axis of the circles (O_1) and (D') (4, 5), and the common tangent D_1L at D_1 is the radical axis of the circles (O) and (D'),

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[1] Nathan Altshiller-Court, *College Geometry*, Johnson Publishing Company, Richmond, Va., 1925, p. 217.

[2] *Ibid.*, p. 168.

[3] A number in parentheses refers to the corresponding section.

[4] *Ibid.*, p. 85.

hence the trace L of D_1L on BC is a point on the radical axis of the circles (O), (O_1), *i. e.*, on the de Longchamps axis of the triangle ABC (3b). Consequently: *The common tangents to the circumcircle of the basic triangle and the three bisecting circles* (5) *meet the corresponding sides of the basic triangle on the de Longchamps axis of this triangle.*

7. The lines B_1C_1 and E_1F_1 are the radical axes of the circle (A, a) with the circles (O_1) and (O) respectively, hence the point $P \equiv (B_1C_1,\ E_1F_1)$ is a point on the radical axis of the circles (O) and (O_1), *i. e.*, on the de Longchamps axis of the triangle ABC (3b). Thus P is the point of intersection of the radical axes d and B_1C_1 of the circle (O_1) with the circles (H_1) (3) and (A, a) respectively, hence P lies on the radical axis of (H_1) and (A, a).

The four points B_1, C_1, E_1, F_1 determine a complete quadrangle inscribed in the circle (A, a), hence the diagonal points $A_1 \equiv (B_1F_1,\ C_1E_1)$, $H_1 \equiv (B_1E_1,\ C_1F_1)$, $P \equiv (B_1C_1,\ E_1F_1)$ determine a triangle A_1H_1P conjugate with respect to (A, a), and therefore A is the orthocenter of A_1H_1P. Thus the line A_1P is perpendicular to the line of centers AH_1 of the circles (A, a), (H_1) and passes through the point P of their radical axis, hence A_1P coincides with the radical axis of these two circles. Consequently: *The lines joining the vertices of the anticomplementary triangle to the traces of the de Longchamps axis on the opposite sides of this triangle are the radical axes of the de Longchamps circle with the circles* (A, a), (B, b), (C, c).

8. The radical axes A_1P, B_1Q, C_1R of (H_1) with the circles (A, a), (B, b), (C, c) (7) form a triangle whose sides meet the sides B_1C_1, C_1A_1, A_1B_1 of $A_1B_1C_1$ in the points P, Q, R, of the de Longchamps axis d of ABC (7). On the other hand the point of intersection of the radical axes B_1Q, C_1R of (H_1) with (B, b), (C, c) respectively lies on the radical axis $A_1H_1D_1$ of the last two circles, hence: *The anticomplementary triangle and the triangle formed by the radical axes of the de Longchamps circle with the circles* (A, a), (B, b), (C, c) *are homological, the center and axis of homology being the de Longchamps point and the de Longchamps axis of the basic triangle.*

9. The line A_1A joining the vertex A_1 of the triangle A_1H_1P to its orthocenter A (7) is perpendicular to PH_1. On the other hand the polar of P with respect to (H_1) passes through the pole A_1 of B_1C_1 with respect to (H_1) and is perpendicular to the line PH_1 joining P to the center H_1 of (H_1), hence this polar coincides with A_1A. Consequently: (a) *The medians of the basic triangle are the polars, with respect to the de Longchamps circle, of the traces of the de Longchamps axis on the sides of the anticomplementary triangle.* (b) *The centroid of the basic triangle is the pole of the de Longchamps axis with respect to the de Longchamps circle of the triangle.*

10. Let (L), (M), (N) be the circles of similitude of the circle (A,a), (B,b), (C,c) taken in pairs. The circle of similitude (L) of the circles (B,b), (C,c) is coaxal with these circles. Then the three circles of similitude (L) (M), (N) are orthogonal to the circles (O), (O_1), (H_1),[1] and therefore coaxal, their radical axis being the line of centers OH_1 of the circles (O), (H_1), *i. e.*, the Euler line of ABC. The centers L, M, N, of (L), (M), (N) are the points of intersection of the de Longchamps axis with the sides of the basic triangle ABC (6).

The circles (M), (N) pass through the points B_1, C_1, respectively. Since they are orthogonal to (O_1), they are tangent to the radii HB_1, HC_1 of (O_1). Now the point H lies on the radical axis of (M), (N), hence B_1, C_1, are antihomologous points on these two circles,[2] and the line B_1C_1 meets the line of centers $d \equiv MN$ in a center of similitude of (M), (N), *i. e.*, this center of similitude coincides with P (7) and lies on the radical axis PA of the two circles (A,a), (H_1).

The line AP is the harmonic conjugate, with respect to AB, AC, of the line AA' joining A to the mid-point A' of BC, hence the line AA' meets d in the harmonic conjugate of P with respect to the centers M, N of the circles (M), (N), *i. e.*, in the second center of similitude of these two circles. Consequently: *The three circles of similitude of the three circles* (A,a), (B,b), (C,c) *taken in pairs have for their centers of similitude the traces of the de Longchamps axis on the medians of the basic triangle and on the three radical axes of the de Longchamps circle with the circles* (A,a), $(B.b)$, (C,c).

These three pairs of centers of similitude are pairs of conjugate points with respect to the de Longchamps circle (9a).

11. The centers of similitude U, U' of the two circles (B,b), (C,c) divide the line of centers BC harmonically in the ratio $b:c$, hence these points are the traces on BC of the bisectors of the angle D_1 of the triangle D_1BC. Now the circle of similitude (L) of the circles (B,b), (C,c) has for diameter the segment UU' and passes through the point D_1, hence (L) is the circle of Apollonius of D_1BC corresponding to the vertex D_1. On the other hand the triangle D_1BC is the symmetric of the basic triangle ABC with respect to the circumdiameter of ABC perpendicular to BC. Similarly for the circles (M), (N). Therefore: *The three circles of similitude of the circles* (A,a), (B,b), (C,c) *are the symmetrics of the circles of Apollonius of the basic triangle with respect to the perpendicular bisectors of the corresponding sides of this triangle.*

12. The triangle A_1BC is the symmetric of the triangle D_1BC with respect to BC, and since the circle (L) (11) is its own symmetric with respect to BC,

[1] J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, p. 107.

[2] *College Geometry*, p. 166.

(L) is also the Apollonian circle of A_1BC passing through A_1. Consequently the lines A_1U, A_1U' are the two bisectors of the angle BA_1C. But the angle BA_1C is identical with the angle $B_1A_1C_1$, hence: *The sides of the basic triangle are met by the bisectors of the corresponding angles of the anticomplementary triangle in the centers of similitude of the circles* (A,a), (B,b), (C,c) *taken in pairs.*

13. The triangles A_1BC and $A_1B_1C_1$ are homothetic, the point A_1 being the homothetic center and the ratio being $1/2$, hence the Apollonian circle (L) of A_1BC (12) corresponds in this similitude to the Apollonian circle (A_1) of $A_1B_1C_1$ passing through A_1. Thus the midpoint of the segment intercepted by (A_1) on the symmedian of $A_1B_1C_1$ passing through A_1 lies on (L). But this midpoint is a vertex of the second Brocard triangle of $A_1B_1C_1$,[1] hence: *The circles of similitude of the three circles* (A,a), (B,b), (C,c) *taken in pairs pass through the respective vertices of the second Brocard triangle of the anti-complementary triangle.*

14. Let A_m be the second point of intersection of the median AA' of the triangle ABC with its circumcircle (O). The areas of the two triangles BAA_m, CAA_m having the common base AA_m, are to each other as the segments BA', CA', hence these areas are equal, and since the angles ABA_m, ACA_m are supplementary, we have

$$AB \cdot BA_m = AC \cdot CA_m, \quad \text{or} \quad A_mB : A_mC = b : c,$$

hence the point A_m belongs to the circle of similitude (L) of (B,b), (C,c). Similarly for the analogous points B_m, C_m. Thus: *The circles of similitude of the circles* (A,a), (B,b), (C,c) *taken in pairs pass through the second points of intersection of the circumcircle of the basic triangle with the corresponding medians of this triangle.*

15. The circle of Apollonius (L) of D_1BC is orthogonal to the circumcircle (O) of this triangle, hence their common chord D_1A_m (14) is the polar of the center L of (L) with respect to (O), and since L lies on BC, the polar D_1A_m will pass through the pole of BC with respect to (O), *i. e.*, through the point of intersection X of the tangents BX, CX to (O) at the points B, C. On the other hand D_1BC being the symmetric of ABC with respect to the diameter $OA'X$ of (O), the line D_1A_mX is the symmetric, with respect to $OA'X$, of the common chord of (O) and the Apollonian circle of ABC passing through A (11), *i. e.*, the symmedian AX of ABC.[2] Similarly for the analogous lines E_1B_mY, F_1C_mZ.

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[1] *College Geometry*, p. 247.

[2] *College Geometry*, p. 236.

The diameter OX of (O) is the bisector of the angle $BXC \equiv ZXY$, hence the line XA_mD_1 is the isogonal conjugate of the line XA in the tangential triangle XYZ of ABC. Since the symmedians XA, BY, CZ of ABC meet in the Lemoine point of ABC, (which point is the Gergonne point of the triangle XYZ), therefore their isogonal conjugates XD_1, YE_1, ZF_1, are also concurrent in a point K'. Now the poles of these lines with respect to (O) are the centers L, M, N, of the circles of similitude (L), (M), (N), hence K' is the pole of the line $d \equiv LMN$ with respect to (O) and lies on the line OH. Thus :*The isogonal conjugate of the symmedian point of a triangle with respect to the tangential triangle lies on the Euler line of the basic triangle and is the pole of the de Longchamps axis of this triangle with respect to its circumcircle.*

16. The de Longchamps circle (H_1) of ABC is the conjugate circle of the triangle $A_1B_1C_1$ (3), hence (H_1) is real or imaginary according to whether $A_1B_1C_1$ is obtuse angled or acute angled, and since the triangles $A_1B_1C_1$ and ABC are similar, the triangle ABC may be substituted for $A_1B_1C_1$ in this condition. The preceding considerations, however, remain valid, as far as the circle (H_1) is involved, in either case. This is due to the fact that no use has been made of any point on the circle (H_1), while the properties dealt with were those of poles and polars, and of the radical axes. These properties depend upon the square of the radius of (H_1), a quantity which is real whatever the shape of ABC may be.[1]

Let us now assume that ABC is acute angled. In this case the circles (O), (O_1), (H_1) form a non-intersecting system of coaxal circles, and therefore the circles (L), (M), (N) of the conjugate coaxal system have two real points S, S' in common on their common radical axis OH_1, *i. e.*, on the Euler line of the triangle ABC. The points S, S' are the limiting points of the first coaxal system and therefore conjugate with respect to every circle of this system, and also symmetric with respect to the trace of the line $d \equiv LMN$ on OH_1. Consequently: *In the plane of an acute angled triangle there are two, and only two points each of which has the property that its distances from the vertices of the triangle are proportional to the sides of the triangle opposite the vertices considered. The two points lie on the Euler line of the triangle, are conjugate with respect to its circumcircle, and are symmetric with respect to the trace of the de Longchamps axis on the Euler line.*[2]

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[1] M. Chasles, *Traité de Géométrie Supérieure*, second edition, pp. 502 etc., Gauthier-Villars, Paris, 1880.

[2] *Cf.* R. Lachlan, *Modern Pure Geometry*, p. 192, ex. 5, Macmillan, 1893.

QUESTIONS AND DISCUSSIONS

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. MATHEMATICAL INDUCTION

By G. E. RAYNOR, Wesleyan University

In the MONTHLY for November 1920, Professor E. T. Bell has criticized the principle of mathematical induction from a logical point of view. His paper is preceded by some remarks by the editor and, in the April 1922 number, Mr. R. S. Hoar and the editor have discussed it further. However none of these gentlemen have attempted to give a specific reply to Professor Bell's closing sentence in which he asks in regard to mathematical induction " . . . . where is either a proof of it or its explicit statement as a postulate of logic to be found?" It is the purpose of this note to attempt to answer the above question and perhaps to reopen the discussion of mathematical induction.

Bertrand Russell in his *Introduction to Mathematical Philosophy*, published in 1919, in the chapter devoted to mathematical induction, states rather dogmatically that this principle must be taken as a definition. However, he completely ignores a paper published in 1913 by A. Padoa in the *Proceedings of the Fifth International Congress of Mathematicians*, Vol. II, in which Padoa shows that the principle may play other rôles in the deductive science of arithmetic. He reminds us that in any deductive science each proposition must have one of three rôles, it must be either a postulate or a definition or a theorem. He then outlines various systems in which the principle has played a part, with reference to works in which complete discussions may be found.

In Peano's *Arithmetices principia nova methodo exposita* published in 1899, mathematical induction plays the part of a postulate along with four others. In Russell's *The Principles of Mathematics* and later in Russell and Whitehead's *Principia Mathematica*, and also in the more recent work by Russell referred to above, it enters explicitly in the definition of inductive numbers. In 1900, Padoa succeeded in reducing Peano's five postulates to four, one of which, although not the principle of mathematical induction, is very nearly it. In

1908, M. Pieri gave four postulates from which the principle of mathematical induction as well as the rest of arithmetic may be deduced. Finally, in the paper which we are partially outlining, Padoa gives another set of postulates and definitions for arithmetic from which mathematical induction follows as a theorem.

It thus appears from Padoa's work and from that of Peano and Russell that the foundations of arithmetic may be so laid that the principle of mathematical induction may appear in the deductive science which arises from them either as a postulate, as a definition, or as a theorem.

It may also be mentioned that in his *Theory of Functions of a Real Variable* 2nd edition, vol. I, E. W. Hobson has given a proof of mathematical induction which is open to objection. His proof is based on a so-called definition of a "simply infinite ascending aggregate" but he does not show that the proof of the existence of such an aggregate is itself independent of the principle he is trying to prove.

II. A New Curve Connected with Two Classic Problems

By G. M. Juredini, Syracuse University

The curve is the locus of a point on a defined triangle which revolves about a circle in a particular manner. In Fig. 1, it is the path traced by the point P, which is the foot of the perpendicular let fall onto the hypotenuse of a right-angle triangle, RQX, the triangle revolving about a circle in such a manner that one of its sides, RQ, is always equal to and an extension of a radius of the circle, and the other side, QX, always reaches a fixed diameter of the circle, OX, as initial line, produced.

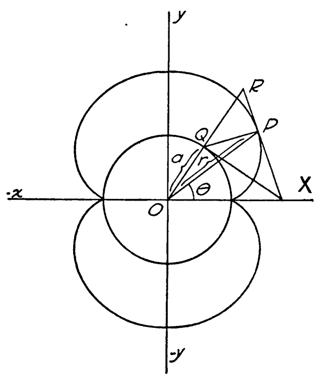

Fig. 1

The equation for the curve may be deduced as follows: In Fig. 1, we have angle $ROX=$ angle $ORX=\phi$. Let angle $POX=\theta$, radius of circle, or $RQ=a$, and line $PO=r$. Then, angle $RXO=180°-2\phi$, $OX=a\cdot\sec\phi$, $RP=a\cdot\cos\phi$, $PX=a\,(\sec\phi-\cos\phi)$, so that, in triangle PXO, we have $\sin\,(180°-2\phi)/r=\sin\,\theta/a(\sec\phi-\cos\phi)$, and from this it follows readily that

$$\sin\theta=2(a/r)\,(\sin^3\phi)\,. \tag{1}$$

Also, in triangle PRO, we have $r^2=4a^2+a^2\cos^2\phi-4a^2\cos^2\phi$, $\cos^2\phi=(4a^2-r^2)/3a^2$,

$$\sin^2\phi=1-(4a^2-r^2)/3a^2=(r^2-a^2)/3a^2. \tag{2}$$

From (1) and (2), we get

$$\sin\theta = 2a/r\left(\frac{r^2-a^2}{3a^2}\right)^{3/2}, \qquad \text{or} \qquad \sqrt[3]{y^2} = \frac{\sqrt[3]{4a^2}}{3a^2}(x^2+y^2-a^2) \tag{3}$$

which is the equation of the curve.[1] The graph takes the form indicated in Fig. 1.

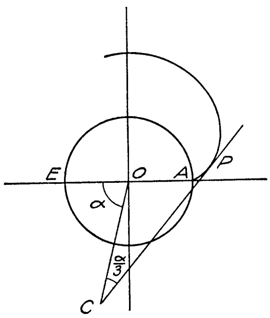

FIG. 2a

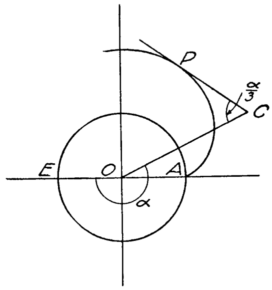

FIG. 2b

Application to Trisection of an Angle: In Fig. 2, let A be the starting point of the curve, and let EOC be the angle required to be trisected. Place OE on the side of the initial line that is opposite A, and measure off OC equal to diameter of circle. From C draw tangent to curve at P. Then PCO is one-third of EOC.

Application to Duplication, or Dimidiation of a Cube: In Fig. 1, let $QP=b$, and let P be at a distance $y=a$ from the line OX. From the right triangle QPR we have $\sin\varphi=(b/a)$, and when this value is inserted in (1) there results, after easy simplification,

$$2b^3=a^3. \tag{4}$$

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[1] The curve of this note is a two-cusped epicycloid as may be verified by eliminating the parameter φ from the equations:

$$2x=a(3\cos\varphi-\cos 3\varphi), \qquad 2y=a(3\sin\varphi-\sin 3\varphi).$$

For such a curve, $(dy/dx)=\tan 2\varphi$, and the given construction is a consequence.

Thus the author has added a new curve to the galaxy of trisectrices already known. Concerning such curves Henri Brocard (*Notes de Bibliographie des Courbes Géométriques*, 1897, p. 290) says

"Trisectrices are infinite in number. One may mention: the quadratrix of Dinostratos, the spiral of Archimedes, the limaçon $r=1+2\cos\theta$, the conchoid of Nicomedes, the equilateral hyperbola, Maclaurin's trisectrix, and a host of others listed by Aubry, *Journal de Mathématiques Spéciales*, 1896, pp. 76–84, 106–112."

N. ANNING

III. Note on a Linear Diophantine Equation

By Elizabeth B. Cowley, Vassar College

1. One of the arithmetical puzzles that date back to mediaeval times is the measuring problem of the three vases, where it is required to divide into two equal parts the contents of an 8-ounce vase if the only empty vases hold 5 ounces and 3 ounces respectively. Various modifications of this specific form of the problem are to be found in collections of mathematical recreations. This problem, which was recently included in a set of psychological test questions, is also interesting because of its connection with certain linear Diophantine equations.

In the fifth edition[1] of Bachet's book the editor uses A, B, and C instead of particular numbers and notes that the various methods of solution are of two types. By one method the B-vase is filled directly from the A-vase and then the C-vase is filled and refilled from the B-vase (each time the C-vase is filled its contents are emptied into the A-vase), until the contents of the B-vase are $A/2$ or are less than C. In the latter case, the contents of the B-vase are emptied into the C-vase and then the B-vase is filled again from the A-vase. If after m fillings of the B-vase its contents are $A/2$, we have the equaton $mB-nC=A/2$, a condition which can always be satisfied, provided that B and C are prime to each other, or that $A/2$ contains all factors common to B and C. That condition is also sufficient if $A \nless (B+C)$. But if $A<(B+C)$, there may not always be enough in the A-vase to refill the B-vase. The editor cites two examples, 20,13,9 and 16,12,7 and says that the first is possible and the second not possible but that these facts are not known *a priori*.

By the second method the C-vase is filled and refilled from the A-vase and emptied into the B-vase until the B-vase is full. Then it is emptied into the A-vase and the remaining contents of the C-vase are turned into the B-vase. If, after m' fillings of the B-vase, the contents of the A vase are $A/2$, we have $A-n'C+m'B=A/2$ or $n'C-m'B=A/2$. The conditions for this method (the C method) are the same as those for the other method (the B-method).

I shall find the conditions that determine *a priori* whether a solution is possible if $A<(B+C)$ and I shall apply to this problem the graphical method of solution employed by C. A. Laisant[2] for certain types of equations.

These conditions are arranged in two tables, called the B-table and the C-table. In order to avoid trivial cases, the following assumptions are made:

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[1] *Problèmes Plaisantes & Délectables qui se font par les nombres* par Claude-Gaspar Bachet sieur de Mézeriac, cinquième édition, revue, simplifiée et augmenteé par A. Labosne, Professeur de Mathématiques. Paris, 1884.

[2] *Association franç. av. sc.*, 1887, 218–235.

I. $A > B > C$ (where A, B, C are positive integers),
II. A is an even number,
III. $A/2 \neq nC$ (where n is an integer),
IV. B and C are prime to each other. (If B and C contain common factors which are in $A/2$, the problem is practically the repetition of a simpler problem; example, 16,10,6 and 8,5,3). Hence, $B = sC + R$, where $R < C$ and R and C are prime to each other.

2. Method of obtaining the B-table. The first filling of the B-vase is always possible, since $A > B$. Then the contents of the three vases are $A-B, B, 0$. Next they become $A-B+sC, 0, R$, or $A-R, 0, R$.

The second filling of the B-vase is impossible if $A-R<B$. If $A-R=B$, the B-vase can be filled but the A-vase is emptied in the process. In either case the combined contents of the other two vases is A. If we continue, we are really using the C-method. Hence we say that the B-method fails in the second attempt at filling the B-vase if $A-R \leqq B$. But if $A-R>B$, the contents of the three vases are $A-R-B$, B, R. The number of fillings of the C-vase to be had from the combined contents of the B and C vases is s if $R<C/2$ and it is $(s+1)$ if $R>C/2$. (It must be noted that $R \neq C/2$, since R and C are prime to each other). If $R<C/2$, the contents become $A-2R$, 0, $2R$. Then the third filling is possible (without emptying the A-vase) if and only if $A-2R>B$. If $R>C/2$, the contents become $A-(2R-C), 0, 2R-C$; and the B-vase can always be filled without emptying the A-vase (since $A-B>R$ and $C>R$, then $A-B+C>2R$ or $A-2R+C>B$). Hence, the A-vase is emptied in the third filling of the B-vase when and only when $A-B \leqq 2R$ with $R<C/2$. This line of reasoning may be continued.

3. Method of obtaining the C-table. Since $B>sC$, the C-method will fail in the first attempt at filling the B-vase if $A \leq (s+1)C$. If $A>(s+1)C$, let $A=(s+1)C+r$, where r must be less than R. In this case the contents are r, B, $C-R$ and then $B+r$, 0, $C-R$. In order to obtain the second filling of the B-vase, we shall need s fillings of the C-vase if $R<C/2$. We can always get these fillings from the A-vase; then the contents are $R+r$, B, $C-2R$ and then $B+R+r$. 0, $C-2R$. But if $R>C/2$, we shall need $(s+1)$-fillings of the C-vase; and they can be obtained if and only if $B+r>(s+1)C$, *i.e.*, if $R+r>C$ Hence, the C-method fails in the second attempt to fill the B-vase if $R>C/2$ and $(R+r) \leq C$. This line of reasoning can be continued.

4. B-table. Conditions for failure at each filling of the B-vase:

I.	The method never fails.	
II.	$A-R \leqq B$.	
III.	$A-2R \leqq B$ with $R<C/2$.	
IV.	$A-3R \leqq B$	with $R<C/3$,
	$A-(3R-C) \leqq B$	with $C/3<R<2C/3$.
V.	$A-4R \leqq B$	with $R<C/4$,
	$A-(4R-C) \leqq B$	with $C/4<R<C/2$,
	$A-(4R-2C) \leqq B$	with $C/2<R<3C/4$.

5. C-table. Conditions for failure at each filling of B-vase:

I.	$A \leqq (s+1)C$.	
II.	$R+r \leqq C$	with $R>C/2$.
III.	$2R+r \leqq C$	with $C/3<R<2C/3$,
	$2R+r \leqq 2C$	with $2C/3<R$.
IV.	$3R+r \leqq C$	with $C/4<R<2C/4$,
	$3R+r \leqq 2C$	with $2C/4<R<3C/4$,
	$3R+r \leqq 3C$	with $3C/4<R$.

V. $4R+r \leqq C$ with $C/5 < R < 2C/5$,
$4R+r \leqq 2C$ with $2C/5 < R < 3C/5$,
$4R+r \leqq 3C$ with $3C/5 < R < 4C/5$,
$4R+r \leqq 4C$ with $4C/5 < R$.

6. Graphical methods of solution. The equations for the B-method and the C-method are respectively $mB - nC = A/2$ and $n'C - m'B = A/2$. If these are written

$$Cx' - Bz' = -A/2 \tag{1}$$

and

$$Cx - Bz = A/2, \tag{2}$$

they are examples of the types of equation $rx - pz = \pm a$ (r and p prime to each other and a and r less than p) which Laisant solved by a lattice of points. For equation (2), construct the points whose abscissas are $1, 2, 3, \ldots, B$ and whose ordinates are the corresponding residues, less than p, modulo p, of C, $2C$, $3C$, $\ldots\, BC$. These points are contained in a square of side B and resting on the x- and y- axes. In the square, draw the (C) line-segments of slope C, through the points. To solve the equation for known values of A, B, C, locate the point whose ordinate is $A/2$. Its abscissa equals x and its z equals the number of line segments to the left of the segment on which it lies. Laisant also has a method for equation (1); but I find that a slightly different method is preferable here. Since $x+x'=B$ and $z+z'=C$, then x' equals the distance of the point from the right side of the square and z' equals $1+$ the number of line-segments to the right of the segment on which the point lies.

The points with which we are especially concerned here lie within a horizontal band bounded by the lines $y=B/2$ and $y=(B+C)/2$. If a point of this band lies on or below the line $y=(s+1)C/2$, it fails by CI. Points on or below the line $y=(B+R)/2$ fail by BII. Hence points on or below both these lines fail by both the B and the C methods. Any point on the left-most line segment has its $A/2=kC$; while a point on the right-most line segment has its $A/2=B-lC$. Other points in the band must be further tested.

EXAMPLE: $B=61$, $C=24$. Points $A/2=31$, 32, 33, 34, 35, 36, fail by both methods. $A/2=37$ lies on the right-most line. Since $R>C/2$, points 38,39, 40, 41, 42 cannot fail by $BIII$. Point 39 succeeds by the B-method, since its $z'=3$. It is easy to find that point 41 succeeds and points 38, 40, and 42 fail. Trying the C-method, since $R>C/2$, values of $A/2$ for which $r \leqq 11$ fail by $C\,II$. This excludes all points except 42. This point succeeds by the C-method. In the cases cited by Labosne, 20, 13, 9 has $z=2$ and avoids CI and CII, and hence succeeds by the C-method, but fails by $BIII$; and 16, 12, 7 has $z=4$ and fails by CII and also fails by BII. Some other sets for which both methods fail are: 24, 19, 8; 34, 31, 10; 42, 36, 11; 44, 37, 10; 44, 35, 24.

The devices used in the analysis of time series are empirical to a high degree, and there is considerable difference of opinion on certain points such as the measurement of seasonal variation by the median-link-relative method. While it is to be hoped that improvements will be effected in these devices, those presented in the book are important tools in the quantitative description of time series.

On p. 297, line 18 from bottom, p_i/q_0 should read p_iq_0.

H. L. RIETZ.

Gli Elementi d'Euclide e la Critica Antica e Moderna. Edited by Federigo Enriques col concorso di diversi collaboratori. Libri I–IV. Rome, A. Stock, 1925. 325 pages. Price 25 lire.

This is the first volume of a series "Per la Storia e la Filosofia delle Matematiche," published under the auspices of the Istituto Nazionale per la Storia delle Scienze Fisiche e Matematiche, and edited by Professor Enriques of the University of Bologna. It is the outcome of a suggestion made by certain leaders in the training of teachers of mathematics and it proceeds upon the principle that it is only by a historic survey of the classical textbooks of geometry that a sound basis can be laid for a critical study of the subject. The result of the abandoning of Euclid as a text has, in the opinion of Professor Enriques, been a serious loss in the equipment of the teacher.

He begins his work with a brief survey of the history of ancient geometry, followed by a bibliography of some of the leading editions of the *Elements*. He then gives the text of Euclid, in Italian translation, based upon the Heiberg edition. In the translation and editorial work he was assisted by Dr. Maria Teresa Zapelloni. As in the case of the Heath edition, the definitions and propositions are followed by critical notes containing much historical material, the one-line definition of *point*, for example, having a full page of commentary, and similarly for *line*, *plane surface*, *angle* (3 pp.), and others. There is also an extended discussion of the postulates and axioms, and the commentary relating to the Pythagorean Theorem (I, 47) fills six pages.

It will therefore be seen that, with such an editor as Professor Enriques to assure the scholarship of the commentary, the work will be helpful to any student of the subject and to any reader of the most influential textbook on mathematics ever written. It is significant that the teachers of Italy, which now ranks among the half dozen leading nations in mathematical activity, should feel the need of such an edition of Euclid at a time when some of our American educators are proclaiming the uselessness and indeed the happy death of the science of geometry.

DAVID EUGENE SMITH.

Matematiche, Scienze Naturali e Medicina nell' Antichita Classica. By J. L. HEIBERG, Traduzione di Gino Castelnuovo. Rome, A. Stock, 1924. 188 pages. Price 12.60 lire.

This is the second volume of the series "Per la Storia e la Filosofia delle Matematiche," published under the editorial direction of Professor Enriques of Rome. It is a translation of Professor Heiberg's *Naturwissenschaften und Mathematik im Klassischen Altertum*, Leipzig, 1912. Besides the translation there are numerous footnotes, largely referring to articles or comments by Professor Enriques, or to notes in Dr. Singer's English edition. There is a brief bibliography,—too brief for those needing generous assistance.

Heiberg's work is so well known to historians of mathematics that no extended description is desirable. It is a general essay rather than a detailed history, but it has scientific value because of the fact that the author stands out as probably the best-versed scholar in the field of Greek mathematical texts of any man of modern times. The topics considered are as follows: (1)Ionic natural philosophy; (2) The Pythagoreans; (3) Medicine in the 5th century B.C.; Hippocrates; (4) Mathematics in the 5th century B.C.; (5) Plato and the Academy; (6) Aristotle; the Peripatetics; (7) The Alexandrian School; (8) The Epigones [successors; that is, later Greek writers]; (9) The Romans; (10) Greek scientific literature of the Byzantine period.

DAVID EUGENE SMITH.

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Proceedings of the National Academy of Sciences, volume 12, no. 4, April 1926: "On Laguerre's Series" by E. Hille, 261-268; "Groups containing a relatively small number of Sylow subgroups" by G. A. Miller, 269–273. Volume 12, no. 5, May, 1926: "On Laguerre's Series, Third Note" by E. Hille, 348–351; "On conformal geometry" by T. Y. Thomas, 352–358; "Concerning indecomposable continua and continua which contain no subsets that separate the plane" by R. L. Moore, 359–363.

Transactions of the American Mathematical Society, volume 28, no. 1, January, 1926: "Intersections and transformations of complexes and manifolds" by S. Lefschetz, 1–49; "Divergent double sequences and series" by G. M. Robison, 50–73; "On certain families of orbits with arbitrary masses in the problem of three bodies" by F. H. Murray, 74–118; "Existence theorems for a linear partial difference equation of the intermediate type" by C. R. Adams, 119–128; "An algebra of sequences of functions, with an application to the Bernoullian functions" by E. T. Bell, 129–148; "Bundles and pencils of nets on a surface" by E. P. Lane, 149–167; "On the momental constants of a summable function" by R. E. Langer, 168–182; "Fundamental systems of formal modular protomorphs of binary forms" by W. L. G. Williams, 183-197; "Non-synchronized relative invariant integrals" by K. P. Williams, 198–206.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in solution of such problems.

3206. Proposed by D. H. Lehmer, University of California.

Prove the following theorems and show how they may be used in finding the factors of R:

THEOREM 1. Let R be a non-square integer of the form $8n+k$ and let $2^p(2m+1)$ be any even denominator of a complete quotient occurring in the expansion of $\sqrt{R}$ in a continued fraction, then, if $k=1$, $p\geqq 3$; if $k=4$ or 0, $p\geqq 2$; if $k=5$, $p=2$; if $k=2$, 3, 6, or 7, $p=1$.

THEOREM 2. If R contains a square factor, k^2, then every multiple of k appearing as a denominator of a complete quotient must also contain a factor k^2.

3207. Proposed by C. N. Mills, Normal, Illinois.

Prove that $\frac{1}{4}m^2$ is the maximum area of a triangle which can be formed with the lines a, b, c, subject to the condition that $a^3+b^3+c^3=3m^3$.

3208. Proposed by V. M. Spunar, Chicago, Illinois.

Disprove the following: If a,n are any integers and $a^x\equiv 1 \pmod{n}$ for $x=n-1$ but not when x is an aliquot part of $n-1$, the integer n is a prime. (Lucas, *Theory of Numbers*, I, p. 441.)

3209. Proposed by M. Zametkin, Jamaica, N. Y.

Given $4x^3-21x+14=0$, show that, if $A=10°/7$, $x_1=\sin 41A+\sin 37A+\sin 25A+\sin 13A-\sin 19A-\sin 11A$. Determine the other two roots, x_2 and x_3, in like sums of sines.

3210. Proposed by Thurman Andrew, University of North Dakota.

Find the general solution of the probability of throwing any number with any number of dice. Dice are here taken to be polyhedrons, with any number, k, of faces numbered consecutively from 1 to k. (Assume all faces equally likely to turn up.)

3211. Proposed by J. A. Bullard, U. S. Naval Academy.

Find by integration the area of the ellipse $ax^2+2hxy+by^2+2gx+2fy+c=0$.

SOLUTIONS

2827 [1920, 186]. Proposed by B. F. Finkel, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle having a given base and a given altitude.

I. SOLUTION BY OTTO DUNKEL, Washington University.

Let the rectangular axes be so taken that the extremities of the base are $(-c,0)$, $(c,0)$ and the ordinate of the variable vertex is h, then the coordinates (X,Y) of the center C of the circle satisfy the equation

$$X^2 = c^2 + \frac{hY^2}{2Y-h}\,. \tag{1}$$

Let the circle C be tangent to the base, or the base extended if the circle is of the escribed kind, at M and to the envelope at $P(x,y)$, then P is the reflection of M on the tangent to (1) at C (see **2691**, *1919*, 131; also *1921*, p. 183). Let the inclination of the tangent be τ, then the figure shows that

$$y = 2Y\cos^2\tau, \qquad x = X - y\tan\tau\,. \tag{2}$$

Hence

$$y = \frac{2Y\cot^2\tau}{1+\cot^2\tau}\,. \qquad x = X - \frac{2Y\cot\tau}{1+\cot^2\tau}\,. \tag{3}$$

From (1) we can obtain $\cot\tau = dX/dY$ in terms of Y alone and also X in terms of Y. When these results are inserted in (3) there results the equations of the envelope in terms of the parameter Y. The elimination of this parameter will give the equation in x and y. These details are left to any reader to whom the results would be a source of pleasure or inspiration.

3118 [1925, 95, 522]. Proposed by Harry Langman, New York City.

If $n>2$ and e is a primitive root of $e^n=1$, show that the determinant $|a_{ij}|$, of order $n-1$ and having the element a_{ij} equal to e^{ij}, has the value

$$(-1)^{(n-1)(n-2)/4}n^{(n-2)/2}.$$

II. Solution[1] by Norman Anning, University of Michigan.

Call the determinant D and observe that, by bordering, D can be written

$$\begin{vmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 1 & e & e^2 & . & . & . & e^{(n-1)} \\ 1 & e^2 & e^4 & . & . & . & e^{2(n-1)} \\ . & . & . & . & . & . & . \\ 1 & e^{(n-1)} & . & . & . & . & e^{(n-1)(n-1)} \end{vmatrix}.$$

We can build up the square of D in such a way that $D^2=|b_{ij}|$, $i,j=1,2,3,\cdots, n$, where the element $b_{ij}=1$ whenever either i or j is equal to 1 and, in all other cases,

$$b_{ij}=1+e^k+e^{2k}+e^{3k}+\cdots+e^{nk-k}, \qquad k=i+j-2.$$

Now, by well-known properties of primitive roots, this latter expression is equal to n when k is a multiple of n and is equal to 0 for all other values of k. Consequently

$$D^2=\begin{vmatrix} 1 & 1 & 1 & . & . & . & 1 & 1 \\ 1 & 0 & 0 & . & . & . & 0 & n \\ 1 & 0 & 0 & . & . & . & n & 0 \\ . & . & . & . & . & . & . & . \\ 1 & 0 & n & . & . & . & . & . \\ 1 & n & 0 & . & . & . & . & 0 \end{vmatrix} = n^{n-2}\begin{vmatrix} n & 1 & 1 & . & . & . & 1 & 1 \\ 1 & 0 & 0 & . & . & . & 0 & 1 \\ 1 & 0 & 0 & . & . & . & 1 & 0 \\ . & . & . & . & . & . & . & . \\ 1 & 0 & 1 & . & . & . & . & . \\ 1 & 1 & 0 & . & . & . & . & 0 \end{vmatrix},$$

the second determinant being obtained from the first by multiplying the first column by n and then dividing all the rows except the first by n.

The coefficient of $n^{(n-2)}$ is a determinant of the nth order which we may call u_n. Then, if we subtract the second row from the first and expand in terms of the elements of the last column,

$$u_n=(-1)^n u_{n-1}.$$

Similarly,

$$u_{n-1}=(-1)^{n-1}u_{n-2}, \text{ etc.}$$

Finally,

$$u_4=(-1)^4u_3=(-1)^4\begin{vmatrix} 3 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}=-1.$$

It follows that $u_n=(-1)^{(n-1)(n-2)/2}$. Then $D^2=n^{(n-2)}(-1)^{(n-1)(n-2)/2}$, and the theorem follows, except for an ambiguity in sign.

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[1] A solution by J. J. Nassau appeared in the Monthly for December, 1925.

Also solved by J. F. REILLY who pointed out that the statement of the problem is faulty since, in certain cases, the given determinant may have either of two values which are equal in magnitude but opposite in sign depending upon the choice of the primitive root e.

3138 [1925, 261]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The vertex of a triangle, whose base is fixed, moves along a straight line coplanar with the line of the base. Find (1) the locus of the orthocenter of the triangle; (2) the envelope of the line joining the feet of the two altitudes dropped from the two fixed vertices of the triangle. See solution of **3152** below.

3152 [1925, 481]. Proposed by Otto Dunkel, Washington University.

From two fixed points, A and B, of a given conic two chords, AC and BD, are drawn intersecting on the fixed chord, IJ, of the same conic and determining another chord, CD. Determine the envelope of the chord CD and a method for locating points on the envelope without the use of equations.

SOLUTION BY OTTO DUNKEL, Washington University.

This solution includes a generalization of **3138 [1925, 261]**. In that problem the conic is a circle with AB as a diameter, while the variable vertex Q of the triangle QAB moves on a fixed line $l \equiv IJ$. The feet C and D of the altitudes BC and AD must lie on this fixed circle. Here the circle and its diameter are replaced by any given conic and any one of its chords. The line l in this discussion may or may not cut the conic in real points.

Consider two points Q_1 and Q on l and the two sets of lines Q_1C_1A, Q_1D_1B and QCA, QDB which determine, respectively, the chords C_1D_1 and CD of the given conic α. Denote by U the intersection of AQ_1 and BQ; by V the intersection of AQ and BQ_1; by P' the intersection of C_1D_1 and CD. Then from the inscribed hexagon $D_1C_1ACDBD_1$, it follows that U, P', V lie on a straight line cutting AB in E. If l cuts AB in S, then from the complete quadrilateral determined by U, Q_1, V, Q it is seen that S and E are harmonically separated by A and B. Suppose now that Q_1 is fixed, then C_1D_1 is fixed. Let CD cut AB in R. Since Q and R are conjugate points with respect to α, they describe projective ranges on l and AB, respectively. The range Q on l is projective with the range U on AC_1, since UQ passes through B. Also the range U is projective with the range P' on C_1D_1, since UP' passes through E. Hence CD cuts the two fixed lines C_1D_1 and AB in corresponding pairs of points P' and R of two projective ranges; it therefore envelopes a conic β tangent to AB and C_1D_1. When Q is at S, R is at E. Hence E, the limit point of the intersection of two tangents, is the point of tangency of AB.

Now let CD coincide with C_1D_1; then P' becomes the point of tangency P_1 of C_1D_1. Hence P_1 is the intersection of C_1D_1 with Q_1E, and thus we have a simple construction for the point of tangency P of any chord CD with β.

If α and β intersect in any point P, this point must lie on a chord CD and the corresponding line QE; but this is impossible unless P and Q coincide. Hence α and β can intersect only where l cuts α. If l cuts α in two distinct points, then α and β have double contact at these points. This is useful in a discussion of the nature of β.

Having constructed two chords C_1D_1, CD and their points of tangency P_1, P, and the point of tangency E of AB, we may abandon the rest of the figure and construct points of β as follows. Let C_1D_1 and CD cut AB in X and Y, respectively, and draw the line PP_1. Draw any line YX_1 cutting XP_1 in X_1 and PP_1 in Z; draw XZ cutting YP in Y_1; then X_1Y_1 is a tangent to β and the point E_1 in which it is cut by EZ is the point of tangency. This follows easily from Brianchon's theorem.

Let the tangents to α at A and B be TAT_1 and TBT_2, where T is their intersection and T_1 and T_2 lie on l; let T_1B and T_2A meet α in A_1 and B_1. Then AA_1A_2 and BB_1B_2 are tangents to β cutting l in A_2 and B_2; their points of tangency P_a and P_b are their intersections, respectively, with T_1E and T_2E. Then it follows that $AP_aA_1A_2$ and $BP_bB_1B_2$ are each harmonic sets. Also, if T_1B and T_2A meet in M, and AA_1 and BB_1 meet in N, then N, M, T and E lie on a straight line, as shown above, and this line is the polar of S with respect to α. This line contains also a fifth point L, the pole of l. It then follows

that A_1B_1 passes through S and hence P_aP_b also passes through S. Hence S and TE are also pole and polar with respect to β.

Returning now to the figure determined by the variable vertex Q, let BC and AD meet in K, then the polar of R, TQ, passes through K, and also the polar of Q passes through L, K, and R. The figure shows that the pencils $A(D)$, $B(C)$, $L(R)$, $T(Q)$ are projective, and hence the locus of K is a conic γ passing through A, B, L and T. If l cuts α, then γ passes through the points of intersection. It follows also that AA_1 and BB_1 are tangents to γ at A and B; also that SL and ST are tangents at L and T. Hence S and TL are pole and polar for γ.

Problem **3138** was also solved by THEODORE BENNETT, H. W. BAILEY, MICHAEL GOLDBERG, W. J. PATTERSON, AUGUST SÖRENSON, MABEL M. YOUNG, and the PROPOSER (two solutions).

3142 [1925, 315]. Proposed by Harry Langman, New York City.

Cut a rectangle 1×2 into three pieces which will fit into a Maltese cross.

SOLUTION BY ABIGAIL E. JOHNSON, Morristown, N. J.

Let XY be the base of length 2 of rectangle $XYZW$. Using X as the origin and XY and XW as axes of coordinates, divide the base into ten and the altitude into five equal units, and connect the following points in order: (0, 0); (1, 3); (4, 2); (5, 5); (8, 4); (7, 1); (10, 0). The resulting three pieces can be formed into a Maltese cross.

3143 [1925, 315]. Proposed by Edward Condon, University of California.

Prove that

$$\frac{\sin (n-1)\alpha}{\sin n\alpha} = \frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\cdots,$$

in which the continued fraction terminates when $2\cos\alpha$ has appeared $n-1$ times. Prove also the corresponding formula for the hyperbolic sines and cosines.

SOLUTION BY R. H. SCIOBERETI, Berkeley, California.

The proof of this relation may be based on a classic recursion formula of trigonometry, namely:

$$\sin(n+1)\alpha = 2\cos\alpha\cdot\sin n\alpha - \sin(n-1)\alpha,\quad n=2, 3\cdots, \tag{1}$$

from which we derive by division by $\sin n\alpha$,

$$\frac{\sin (n+1)\alpha}{\sin n\alpha} = 2\cos\alpha - \cfrac{1}{\cfrac{\sin n\alpha}{\sin (n-1)\alpha}}.$$

Applying equation (1) to the ratio $\sin n\alpha/\sin(n-1)\alpha$ and repeating this process n times in succession, we shall have converted the quotient $\sin(n-1)\alpha/\sin n\alpha$ into the proposed terminating continued fraction. The given formula can now be proved by induction.

The recursion formula

$$\cos (n+1)\alpha = 2\cos\alpha\cdot\cos n\alpha - \cos(n-1)\alpha \tag{2}$$

shows that the ratio $\cos(n-1)\alpha/\cos n\alpha$ can be converted into a terminating continued fraction of the same form; the last partial quotient is, however, equal to $\cos\alpha$. Hence

$$\frac{\cos (n-1)\alpha}{\cos n\alpha} = \frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\cdots\frac{1}{-2\cos\alpha}\,\frac{1}{-\cos\alpha},$$

where $\cos\alpha$ appears n times.

As for the hyperbolic functions $\cosh\alpha$ and $\sinh\alpha$, it will be sufficient to observe that the two equations (1) and (2) will be exactly of the same form, since if (1) is multiplied by $i=\sqrt{-1}$ and if α is changed into $(i\alpha)$ in both, then the desired relations follow at once:

$$\sinh(n+1)\alpha = 2\cosh\alpha\sinh n\alpha - \sinh(n-1)\alpha,$$
$$\cosh(n+1)\alpha = 2\cosh\alpha\cosh n\alpha - \cosh(n-1)\alpha.$$

Also solved by MICHAEL GOLDBERG and N. PETROFF.

3145 [1925, 433]. Proposed by W. D. Cairns, Oberlin College.

The center of gravity of any zone of a certain surface of revolution lies midway between the bases of the zone. What is the surface?

SOLUTION BY T. E. STERN, Princeton University.

Consider the surface as generated by the rotation of a curve about the x axis. Take as origin the projection on the x axis of any point on the curve. Consider the curve in the interval Ox. By the given conditions

$$\int_0^x xy\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx = \frac{x}{2}\int_0^x y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx \cdot \tag{1}$$

Differentiating with respect to x and reducing,

$$\frac{y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}}{\int_0^x y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx} - \frac{1}{x} = 0, \tag{2}$$

where the numerators are the x-derivatives of the denominators. Integrating,

$$\int_0^x y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx = Kx \tag{3}$$

where k is a constant. Differentiating, and reducing, we get

$$\frac{dy}{dx} = \left[\frac{k^2}{y^2}-1\right]^{1/2}. \tag{4}$$

If y is a constant then $y^2=k^2$. If y is not a constant, then by solving (1), $(x-c)^2+y^2=k^2$, c being a constant. Thus the surface must be either cylindrical or spherical.

Also solved by R. H. SCIOBERETI.

NOTES AND NEWS

Readers are invited to contributed to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

In recognition of the greetings sent by the Mathematical Association to MITTAG-LEFFLER on the occasion of his eightieth birthday anniversary last winter, he has sent to the Association his picture with the inscription: *Propter honorem, quem mihi octoginta annos nato ante diem XVI. Kal. Apr. A.D. MCMXXVI benigne tribuisti, cuiusque memoria animo meo numquam excidere poterit, maximas tibi gratias ago.*

Assistant Professor F. W. OWENS, of Cornell University, has been appointed head of the mathematics department at Pennsylvania State College.

Dr. B. H. CAMP, professor of mathematics at Wesleyan University, was a non-resident lecturer in mathematical statistics in the University of Michigan Summer Session. He gave an elementary and an advanced course in this subject. Dr. Camp recently completed a year of study under Karl Pearson.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7-8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Annapolis, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE MAY MEETING OF THE ILLINOIS SECTION

The seventh annual meeting of the Illinois Section of the Mathematical Association of America was held at James Millikin University, Decatur, on May 7–8, 1926. Chairman E. B. Lytle presided. A short address of welcome was given by President E. Penney of James Millikin University.

The attendance was thirty-nine including the following twenty-eight members of the Association:

Beulah Armstrong, R. D. Carmichael, C. E. Comstock, D. R. Curtiss, Arnold Emch, A. E. Gault, Mary G. Haseman, Mabel M. Herren, Mildred Hunt, E. C. Kiefer, E. B. Lytle, W. D. MacMillan, Martha P. McGavock, R. M. Mathews, Bessie I. Miller, C. N. Mills, E. J. Moulton, Rev. Paul Muehlman, Mary W. Newson, H. P. Pettit, Theresa M. Renner, G. T. Sellew, H. A. Simmons, H. E. Slaught, C. J. Stowell, Mildred E. Taylor, M. E. Wescott, F. E. Wood.

The following officers were elected for the next year: Chairman, E. C. KIEFER, James Millikin University; Vice-chairman, H. P. PETTIT, Illinois Wesleyan University; Secretary-Treasurer, BESSIE I. MILLER, Rockford College. The executive committee was advised to accept the invitation of Professor Pettit to have the next meeting of the section at Illinois Wesleyan University. The secretary was instructed to extend to Professor Kiefer and to James Millikin University a unanimous vote of thanks for the hearty welcome given the Section and for the very pleasant arrangements which had been made for the holding of the meeting.

The following program was presented:

1. "Diophantine problems in weighing" by Professor H. A. SIMMONS, Northwestern University.
2. "Mathematics of art and literature" (Illustrated) by Professor ARNOLD EMCH, University of Illinois.
3. "College geometry" by Professor R. M. MATHEWS, University of Illinois.
4. "Discussion: Undergraduate courses in geometry" led by Professor BESSIE I. MILLER, Rockford College.
5. "The Einstein theory" by Professor R. D. CARMICHAEL, University of Illinois.
6. "The second Carus monograph" by Professor D. R. CURTISS, Northwestern University.
7. "What constitutes a good teacher of mathematics?" by Professor MARY W. NEWSON, Eureka College.

8. "Freshman mathematics for students with one year of high school algebra" by Dr. JOSEPHINE B. GLASGOW and Dr. MARY G. HASEMAN, University of Illinois.

9. "A survey course in mathematics for freshmen" by Professor H. E. SLAUGHT, University of Chicago.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. This paper exhibited a method of attacking, by means of a lever and a system of n weights, a certain Diophantine problem. Under a special definition of the term "consecutive," it was shown that there exists a maximum consecutive weighing capacity for n weights. The n weights which have this maximum consecutive capacity have the n values which constitute a solution of the Diophantine equations in question.

2. Professor Emch's address, which was illustrated by a large number of lantern slide projections, dealt with instances of the influence of mathematics upon literature and the arts at various periods of human history. Referring to the mysticism which has in all ages been woven around certain numbers like 3 and 7, he traced this, by means of numerous examples, back to the Pythagoreans who found in harmony and in number the music of the spheres. The cosmology of Dante's "Divine Comedy" was then described and its system of revolving concentric heavenly spheres was compared with the universe as conceived by Aristotle. "From the modern standpoint of relativity, Dante's conception of the world is a sort of Riemannian manifold whose metric is impressed upon it by a central source." Tintoretto's colossal "Paradise" in the Doge's palace in Venice corresponds exactly to Dante's cosmology with its peculiar non-Euclidean construction.

The wide field of applications of perspective to painting, sculpture, panoramas, etc., was traversed and reference was made to Claude Bragdon's recent use in the decorative arts of four-dimensional geometry and of the configurations of magic squares and of the chess-board (*Projective Ornament*, Manas Press, Rochester, N. Y., 1915). Professor Emch himself (*Scientific Monthly*, 1918, pp. 273–281) has made similar application of the principle of involution, of the bicircular points at infinity and of continuous crinkly curves. He closed with the following quotation from Bragdon:

"The outworn beauty is the beauty of mere appearances. The new beauty, which corresponds to the new knowledge, is the beauty of principles: not the world aspect, but the world order. The world order is most perfectly embodied in mathematics. This fact is recognised in a practical way by the scientist, who increasingly invokes the aid of mathematics. It should be recognized by the artist, and *he* should invoke the aid of mathematics too."

3. With the Greeks geometry *was* mathematics; today the dominance of analysis is indicated by the preponderance of theses and papers in analysis and by the remarks of those who insist that geometry is "a natural science like physics." Observing that this dominance extends to the undergraduate curriculum Dr. Mathews made a plea for a course in pure geometry open to sophomores. When the present day college curricula are examined we find that there is very little of new geometric fact and nothing of geometric principle or method taught even in the courses of analytic geometry and calculus. The result is that the undergraduate learns no more geometry systematically. Further work in pure geometry is desirable: (*a*) in fairness to the geometrically minded undergraduates that they may have elections suited to their abilities; (*b*) for the equipment of prospective high-school teachers; (*c*) and for the prospective specialist that he may gain greater *vividness* of subject matter, *enlargement of view* by acquistion of new fundamental principles, and mathematical *power* by application of these. One possible course is an extension of Euclidean methods; this is good for prospective teachers if there be time for it. It is proposed however that the new course give some new fundamental principles, methods and insight and actually develop unsuspected geometric relations. These ends can be reached through a course in the synthetic geometry of projection and section which was briefly described.

4. Professor Miller gave a brief discussion of Dr. Mathews' paper, bringing out the aesthetic values of courses in geometry and also the mathematical values of such courses to the student of analysis.

5. Professor Carmichael was introduced by President Penney and gave an hour's non-technical lecture on the theory of relativity. He spoke of its origin and development, its nature, its influence upon science, and its philosophical implications. He gave also an analysis of the present state of knowledge concerning the verification and the validity of the theory.

6. Professor Curtiss outlined the second Carus monograph, *Analytic Functions of a Complex Variable*, emphasizing its purpose to set forth the principles of the subject with only necessary details and with no digressions. The central idea is that of Riemann, which deduces the properties of functions from definitions that contain as little as possible that is superfluous. The reader is enabled to pursue special branches of the subject further by the references to standard treatises which are given at the end of each chapter.

7. Mrs. Newson discussed informally a number of points connected with the teaching of mathematics and the choice of teachers. A criterion of the success of a given teacher or class of teachers must be based on results. A complete study of the above topic would require extensive statistical study which she hopes to make at some future time. In the meantime there are signs that dissatisfaction with our present teaching of mathematics, especially in

the high schools is leading to a feeling that any change is desirable. In her opinion, the tendency to give the teaching of mathematics to men rather than women will be disastrous to the teaching of preparatory mathematics, in view of two important facts, first that many of the best men are attracted into business or professional life and second that those men who do take up high school teaching are, for the most part, administrative officers or athletes whose first interest is in other lines.

8. Drs. Mary G. Haseman and Josephine B. Glasgow reported the results of certain experiments in content and method of presentation in a year's course for college freshmen who have had the minimum entrance requirements in mathematics. Their object was to prepare such students in ten semester hours of college work to continue with the calculus in their second year. They have found it expedient to omit a formal review of high school algebra. They stress the idea of a function and its graphical representation as the unifying element of the year's work. Since the class of students concerned is usually below the average in mathematical preparation and interest, a special effort has been made to awaken an enthusiasm or at least a liking for the subject.

9. The paper by Professor Slaught discussed a new type of course for freshmen, especially for those who do not intend to pursue mathematics further in college but who would be glad to have an intelligent acquaintance with the actual mathematical processes and the rôle which they have played in developing our present civilization,—provided this could be accomplished within a third or a half of an academic year.

It is believed that such a survey course may provide a desirable addition to the college curriculum for the following groups:

1. Students whose main interests are in other lines but who wish an intelligent comprehension of mathematics and its service to the world.

2. Students who are in doubt as to the choice of their major work and welcome such evidence as this kind of a course affords.

3. Students who intend to take the regular courses in trigonometry, analytics and calculus; they will find this preliminary survey an enlightening and stimulating introduction to the more intensive and prolonged study of these subjects.

4. Students who enter with only two units of high school mathematics; these may profitably take this survey course as their final work in mathematics.

BESSIE I. MILLER, *Secretary-Treasurer.*

THE THIRD MEETING OF THE INDIANA SECTION.

The third meeting of the Indiana Section of the Mathematical Association of America was held May 7 and 8, 1926, at Purdue University.

There were forty present at the meeting including the following thirty-one members of the Association: W. C. Arnold, Gladys L. Banes, C. F. Barr, J. C. Bennett, E. M. Berry, E. P. Blackburn, S. Bolks, G. E. Carscallen, H. T. Davis, S. C. Davisson, J. E. Dotterer, W. E. Edington, P. D. Edwards, E. D. Grant, G. H. Graves, H. E. H. Greenleaf, L. Hadley, C. T. Hazard, Cora B. Hennel, F. H. Hodge, E. N. Johnson, J. J. Knox, Florence Long, Juna M. Lutz, W. Marshall, T. E. Mason, J. C. Nixon, C. K. Robbins, R. B. Stone, H. N. Wright, W. A. Zehring.

On Friday evening the visiting members were entertained at a joint dinner given at the Purdue Union by the Purdue chapters of Sigma Xi and the American Association of University Professors in honor of Professor F. R. Moulton of the University of Chicago, national president of Sigma Xi.

At eight o'clock Professor Moulton gave an illustrated lecture on the subject: The Origin and Evolution of Worlds. The nature and origin of the solar system and the structure of the galaxy were discussed in Professor Moulton's entertaining style. The address concluded with a statement of the possibility of the existence of super galaxies of which the spiral nebulae form atomic particles.

At the session on Saturday morning presided over by Professor F. H. Hodge of Purdue, chairman, a constitution for the section was adopted and the following officers elected: Professor E. N. JOHNSON, Butler College, chairman; Professor J. E. DOTTERER, Manchester College, vice chairman; Professor H. T. DAVIS, Indiana University, secretary-treasurer.

Professor Johnson presented a report by the committee on requirements for high school teachers which was adopted. A discussion of this report, prepared by Professor D. A. Rothrock of Indiana University was read, in the absence of Professor Rothrock, by Professor W. E. Edington of Purdue. Professor S. C. Davisson of Indiana University discussed the question of the segregation of superior and inferior students. Mr. C. F. Barr of Purdue in continuing the discussion showed how the Iowa placement tests in mathematics could be used as a tool in making student classifications at the time of their entrance. A study made by correlating the grades obtained on the placement test and the grades recorded in the mathematics department at Purdue showed a correlation coefficient of $.741 \pm .015$ with the data grouped in five-unit intervals.

The meeting then adjourned to the Purdue Union where a luncheon was served to the members and their guests.

The afternoon program consisted of the following papers:

(1) "The construction and use of orthogonal and biorthogonal functions," by Professor H. R. MATHIAS, Indiana Central College. (Introduced by Professor Davisson).

(2) "A certain general type of contact transformation," by Professor C. K. ROBBINS, Purdue University.

(3) "The summation of series," by Mr. H. A. ZINSZER, Indiana University. (Introduced by Professor Davis).

(4) "The fractional calculus," by Professor H. T. DAVIS, Indiana University.

(5) "The true transition curve and some of its approximations," by Dr. E. M. BERRY, Purdue University.

(6) "Characteristic algebraic errors of college freshmen (second Paper)" by Mr. C. F. BARR, Purdue University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Mathias showed how a set of normalized orthogonal functions could be built up in a given interval from a set of n linearly independent functions and indicated the nature of the expansion problem associated with such sets.

2. Professor Robbins showed that $x'=f_1(x, y, p)$, $y'=f_2(x, y, p)$, $z'=f_3(x, y, p)$ is a contact transformation if the vanishing of $dy'-p'dx'$ is a consequence of the vanishing of $dy-p\,dx$. This condition leads to a set of partial differential equations which can be easily solved in special cases. The ordinary dilation is obtained by a proper specialization of the arbitrary functions involved.

3. Making use of fundamental properties of the operator $\theta=x\ d/dx$, Mr. Zinszer developed several methods for summing series. These methods were concerned with the establishing of a differential equation whose solution is the sum of the given series. Various ways of deriving the differential equation from the given series were discussed and examples given to illustrate the theory.

4. Professor Davis discussed the nature of problems which come naturally under the discipline of a calculus founded on fractional operations. Methods for solving various types of fractional equations were discussed. It was pointed out that many of the expansions obtained in the application of the Heaviside operational calculus to electrical circuit theory are special cases under the calculus of fractional operators.

5. Dr. Berry showed that for a railroad the true transition curve from straight track to circular track is such that the curvature is proportional to the distance traversed, measured from the end of the straight track. The coordinates were obtained as Fresnel's integrals and the curve was shown to be Cornu's spiral found in connection with the theory of the diffraction of light.

Near the origin a cubical parabola and a lemniscate are two approximations; for large distances from the origin a lituus is a good approximation.

6. Mr. Barr's paper was a continuation of a study presented at the last meeting of the Indiana section. The data were collected from a study of 1000 semester examination papers written under twenty instructors. The results indicated that probably too much time was being given to re-mastering material of high school grade and too little time to actual mastery of algebra of college grade.

The time and place of the next meeting were left to be decided by the executive committee.

H. T. DAVIS, *Secretary-Treasurer.*

ORIGINS OF FOURTH DIMENSION CONCEPTS

By FLORIAN CAJORI, University of California

1. From Aristotle to Henry More. Inquiries into the possibility of a fourth dimension of space reach as far back as Greek philosophy. Nevertheless, for 2000 years no one dared to proclaim the existence of such a space. Thus Aristotle in his *Heaven* says that a solid has magnitude "in three ways and beyond these there is no other magnitude because the three are all." This is the record of man's observation and every-day experience in our physical universe. In his *Metaphysics* [1066b32] he speaks of a body as "that which has dimension every way"; in his *Physics* [IV, 1] when considering motion, he regards "dimensions" as six, dividing each of the three into two opposites, "up and down, before and behind, right and left," these terms being taken relatively. More pretentious was the procedure of Ptolemy who was an astronomer, but dealt also with the philosophy of mathematics. He was the first to offer a "proof" of the unprovable "parallel-postulate" of Euclid. In the same way he "disproved" the possibility of more than three dimensions, because, as Simplicius tells us, "it is possible to take only three lines that are mutually perpendicular, two by which the plane is defined and a third measuring depth."[1] The book containing Ptolemy's proof is now lost. Perhaps the first to approach the fourth dimension from the side of physics, was the Frenchman, Nicole Oresme,[2] of the fourteenth century. In a manuscript treatise, he sought a graphic representation of the Aristotelian forms, such as heat, velocity, sweetness, by laying down a line as a basis designated *longitudo*, and taking one of the forms to be represented by lines (straight or circular) perpendicular to

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[1] *Simplicii in Aristotelis De Coelo Commentaria*, ed. Heiberg, Berlin, 1904, 7a, 33.

[2] P. Duhem, *Études sur Léonard de Vinci*, III[e] série, Paris, 1913, p. 388; H. Wieleitner, *Isis*, vol. 7, 1925, pp. 487, 488.

this either as a *latitudo* or an *altitudo*. The form was thus represented graphically by a surface. Oresme extended this process by taking a surface as the basis which, together with the *latitudo*, formed a solid. Proceeding still further, he took a solid as a basis and upon each point of this solid he entered the increment. He saw that this process demanded a fourth dimension which he rejected; he overcame the difficulty by dividing the solid into numberless planes and treating each plane in the same manner as the plane above, thereby obtaining an infinite number of solids which reached over each other. He uses the phrase "fourth dimension" (4^{am} *dimensionem*).

The Italian algebraist Cardan, in the first chapter of his *Ars Magna* (1545), considering the powers of numbers, says that "the first power refers to a line, the square to a surface, the cube to a solid and that it would be fatuous indeed for us to progress beyond, for the reason that it is contrary to nature." From Cardan's book, the German algebraist *Stifel* learned the solution of cubic equations, and it is possible that Stifel's statement on dimensions is an echo of Cardan. The German writer says[1]: "And further (than the cube) the geometric progression cannot advance to more dimensions," although in arithmetic it is quite allowable; he says "that we set down corporeal lines and surfaces, and pass beyond the cube as if there were more than three dimensions, although this is contrary to nature." It has not been noted in histories of mathematics that a proof of the impossibility of a fourth dimension was given by the German mathematician and astronomer *Clavius* who for many years resided in Rome, and was active in framing the Gregorian calendar. Following, at least in part, the proof due to Ptolemy, Clavius[2] endeavors to give a detailed demonstration of the theorem that not more than three concurrent lines can be drawn, each perpendicular to each of the others. The proof was reproduced and approved in 1802, in a German cyclopedia of philosophy.[3]

A second approach to the fourth dimension, from the side of physics, is found in *Descartes*.[4] He was trying to find a graphic representation of the motion of a freely falling body. If the body is acted upon by a single accelerating cause, the motion (distance passed over) is represented by a triangle, if by two causes, it is represented by a triangular pyramid; if by three causes, "by other figures (*alijs figuris*)." But he does not let us into the secret, what those "other figures"

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[1] *Die Coss Christoffs Rudolffs.* Durch Michael Stifel, Königsberg, 1553, fol. 9; H. P. Manning, *Geometry of Four Dimensions*, New York, 1914, p. 2, 3; H. Wieleitner, *Isis*, vol. 7, 1925, p. 486.

[2] *Christophori Clavii in Sphaeram Ioannis de Sacro Bosco Commentarivs.* 3. ed., Venice, 1601, p. 13–15. Clavius' first edition appeared in 1585(?).

[3] G. S. A. Mellin, *Encyclopädisches Wörterbuch der Kritischen Philosophie*, vol. 4, part II, Jena und Leipzig, 1802, Art. "Raum."

[4] *Oeuvres de Descartes*, éd. Ch. Adam et P. Tannery, vol. X, Paris, 1908, pp. 75–78, 219; H. Wieleitner, *Isis*, vol. 7, 1925, p. 488.

were. Evidently, Descartes was repulsed as effectively as Oresme had been repulsed.

A different mode of approach is found in Pascal[1] who considered summations or integrations which may be expressed in modern symbols thus:

$$\sum_0^a x^m \cdot \Delta x = \frac{a^{m+1}}{m+1}.$$

This yields, when $m=1$, a surface; when $m=2$, a solid; but when $m=3$, a "plane-plane, composed of . . . . solids each of which is multiplied by a small division of the axis, forming . . . . small plane-planes of the same altitude, ... And one need not be disturbed by this fourth dimension, because . . . . on taking planes in place of solids, or even lines . . . . the sum of the lines gives a plane which takes the place of this "plane-plane." Thus Pascal like his predecessors sidestepped the fourth dimension; and he did so by letting a line represent a solid numerically.

We have now cited the judgments of eight thinkers, distributed in time over 2000 years and geographically over Greece, Egypt, Italy, Germany, and France. These men rejected the possibility of a space of more than three dimensions as at variance with our external sense-perception. Theirs were arguments based on experience, much like that of the discouraged fat man who was certain there was no fourth dimension, for if there was one, he surely would have it.

2. Henry More's Generalization. The earliest success in the quest for a fourth dimension was recorded by Henry More, a Cambridge Platonist and contemporary of Newton. It was the product of theologic speculation, as found in his book entitled *The Immortality of the Soul* (1659). In it he does not use the term "fourth Dimension," but he speaks of a "fourth Mode" which he calls "Essential Spissitude," from the Latin *spissitudo*, meaning thickness. "The greatest and grossest obstacle to the belief of the Immortality of the Soul," says More,[2] "is that confident opinion in some, as if the very notion of a Spirit were a piece of None-sense." He combatted the Cartesian philosophy according to which nature could be fully explained by mechanical laws alone. More postulated that extension is a property not only of matter but also of spirits.[3] To retain the quality of extension, and yet be able to contract, dilate and change

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[1] *Oeuvres de Blaise Pascal*, éd. L. Brunschvicg, P. Boutroux et F. Gazier, Paris, 1914, vol. 8, p. 357–358, 365–367, also vol. 9, pp. 3–44; H. Bosmans, *Annales de la société scientifique de Bruxelles*, vol. 42, part I, documents et comptes rendus, p. 337; H. Wieleitner, *Isis*, vol. 7, 1925, p. 488.

[2] Henry More, *Immortality of the Soul*, 4. Ed., London, 1713, bk. 1, chap. 2, p. 8.

[3] H. More, *Enchiridion metaphysicum*, London, 1679, §9, p. 321; Joseph Glanvil, *Saducismus Triumphatus: or . . . . Witches and Apparitions*, 2. Ed., London, 1682, More's "*An Appendage to the First Part*," pp. 136, 137, and More's "*Answer to a Letter*," pp. 38.

its form, a spirit was given the power of "redoubling" into itself,[1] as when "a string is doubled and redoubled," so that the "dimension of longitude is in some part lost." This "redoubling" might be an incomplete bending, like that of a finger, length changing into length and breadth, resulting in an increase of dimension. By redoubling a spiritual substance back upon itself, its length may be greatly reduced, also its breadth and thickness. To allow it full freedom of contraction and dilation, yet without loss of extension, the fourth dimension is created. More does not proceed further than the "fourth Mode," for he assumes that this is the state when a spirit "will not easily admit of further redoubling," being in "saturation." In later years, More again discussed the same subject in his manual on metaphysics, the *Enchiridion Metaphysicum*, 1671, and in his additions to the second edition, 1682, of a work by Joseph Glanvil. More is a mystic. His ideas are obscurely presented, so much so, that a German critic contended that More did not arrive at the concept of a fourth dimension of space at all, and should not be mentioned in the history of hyper-space.[2] This criticism has been accepted by some as valid, but it does not seem so to us, for the following reasons: (1) More ascribes extension to material substances and also to the soul; (2) the soul of man has its "chief seat" "in the fourth Ventricle of the Brain,"[3] but is not confined there, it being able to spread throughout the whole body on occasion; when it is contracted, the space occupied possesses not only the three dimensions, but also this fourth dimension or spissitude. (3) More says[4] that "when one part of an Extended Substance runs into another, something both of Longitude, Latitude, and Profundity, may be lost," yet "what is lost here in all (three dimensions), or any of the two dimensions, is kept safe in Essential Spissitude"; (4) More's "redoubling" constituted a mental process by which the passage of a substance from a space of lower dimensions to one of higher, including the fourth dimension, could be effected. (5) In his *Enchiridion Metaphysicum*[5] and his publication of 1682, noted above, he actually uses the term "fourth dimension"; (6) he says[6]: it is "as easy and familiar to my Understanding, as that of the Three dimensions to my Sense or Phancy," thereby explaining that with him the fourth dimension is a matter of the understanding, while the notion of three dimensions is reached through his senses; (7) The spirit has extension and is in the fourth dimension.

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[1] H. More, *Immortality of the Soul*, 1713, p. 6.

[2] R. Zimmermann, Henry More und die vierte Dimension des Raumes, Wien, 1881. See *Jahrbuch über Fortschritte der Mathematik*, vol. 13, Berlin, 1883, pp. 50–53.

[3] H. More, Immortality of the Soul, 1713, Bk. II, Chap. 7, §18, p. 93. See also E. A. Burtt, *The Metaphysical Foundations of Modern Physical Science*, London, 1925, pp. 129, 130.

[4] H. More, *Immortality of the Soul*, 1713, p. 6.

[5] H. More, *Enchiridion metaphysicum*, 1679, chap. 28, part 1, §7, p. 320; J. Glanvil, *op. cit.*, pp. 136, 137.

[6] H. More, Immortality of the Soul, 1713, p. 6.

A paragraph heading, in the *Enchiridion Metaphysicum* reads[1]: "That besides the three dimensions which are filled with all extended material things, a fourth must be admitted, with which coincides the spirit." If More's fourth dimension, the abode of the spirit, did not belong to space, he could hardly have used such phrases[2] as "the spiritual object which we call space."

3. From More to Minkowski. More's fourth dimension does not receive great prominence in his writings. That fact, combined with the obscure presentation and the subtlety of the concept, caused it to be overlooked by his contemporaries. We have found only one writer before the nineteenth century who commented on it—in a tract which itself passed into oblivion. Thus More's intellectual flower bloomed quite unseen till paleontologists of the nineteenth century discovered it in the strata of forgotten ideas. That one writer was John Keill of Oxford who, in the Introduction to his *Examination of Dr. Burnet's Figure of the Earth*, (1698, 2. Ed. 1734), cited More among a score of other writers to prove that philosophers maintain "opinions more absurd than can be found in any of the most Fabulous Poets." "Dr. More", says Keill, "will have Souls, besides the three dimensions which belong to Bodies, to have a fourth, which he calls the Soul's *essential spissitude* by which it can contract or dilate itself when it pleases."

Other writers of the seventeenth and eighteenth centuries who approached the concept of a fourth dimension of space were repulsed as had been writers of earlier centuries. Thus, John Wallis,[3] in his *Algebra*, 1685, says: "A Line drawn into a Line shall make a Plane or Surface; this drawn into a Line, shall make a Solid: But if this Solid be drawn into a Line, or this Plane into a Plane, what shall it make? a Plano-Plane? That is a Monster in Nature, and less possible than a Chimaera or Centaure. For Length, Breadth and Thickness, take up the whole of Space. Nor can our Fansie imagine how there should be a Fourth Local Dimension beyond these Three." Similarly, the Frenchman Ozanam[4] rejects it, because in nature we do not know of any quantity which has more than three dimensions.

Mathematical historians have overlooked, thus far, the arguments presented by Leibniz and Kant against the possibility of a fourth dimension. Leibniz[5]

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[1] H. More, *Enchiridion metaphysicum*, 1679, chap. 28, part 1, §7, p. 320.

[2] H. More, *Opera omnia*, London, 1675–9, vol. 2, p. 171ff; quoted by E. A. Burtt, *op. cit.*, p. 141.

[3] John Wallis, *Treatise of Algebra*, London, 1685, p. 126.

[4] J. Ozanam, *Dictionnaire mathématique*, Amsterdam, 1691, p. 62.

[5] G. W. Leibniz, *Die Theodicee*, von J. H. Kirchmann, Leipzig, 1879, §351, p. 375. The date of this document is 1710. As Dr. D. Mahnke informs me, there is a Leibnizian manuscript which shows that as early as 1673, Leibniz had seen Pascal's letter to Carcavi (*Oeuvres de B. Pascal*, Paris, 1914, vol. 8, pp. 357–358, 365–367) relating to a fourth dimension of space; in his comments thereon, Leibniz rejects the idea of a fourth dimension.

accepts the argument of Ptolemy and Clavius. Kant very properly rejected this proof as reasoning in a circle. In one of his early speculations, Kant[1] suggests that the three dimensions of space arise from the fact that material bodies influence each other according to the law of inverse squares, and our soul is subject to that same law. He did not attempt a formal proof, but added that God might have chosen the law of inverse cubes and thereby have given us a space of higher dimensions. When a non-mathematician of the intellectual force of a Kant will indulge in loose ratiocinations of this sort one realizes more than ever, what a gift mathematics has to offer the world in the nature of processes of rigorous thinking.

The new idea of time as a fourth dimension was first put into print by D'Alembert[2] in 1754, and by Lagrange[3] in 1797. All that D'Alembert says is this: "I stated above that it is impossible to conceive of more than three dimensions. A man of parts, of my acquaintance, holds that one may however look upon duration as a fourth dimension, and that the product of time and solidity is in a way a product of four dimensions. This idea may be challenged but it seems to me to have some merit other than that of mere novelty." Lagrange advanced further in his statement: "Since the position of a point in space depends upon three rectangular coordinates these coordinates in the problems of mechanics are conceived as being functions of t. Thus we may regard mechanics as a geometry of four dimensions, and mechanical analysis as an extension of geometrical analysis."

This idea was neglected for about a century. In 1885, a contributor to *Nature*[4] who signed himself "S," advanced it as a novel proposition. Suggesting that there could be many different fourth dimensions, he spoke of time as *a* fourth dimension, rather than *the* fourth dimension. He called the new space "time-space," and conceived of a "cube and the whole of the three-dimensional space in which it is situated, as floating away in time-space." This idea is skilfully elaborated in H. G. Wells' novel, *The Time Machine*, 1895. This machine conveys the traveller backwards and forwards in time, enabling him to study cosmical changes; he found that millions of years from now, nearly all traces of life will have vanished.

The early hyper-space writings of C. H. Hinton, in England, touch theological thought. His *Scientific Romances*, 1886, contain applications to ethics and metaphysics and the theory of free will. In his *New Era of Thought*, 1888, he argues that hyper-space is "the scientific basis of altruism, and religion." In

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[1] Immanuel Kant, *Werke*, ed. K. Rosenkranz und F. W. Schubert, Leipzig, vol. 5, 1839, §9, pp. 25–27.

[2] J. D'Alembert, *Encyclopédie* (edited by Diderot), 1754, Art. "Dimension." See R. C. Archibald, *Amer. Math. Soc. Bull.*, 1914, vol. 20, pp. 409–412.

[3] J. Lagrange, *Théorie des fonctions analytiques*, 1797, p. 223.

[4] *Nature*, London, 1885, vol. 31, p. 481.

some of his later writings on the fourth dimension, particularly his lecture before the Philosophical Society of Washington[1] in 1901, speculations in physics, electricity and magnetism are the dominating motive; he argues for the recognition of the fourth dimension of space as a physical reality. The fourth dimension appears also in the speculations of spiritualists and theosophists.

Physical speculation entered in a surprising manner in the writings of W. W. R. Ball,[2] when he endeavored, in 1891, to assist physicists who were in a state of bewilderment with regard to the luminiferous ether. This ether was to transmit radiant energy by transverse waves such as can be generated only by an elastic solid; it was to impart energy to material bodies and also to receive energy from them. And yet, this solid ether was to offer not the least resistance to the planets sweeping through it. The explanations of Stokes and Kelvin that the ether was like shoemakers' wax which would vibrate under a sharp blow, yet would be plastic and permit slow motions of a heavy solid through it, never quite appealed to the mass of physicists as entirely satisfactory. Ball tried another avenue of escape. He placed the ether by itself in a fourth dimension of space, but allowed it to make contact on its border with the small particles of bodies in the world of three dimensions. This ether could receive molar vibrations and also could transmit vibrations to molecules. Yet, being in a fourth dimension, the ether would offer no resistance to planets moving in the space of three dimensions, for that space was really empty and therefore offering no resistance.

The imagination of the pure mathematician of the nineteenth century became enamored with the problem of the fourth dimension. The advent of the non-Euclidean geometries of Lobachevski and Bolyai aided the movement. Discarding the limitations set by the senses, he proceeded to create geometries in his mental universe which might have their counterpart in the physical universe, or they might not. He wandered far and wide, and made the most wonderful discoveries. Indeed the bibliography of hyper-space and non-Euclidean geometry fills a large volume by itself. Conservative workers said, "if a fourth dimension did exist in space," then such and such theorems would follow. Others felt that a physical fourth dimension may perhaps exist, but is not visible to them any more than a three dimension could be to a flatlander. Still others spoke of the fourth and higher dimensions as truly existing, as indeed they do exist in their own minds. A literary man in Berkeley once remarked to me: Is it not non-sense for mathematicians to talk about a fourth

[1] C. H. Hinton, *Bulletin of the Philosophical Society of Washington*, vol. 14, 1900–1904, Washington, 1906, pp. 179–203.

[2] W. W. R. Ball, *Messenger of Mathematics*, vol. 21, 1891, p. 20.

dimension and could they not employ their time in some way useful to mankind? His geometrical world was limited to that of physical space; as it had been limited for mathematicians themselves before the emancipation of the mind from its physical environment. As to the utility of hyper-space geometry, it was difficult to satisfy the Berkeley man. The fact pointed out by Möbius[1] in 1827 that if a fourth dimension did exist, symmetrical figures in three dimensions could be made to coincide, did not appeal to the Berkeley critic. George Green's extension,[2] in 1833, of the attraction of an ellipsoid to any number of variables, so as to be "no longer confined as it were to the three dimensions of space" seemed to the Berkelian dull, compared to a game of solitaire. The use of time as a fourth dimension in mechanics was to him simply a play of words; mechanics profited nothing by this strange idea. Felix Klein's untying knots in four dimensional space which were tied fast in our physical space met with the disapproval of our critic; our divorce courts, he said, need no mystic symbol of encouragement. Nor could Sylvester[3] have convinced our Berkeley critic by the statement:

> I know there are many, who, like my honoured and deeply lamented friend, the late eminent Prof. Donkin, regard the alleged notion of generalized space as only a disguised form of algebraical formulisation; but the same might be said with equal truth of our notion of infinity in algebra, or of impossible lines, or lines making a zero angle in geometry, the utility of dealing with which as positive substantiated notions no one will be found to dispute. . . . . If Gauss, Cayley, Riemann, Schläfli, Salmon, Clifford, Kronecker, have an inner assurance of the reality of transcendental space, I strive to bring my faculties of mental vision into accordance with theirs.

Less serious, more playful and to the uninitiated more delightful is the treatment of different spaces in stories like *Flatland*, from the pen of Edwin Abbott Abbott, at one time headmaster of a school in London, a preacher, text-book writer, and Shakespearean scholar. It is a book which deserves to be more widely read by teachers and students.

4. Minkowski and After. More recent advances in physics have brought about a real and vital connection between that science and hyper-space, a veritable "fusion of geometry and physics."[4] To recall this, I need only mention the name Minkowski. What used to be "space and time" separately has now become "space-time" unified. Time by itself means nothing, and space by itself means nothing; the fixation of events requires both time and space. According to Minkowski (1908) "Nobody has ever noticed a place,

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[1] A. F. Möbius, *Der barycentrische Calcul*, Leipzig, 1827, §140, p. 184. See also H. P. Manning, *op. cit.*, pp. 4, 5.

[2] *Mathematical Papers of George Green*, edited by N. M. Ferrers, 1871, p. 188.

[3] J. J. Sylvester, Inaugural Presidential Address to the mathematical and physical section of the British Association at Exeter, 1869. Reprinted in Sylvester's *Laws of Verse*, p. 113.

[4] Bertrand Russell in his introduction to A. V. Vasiliev, *Space Time Motion*, transl. by H. M. Lucas and C. P. Sanger, London, 1924, p. XXI.

except at a time, or a time except at a place."[1] As Wagner says in Parsifal: "You see, my son, to space here shall be time." How did these changes come about? The answer is suggested by the names: Luminiferous ether, Michelson and Morley experiment, Lorentz transformation, Maxwell's equations, Clifford's *Common Sense of the Exact Sciences*, Einstein's relativity. Vasiliev[2] mentions the tradition that the portrait of the Austrian astronomer Littrov, founder of the observatory at Kazan, fell down in the astronomical cabinet of Kazan University at the precise moment when Littrov died in Vienna. Would Einstein admit that the two events could be simultaneous to all accurate observers? Let us suppose that a man sitting in a chair at his home in a town half way between Kazan and Vienna, received radio messages of the respective events from Kazan and Vienna. Suppose the two messages reached him simultaneously. If instead of being at home, he had been travelling on a train going fifty miles an hour from Kazan toward Vienna, and had passed through this midway town at the moment of the death at Vienna, as judged in Vienna, then he would have received the radio message from Vienna before the one from Kazan, and would have concluded that the death at Vienna occurred earlier than the fall of the picture in Kazan. Under the same conditions, a traveller in the opposite direction would infer that the death came after. Thus the Einstein theory rejects simultaneity in itself and makes it depend upon certain conditions of motion. Every reference system has its own time, and, unless we know the reference system employed, there is no meaning in the statement of the time of an event. In this theory time is inextricably connected with space; the space-time fourth dimension is advanced as a physical reality. A re-formulation of physics is in progress under the space-time geometry. Nor does Einstein represent the culmination of speculation in this field. Recently, G. N. Lewis of the University of California advanced a theory of radiant energy making all phenomena of the physical universe reversible in space-time, so that the universe is no longer conceived to be running down like a clock. The fourth dimensional concept of space-time underlies speculations which are altering the very foundations of science. Thus a four dimensional concept of space, persistently laughed out of existence by the common sense of the nineteenth century, is becoming so firmly established that before long it may be embraced by the common sense of the twentieth century.

To sum up: We have seen that the fourth dimension concept was approached as a generalization of the three dimensions or as a graphic representation in algebra and physics, but was rejected during 2000 years, as contrary to sense

[1] *The Principle of Relativity*, by H. A. Lorentz, A. Einstein and others, London, 1923, p. 76.
[2] A. V. Vasiliev, *op. cit.*, p. 149.

experience. The earliest acceptance of a fourth dimension of space was made by a devout theologian providing a suitable abode for the soul and for spirits. In the nineteenth and twentieth centuries it proved an aid to mysticism and occultism. The transcendental formulation of geometric assumption led to extensive studies of geometry and of kinematics in hyper-space. In eighteenth century mechanics, time as a fourth dimension was loosely added to the three dimensions of space. In the twentieth century, transcendentalism is followed by realism; time as a fourth dimension enters studies relating to the very anatomy of physical science. Time and the three dimensional space become inseparable; "they melt in a surprising manner into one another, like sunset tints or the colors of the dying dolphin."

A CLASSIFICATION OF SECOND DEGREE LOCI OF SPACE[1]

By L. J. PARADISO, Ohio State University

1. Introduction. A survey of the methods used in classifying the quadric surfaces reveals the surprising fact that no classification, as far as the writer knows, has been attempted by means of their invariants and covariants alone, similar to that made for second degree curves. In fact, until recently no complete classification of second degree curves has been made, as MacDuffee has shown in a recent paper in this MONTHLY.[2] In his paper, MacDuffee gave a complete classification of second degree curves by means of their invariants and an additional covariant. He also suggested that the methods which he there employed could likewise be applied to second degree loci of space.

In this paper an absolutely complete classification of all second degree surfaces is made, it is believed for the first time, by means of their invariants and covariants alone. It is shown in §3 that a complete system of invariants and covariants of the second degree real polynomial in three variables under euclidean transformations consists of four invariants and three covariants. The four invariants used are the well known I, J, D, Δ given in most treatises on space geometry. The second degree polynomial is one of the covariants. It seems that the other two covariants of this particular system, Θ and Φ, have escaped attention. The equation $\Theta=0$ gives the locus of the intersection of three mutually perpendicular tangent planes to the quadric surface and the equation $\Phi=0$ gives the locus of the intersection of three mutually perpendicular tangent lines.

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[1] Presented to the faculty of the Graduate School of the Ohio State University in candidacy for the degree of Master of Arts, September, 1926.

[2] C. C. MacDuffee, *Euclidean invariants of plane second degree curves*, this MONTHLY, (*1926*, 243-252).

It is shown in §7 that when $\Delta = D = 0$, the corresponding surfaces cannot be classified by the invariants alone. It is in this case that the covariant Θ (whose degree is invariant) reduces to an invariant, thus enabling us to proceed with the classification. A similar situation arises when $J = D = \Delta = \Theta = 0$, in which case Φ becomes an invariant and so we are able to complete the classification. Furthermore, every parameter in every canonical form is expressed in terms of invariants. The results are summarized in a table by means of which one can classify at once any given second degree locus merely by calculating its invariants and covariants and by writing its canonical form.

2. The Lie group. The most general euclidean transformation from one set of mutually perpendicular axes to another set similarly oriented is a six-parameter group composed of six one-parameter groups—three translation groups and three rotation groups—as follows:

$$\begin{array}{lll} x' = x - \alpha, & x' = x, & x' = x, \\ y' = y, & y' = y - \beta, & y' = y, \\ z' = z, & z' = z, & z' = z - \gamma, \\ x' = x, & x' = x \cos\psi + z \sin\psi, & x' = x\cos\theta + y \sin\theta, \\ y' = y\cos\phi + z \sin\phi, & y' = y, & y' = -x\sin\theta + y\cos\theta, \\ z' = -y\sin\phi + z\cos\phi, & z' = -x\sin\psi + z\cos\psi, & z' = z, \end{array} \tag{1}$$

where the parameters $\alpha,\beta,\gamma,\varphi,\psi,\theta$ are independent real variables.

The general second degree polynomial may be written thus:

$$F \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz + d$$

where the coefficients as well as x,y,z are independent real variables.

The Lie generators of the six one-parameter groups (1) leaving F invariant are found[1] to be:

$$\begin{aligned} U_\alpha &= -\frac{\partial}{\partial x} + a\frac{\partial}{\partial l} + h\frac{\partial}{\partial m} + g\frac{\partial}{\partial n} + 2l\frac{\partial}{\partial d}, \\ U_\beta &= -\frac{\partial}{\partial y} + b\frac{\partial}{\partial m} + f\frac{\partial}{\partial n} + h\frac{\partial}{\partial l} + 2m\frac{\partial}{\partial d}, \\ U_\gamma &= -\frac{\partial}{\partial z} + c\frac{\partial}{\partial n} + g\frac{\partial}{\partial l} + f\frac{\partial}{\partial m} + 2n\frac{\partial}{\partial d}, \\ U_\phi &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + 2f\left(\frac{\partial}{\partial b} - \frac{\partial}{\partial c}\right) + (c-b)\frac{\partial}{\partial f} + g\frac{\partial}{\partial h} \\ &\qquad - h\frac{\partial}{\partial g} + n\frac{\partial}{\partial m} - m\frac{\partial}{\partial n}, \end{aligned} \tag{2}$$

[1] The method used is similar to that employed by MacDuffee, *loc. cit.*, pp. 244-247.

$$U_\psi = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + 2h\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial b}\right) + (b - a)\frac{\partial}{\partial h} + f\frac{\partial}{\partial g} - g\frac{\partial}{\partial f} + m\frac{\partial}{\partial l} - l\frac{\partial}{\partial m},$$

$$U_\theta = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + 2g\left(\frac{\partial}{\partial c} - \frac{\partial}{\partial a}\right) + (a - c)\frac{\partial}{\partial g} + h\frac{\partial}{\partial f} - f\frac{\partial}{\partial h} + l\frac{\partial}{\partial n} - n\frac{\partial}{\partial l}\cdot$$

A necessary and sufficient condition that a function be a euclidean invariant or covariant of F under transformations (1) is that it be made to vanish by each of the operators of (2).

The six differential equations obtained by setting each of the operators of (2) equal to zero,

$$U_i = 0, \qquad (i = \alpha,\beta,\gamma,\phi,\psi,\theta), \tag{3}$$

form a complete system[1] of differential equations, since their left members are the generators of a group. They involve thirteen variables and hence have $13-6=7$ functionally independent solutions. These seven solutions form a complete system, from the standpoint of Lie, of invariants and covariants of F under euclidean transformations. If we disregard the terms in x, y, z, in (3), we get six equations in ten variables which have $10-6=4$ functionally independent solutions, the four invariants of F. Thus we have the following:

THEOREM 1. *A complete system of euclidean invariants and covariants of F consists of four invariants and three covariants, all of which are functionally independent.*

3. A complete system of invariants and covariants. Since four independent invariants I, J, D, Δ of F are well known,[2] our problem consists in finding the three covariants. The polynomial F is, of course, a covariant. We need to find two more.

If we obtain a surface, $G=0$, which is related to $F=0$ in a manner independent of the axes, then G must be a multiple of a covariant of F. Many such surfaces can be found, from among which we may select two which have as simple equations as possible and which are functionally independent of F and the invariants. The two which suit our needs best are Φ and Θ, where $\Theta=0$ is the locus[3] of the intersection of three mutually tangent planes from $F=0$, and

[1] Goursat-Hedrick, *Mathematical Analysis*, Ginn, 1917, vol. 2, p. 267.

[2] See, for instance, Snyder and Sisam, *Analytic Geometry of Space*, Henry Holt, 1924, p. 82.

[3] This equation has been found for the special case of the centralized quadric. *Cf.* Snyder and Sisam, *loc. cit.*, p. 93, ex. 6.

$\Phi=0$, is the locus of the intersection of three mutually perpendicular tangent lines[1] from $F=0$; or, what amounts to the same thing, the locus of the vertices of the enveloping cones of $F=0$ which have three mutually perpendicular generators.

To show that these seven invariants and covariants are functionally independent, it is necessary and sufficient to show that their Jacobian does not vanish identically. In particular one can verify that when $f=g=h=x=y=z=0$, then

$$J\left(\frac{I,\ J,\ D,\ \Delta,\ F,\ \Theta,\ \Phi,}{a,\ b,\ c,\ l,\ m,\ n,\ d}\right)\not\equiv 0$$

where I, J, D, Δ, F, Θ and Φ are the expressions given below. Hence it follows that the invariants and covariants are functionally independent; also, since only seven can exist in a complete system, all other invariants and covariants of F under transformations (1) are functions of these. We then have

THEOREM 2. *A complete system of functionally independent euclidean invariants and covariants of F is given by:*

$$I\equiv a+b+c, \qquad J\equiv bc+ab+ac-f^2-g^2-h^2,$$

$$D\equiv\begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c\end{vmatrix},\qquad \Delta\equiv\begin{vmatrix} a & h & g & l\\ h & b & f & m\\ g & f & c & n\\ l & m & n & d\end{vmatrix}, \tag{4}$$

$$F\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2lx+2my+2nz+d,$$

$$\Theta\equiv D(x^2+y^2+z^2)+2Lx-2My+2Nz-Jd+(a+b)n^2$$
$$+(a+c)m^2+(b+c)l^2-2lng-2mnf-2lmh,$$

$$\Phi\equiv(ab+ac-h^2-g^2)x^2+(ab+bc-f^2-h^2)y^2+(bc+ac-f^2-g^2)z^2$$
$$+2(af-hg)yz+2(bg-fh)zx+2(ch-fg)xy+2(bl+cl-hm-gn)x$$
$$+2(am+cm-fn-hl)y+2(an+bn-fm-gl)z+d(a+b+c)$$
$$-(l^2+m^2+n^2),$$

where L, M, N are the minors of l, m, n, respectively in Δ.

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[1] This locus also has been obtained for the special case of the centralized quadric. *Cf.* G. Salmon, *Analytic Geometry of Three Dimensions*, Hodges, Figgis and Co., 1882, p. 100, ex. 6.

4. Properties of the invariants and covariants. When F is multiplied by a positive constant k the locus of $F=0$ is unchanged, while each invariant and covariant is multiplied by k^i where i is its degree in the coefficients of F. Hence one of these invariants does not measure a geometric magnitude but its vanishing does characterize an invariantive geometric property of $F=0$. Such an invariant or covariant is called an *algebraic* invariant[1] of *weight* k^i. In case i is even the sign of the invariant is unalterable.

However, we can form quotients such as I^2/J, D^2/J^3, etc., which are of weight zero and which can be used to measure geometric magnitudes. These are called *geometric* invariants.

It is important to note that the degree of a covariant is an *arithmetic* invariant so that when Θ or Φ reduces to a constant, that constant is an invariant[2] of F.

5. Classification of the generic case and the cones. We shall assume henceforth that F is of degree 2, since otherwise the locus is either a plane or the entire space. It follows that

$$I^2-2J=a^2+b^2+c^2+2f^2+2g^2+2h^2>0,$$

since all the coefficients are real. Hence I and J cannot be zero simultaneously.

The coefficient f in F is transformed into f' by the transformation groups (1) where

$$f' = \sin\phi\cos\phi(c-b) + f(\cos^2\phi - \sin^2\phi).$$

Hence we can always make $f'=0$ by choosing a ϕ which satisfies

$$\tan 2\phi = 2f/(b-c).$$

Similarly, we can always find values of ψ and θ such that $g'=h'=0$, so that, without loss of generality, we can reduce $F=0$ (dropping primes) to

$$ax^2+by^2+cz^2+2lx+2my+2nz+d=0\ ;\ (a,b,c)\neq(0,0,0). \tag{5}$$

The invariants in this case become

$$\begin{aligned} I &= a+b+c, \quad J = ab+bc+ac, \quad D = abc, \\ \Delta &= abcd - abn^2 - acm^2 - bcl^2. \end{aligned} \tag{6}$$

We shall now consider the different cases.

CASE I. $D\neq 0$. From (6) we find that $abc\neq 0$, and the transformation

$$x = x' - \frac{l}{a}, \qquad y = y' - \frac{m}{b}, \qquad z = z' - \frac{n}{c}, \tag{7}$$

is valid, reducing (5) to

$$ax^2+by^2+cz^2+d=0\ ; \qquad abc\neq 0. \tag{8}$$

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[1] MacDuffee, *loc. cit.*, p. 247.

[2] *Loc. cit.*, p. 247.

The invariants now become

$$I = a + b + c, \qquad J = ab + bc + ac, \qquad D = abc, \qquad \Delta = abcd, \tag{9}$$

so that $d=0$ when and only when $\Delta=0$.

CASE I (A). $D\neq 0$, $\Delta\neq 0$. We have F reduced to (8) with $abcd\neq 0$, which is the generic case, since (8) can be transformed into

$$\frac{x^2}{\xi} + \frac{y^2}{\eta} + \frac{z^2}{\zeta} = 1,$$

where

$$\xi = -\frac{d}{a}, \quad \eta = -\frac{d}{b}, \quad \zeta = -\frac{d}{c}, \quad d \neq 0.$$

The invariants in this case reduce to those in (9) with the aid of which we find that

$$\begin{aligned} \xi + \eta + \zeta &= -d\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = -\frac{J\Delta}{D^2}, \\ \xi\eta + \xi\zeta + \eta\zeta &= d^2\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) = \frac{I\Delta^2}{D^3} \\ \xi\eta\zeta &= -\frac{d^3}{abc} = -\frac{\Delta^3}{D^4}. \end{aligned} \tag{10}$$

It is to be noticed that these quotients are geometric invariants. We now have that ξ, η, ζ are the roots of the cubic

$$u^3 + \frac{J\Delta}{D^2}u^2 + \frac{I\Delta^2}{D^3}u + \frac{\Delta^3}{D^4} = 0. \tag{11}$$

In order to characterize the ellipsoids and hyperboloids by the invariants, we shall consider the three possible cases which arise.

a. $I\neq 0$, $J\neq 0$. No coefficient of (11) is zero for this case, and also all three roots are real. The coefficient of u^2 has the sign of $J\Delta$, the coefficient of u has the sign of ID and the constant term has the sign of Δ. By Descartes' rule of signs,[1] the number of positive roots is exactly equal to the number of variations in signs of the terms.

For the ellipsoid, three roots are positive, and hence the signs of the terms of (11) alternate. That is, $\Delta<0$, $J>0$, $ID>0$.

There is no locus (sometimes called imaginary ellipsoid) when there are no positive roots, i.e., $\Delta>0$, $J>0$ and $ID>0$.

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[1] See Dickson, L. E., *First Course in Theory of Equations*, Wiley, 1922, p. 75, ex. 15.

For the hyperboloid of one sheet, two roots are positive, and there are two variations in sign. That is, $\Delta > 0$ and either $J < 0$ or $ID < 0$.

For the hyperboloid of two sheets, one root is positive, and there is one variation in sign. So that in this case $\Delta < 0$ and either $J < 0$ or $ID < 0$.

b. If $I = 0$, then $J \neq 0$ by §5. Equation (11) becomes

$$u^3 + \frac{J\Delta}{D^2} u^2 + \frac{\Delta^3}{D^4} = 0.$$

The coefficient of u^2 has the sign of $J\Delta$ and the constant term has the sign of Δ. We cannot have three positive roots or three negative roots, since the coefficient of u is an elementary symmetric function of the roots and is in this case zero. Hence there are one or two variations in sign.

When $\Delta < 0$ there is one variation in sign no matter whether $J > 0$ or $J < 0$ so that we have the hyperboloid of two sheets in either case.

When $\Delta > 0$ and $J < 0$ there are two variations in sign and hence we have the hyperboloid of one sheet. When $J > 0$ the equation which has its roots the negative of those of the equation above has one variation in sign so that for this case also we have the hyperboloid of one sheet.

c. If $J = 0$, then $I \neq 0$ by §5. Equation (11) now becomes

$$u^3 + \frac{I\Delta^2}{D^3} u + \frac{\Delta^3}{D^4} = 0.$$

The coefficient of u has the sign of ID and the constant term the sign of Δ. We cannot have three positive roots or three negative roots, since the coefficient of u^2 (which is the sum of the roots) is an elementary symmetric function of the roots and is zero. Hence there are one or two variations in sign.

When $\Delta < 0$ we have the hyperboloid of two sheets since this gives one variation in sign regardless of the sign of ID.

Also, when $\Delta > 0$ and $ID < 0$ we have the hyperboloid of one sheet since this gives two variations in sign. When $ID > 0$ the equation having its roots the negative of those in the equation above has one variation in sign so that we again have the hyperboloid of one sheet.

We may summarize the three cases as follows:

THEOREM 3. *When $D \neq 0$ and $\Delta \neq 0$, we have the ellipsoid if $\Delta < 0$, $J > 0$, and $ID > 0$, no locus if $\Delta > 0$, $J > 0$, $ID > 0$, hyperboloid of one sheet if $\Delta > 0$ and either $J \leqq 0$ or $ID \leqq 0$, and finally we have the hyperboloid of two sheets if $\Delta < 0$ and either $J \leqq 0$ or $ID \leqq 0$.*

Note that these cases are mutually exclusive.

CASE I (*B*). $D \neq 0$, $\Delta = 0$. For this case we have (8) with $d = 0$, i.e.,

$$ax^2 + by^2 + cz^2 = 0 ; \qquad abc \neq 0. \tag{12}$$

The invariants for this case reduce to

$$I = a+b+c, \qquad J = ab+bc+ac, \qquad D = abc, \qquad \Delta=0\,,$$

so that a, b, c are the roots of the cubic

$$u^3 - Iu^2 + Ju - D = 0. \tag{13}$$

Since I, J, D are algebraic invariants of weights 1, 2, 3 respectively, the roots are algebraic invariants of weight 1. We shall therefore consider (12) to be our canonical form. It involves only two geometric invariants, the ratio of the roots, but to divide by one of the coefficients would destroy the symmetry.

Since $\Delta=0$ and $D\neq 0$, (11) represents a cone.[1] Conversely, when in F, $l=m=n=d=0$ (which is the condition that $F=0$ be a cone), then $\Delta=0$. Hence, a necessary and sufficient condition that $F=0$ be a cone is that $\Delta=0$ and $D\neq 0$.

We note from (12) that for this case three positive roots of (13) is the same case as three negative roots. Also two negative roots and one positive root of (13) is the same case as one negative root and two positive roots.

In (13) the sign of u^2 is that of $-I$, the sign of u is that of J and the sign of the constant term is that of $-D$. Let us consider

a. $I\neq 0$, $J\neq 0$. All the roots are real since they are irrational invariants and a, b, c are real. Using Descartes' rule of signs as in the previous case we find that the condition for three positive roots is $I>0$, $J>0$, $D>0$. However, since the signs of I and D are not invariant (§4) we may write this condition as $J>0$ and $ID>0$, where the sign of ID is now invariant. The condition for no positive root is $I<0$, $J>0$, and $D<0$. This being the same case as three positive roots we have that in both cases the condition for a point locus is $J>0$ and $ID>0$. Also since two positive roots and one negative root is the same case as one positive root and two negative roots, we have in both of these cases the real cones when $J<0$ or $J>0$ and $ID<0$.

b. $I=0$ and $J\neq 0$ (§5). In this case (13) becomes

$$u^3 + Ju - D = 0,$$

from which it is evident that we can have neither three positive nor three negative roots and hence from (12) we see that only real cones are possible.

c. $J=0$, so that by §5 $I\neq 0$, and (13) reduces to

$$u^3 - Iu - D = 0,$$

and by reasoning as in Case **b** we find that only the real cones are possible for this case also.

We may summarize the foregoing cases by the following theorem:

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[1] Snyder and Sisam, *loc. cit.*, art. 67.

THEOREM 4. *When* $\Delta=0$, *and* $D\neq 0$, *the locus of* (12) *is a single point if* $J>0$ *and* $ID>0$, *and in all other cases we have real cones.*

6. The paraboloids. We shall now consider

CASE **II.** $D=0$. From (6) we see that $abc=0$. Since at least one of the coefficients a, b, c of (5) is not zero (§5) we may assume that $a\neq 0$, for the other cases may be brought to this; *e.g.*, if $c\neq 0$, then the transformation (obtained by putting in (1) $\psi=\pi/2$),

$$x' = z, \qquad y' = y, \qquad z' = -x,$$

will make $a'=c$.

Now either $b=0$ or $c=0$. We shall assume that $c=0$, the other case being reducible to this if we substitute in (1) $\phi=\pi/2$, getting the transformation $x'=x$, $y'=z$, $z'=-y$.
Now (5) becomes

$$ax^2 + by^2 + 2lx + 2my + 2nz + d = 0\,; \qquad a \neq 0. \tag{14}$$

The invariants now are

$$I = a + b, \quad J = ab, \quad D = 0, \quad \Delta = -abn^2.$$

Hence $b=0$ when and only when $J=0$.

CASE II (A). $D=0$, $J\neq 0$. We have (14) with $ab\neq 0$ so that

$$x = x' - \frac{l}{a}, \qquad y = y' - \frac{m}{b}, \qquad z = z',$$

is valid and reduces (14) to

$$ax^2 + by^2 + 2nz + d = 0\,; \qquad ab \neq 0. \tag{15}$$

We note that $n=0$ when and only when $\Delta=0$.

CASE II A (1). $D=0$, $J\neq 0$, $\Delta\neq 0$. We have (15) with $n\neq 0$, so that

$$z = z' - \frac{d}{2n}, \qquad x = x', \quad y = y',$$

is valid and reduces (15) to the canonical form,

$$ax^2 + by^2 + 2nz = 0\,; \qquad abn \neq 0. \tag{16}$$

The invariants become

$$I = a + b, \quad J = ab, \quad \Delta = -abn^2,$$

so that n is a root of the equation $Jn^2+\Delta=0$ and a, b are roots of

$$u^2 - Iu + J = 0. \tag{17}$$

The roots of both these equations are algebraic invariants of weight 1, so that (16) is a canonical form whose parameters are algebraic invariants.

The roots of (17) are like or unlike in sign according as $J>0$ or $J<0$. Hence

THEOREM 5. *For* $J>0$ (16) *is the equation of an elliptic paraboloid, for* $J<0$ (16) *is the equation of a hyperbolic paraboloid.*

7. The quadric cylinders and planes. We now consider

CASE II A (2). $D=0$, $J\neq 0$, $\Delta=0$. In this case (15) becomes

$$ax^2+by^2+d=0\,,\qquad ab\neq 0\,, \tag{18}$$

and invariants reduce to

$$I=a+b,\quad J=ab,\quad D=\Delta=0.$$

Hence it is evident that in (18) d cannot in any way be determined from the invariants. Here the invariants fail us for the first time and the classification would terminate were it not for the covariants Θ and Φ to which we must resort. The covariant Θ in this case reduces to a constant, $-Jd$. Conversely, if $\Theta=k$ it follows from (4) that $D=L=M=N=0$. But since

$$\Delta = lL - mM + nN - dD,$$

we have $\Delta=0$. Hence

THEOREM 6.[1] *A necessary and sufficient condition that a second degree locus be a cylinder* (*quadric cylinder or plane*) *is that* Θ *reduce to a constant.*

CASE II (A) 2(a). $D=0$, $J\neq 0$, $\Delta=0$, $\Theta=k\neq 0$. Since $\Theta=-Jd$, we have (18) with $d\neq 0$. Then d is a root of $Jd+\Theta=0$ and a, b are the roots of

$$u^2-Iu+J=0. \tag{19}$$

Evidently a, b, d are algebraic invariants of weight 1 and (18) is a canonical form.

The roots of (19) are like or unlike in sign according as $J>0$ or $J<0$. If $J<0$ (18) is the equation of a hyperbolic cylinder. If $J>0$ (18) is the equation of the elliptic cylinder or has no locus according as $I\Theta>0$ or $I\Theta<0$.

Furthermore, if in (19) $I=0$, then we have $u^2+J=0$, from which it follows that if $J<0$ we have the hyperbolic cylinder but if $J>0$ there is no locus. We may now summarize the above by

THEOREM 7. *When* $J<0$ *equation* (18) *represents a hyperbolic cylinder and when* $J>0$ *equation* (18) *represents an elliptic cylinder or represents no locus according as* $I\Theta>0$ *or* $I\Theta\leqq 0$.

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[1] *Cf.* B. Niewenglowski, *Cours de Géométrie Analytique*, Gauthier-Villars, 1896, vol. 3, p. 272. The constant term of Θ is called by Niewenglowski the "*invariant des cylindres*," and is attributed to Darboux

CASE **II** (A) **2**(b). $D=0$, $J\neq 0$, $\Delta=0$, $\Theta=0$. Since $\Theta=0$ we have $d=0$ and (18) becomes

$$ax^2+by^2=0, \qquad ab\neq 0. \tag{20}$$

The invariants of (20) are $I=a+b$, $J=ab$, from which a, b are the roots of $u^2-Iu+J=0$. It is seen from this that the roots are like or unlike in sign according as $J>0$ or $J<0$. Hence we have

THEOREM 8. *If $J<0$ then* (20) *is the equation of two intersecting real planes, and if $J>0$ then* (20) *becomes the equation of a single real line* ($x=0$, $y=0$).

It is to be noted here that the case **II** (B) **1** for which $D=J=0$, $\Delta\neq 0$, is impossible since $D=0=J$ implies $\Delta=0$.

CASE **II** (B) **2**. $D=0, \Delta=0$, $J=ab=0$. We have $b=0$, and (14) reduces to

$$ax^2+2lx+2my+2nz+d=0, \qquad a\neq 0. \tag{21}$$

The transformation $x=x'-l/a$, $y=y'$, $z=z'$ reduces equation (21) to

$$ax^2+2my+2nz+d=0, \qquad a\neq 0. \tag{22}$$

The invariants now are

$$I=a, \quad J=D=\Delta=0, \quad \Theta=a(n^2+m^2).$$

Hence $\Theta=0$ when and only when $n=m=0$.

CASE **II** (B) **2**(a). $D=J=\Delta=0$, $\Theta=k\neq 0$. We have $(m,n)\neq(0,0)$. In (22) let us assume that $m\neq 0$, the other case being brought under this by the transformation obtained by setting in (1) $\phi=\pi/2$, $x=x'$, $y=-z'$, $z=y'$. The transformation $x=x'$, $y=y'$, $z=z'-(d/2m)$, reduces (22) to

$$ax^2+2my+2nz=0, \quad (m,n)\neq(0,0), a\neq 0. \tag{23}$$

Since $m^2+n^2\neq 0$, the transformation

$$y=\frac{m}{\sqrt{m^2+n^2}}y'-\frac{n}{\sqrt{m^2+n^2}}z', \quad z=\frac{n}{\sqrt{m^2+n^2}}y'+\frac{m}{\sqrt{m^2+n^2}}z',$$

is valid and reduces (23) to

$$ax^2+2my=0, \qquad am\neq 0. \tag{24}$$

The invariants of (24) are

$$I=a, \quad J=D=\Delta=0, \quad \Theta=am^2,$$

so that m is a root of $m^2I-\Theta=0$. Thus a and m are algebraic invariants of weight 1 and (24) is the canonical form of the parabolic cylinder.

CASE **II**(B) 2(b). $D=J=\Delta=\Theta=0$. We see that $\Theta=0$ implies that $m=n=0$, so that (22) becomes

$$ax^2+d=0\,,\qquad a\neq 0. \tag{25}$$

Here again we can no longer classify d in (25) by means of the invariants used hitherto, so that we must now resort to the covariant Φ. In this case $\Phi=ad=k$ so that $I=a$, $Id=\Phi$ and both a and d are algebraic invariants of weight 1. We now have

THEOREM 9. *If $\Phi>0$ equation* (25) *gives no locus, if $\Phi<0$* (25) *is the equation of two parallel planes, and if $\Phi=0$ it becomes $ax^2=0$ $(a\neq 0)$ which represents two coincident planes.*

In the case immediately above we found that when $D=J=\Delta=\Theta=0$, then $\Phi=k$. Conversely, if $\Phi=k$ then, since assuming $f=g=h=0$ implies no real restriction on $F=0$ (§5), we can use the invariants (6) and the expression for Φ in (4) and find that we must have the following conditions:

$$\begin{aligned} ab+ac&=0, &\qquad n(a+b)&=0,\\ ab+bc&=0, &\qquad m(a+c)&=0,\\ bc+ac&=0, &\qquad l(b+c)&=0. \end{aligned}\tag{26}$$

With the aid of the invariants of (6) we get by adding the first set of equations (26) that $J=0$. Also from the same set we have that $ab=-ac$ and hence $bc-ac=0$, so that we therefore get that $ac=0$, $bc=0$. Assume that $a\neq 0$ (§5) and thus we have that $b=0$ and $c=0$. Hence $D=0$ and $\Delta=0$ from (6). With the aid of the second set of equations (26) we find from (4) that $\Theta=0$. Hence we have

THEOREM 10.[1] *A necessary and sufficient condition that the second degree locus reduce to planes is that $\Phi=k$.*

We have now classified all the possible cases of the second degree loci of space according to their invariants and covariants alone. The following table summarizes the above results:

8. Summary.

I A. $D\neq 0, \Delta\neq 0$. *Ellipsoids and hyperboloids.*

1. $\Delta<0, J>0, ID>0$, ellipsoid.
2. $\Delta>0, J>0, ID>0$, no locus (imaginary ellipsoid).
3. $\Delta<0$, and $J\leqq 0$ or $ID\leqq 0$, hyperboloid of two sheets.
4. $\Delta>0$, and $J\leqq 0$ or $ID\leqq 0$, hyperboloid of one sheet.

B. $D\neq 0, \Delta=0$. *Cones.*

1. $J>0, ID>0$, point locus.
2. $J\leqq 0$ or $J>0$ and $ID\leqq 0$, real cones.

[1] Cf. Niewenglowski, *loc. cit.*, p. 274. Niewenglowski obtains the constant term of Φ and calls it "the invariant of two parallel planes."

II *A*. $D=0, J\neq 0$. *Paraboloids, cylinders, planes.*

1. $D=0, J\neq 0, \Delta\neq 0$.
   - (*a*) $J>0$, elliptic paraboloid.
   - (*b*) $J<0$, hyperbolic paraboloid.
2. $D=\Delta=0, J\neq 0$.
   - (*a*) $\Theta=k\neq 0$.
     - (1) $I\Theta>0, J>0$, elliptic cylinder.
     - (2) $I\Theta\leqq 0, J>0$, no locus.
     - (3) $J<0$, hyperbolic cylinder.
   - (*b*) $\Theta=0$.
     - (1) $J<0$, intersecting real planes.
     - (2) $J>0$, single line ($x=0, y=0$).

B. $D=0, J=0$. *Cylinders, planes.*

1. $D=0, J=0, \Delta\neq 0$. Vacuous.
2. $D=0, J=0, \Delta=0$.
   - (*a*) $\Theta=k\neq 0$, parabolic cylinder.
   - (*b*) $\Theta=0$.
     - (1) $\Phi=k\neq 0$.
       - (I) $\Phi>0$, no locus.
       - (II) $\Phi<0$, real parallel planes.
     - (2) $\Phi=0$, coincident planes.

QUESTIONS AND DISCUSSIONS

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. THE COMPLEX VARIABLE IN THE SOLUTION OF PROBLEMS IN ELEMENTARY ANALYTIC GEOMETRY.

By G. A. BINGLEY, St. John's College, Annapolis, Md.

It is the experience of every teacher of the freshman course in college mathematics, that his students regard the introduction of the complex variable with a certain amount of suspicion and incredulity. In this attitude the student is of course doing nothing more than repeating the reaction which the human race as a whole manifested not so many generations ago. The extension of the fundamental operations on real numbers to complex numbers does not have

sufficiently convincing applications in the material of the traditional course in freshman college algebra. The formal presentation of complex numbers from the point of view of operations on pairs of reals, although elegant, is pedagogically inadvisable. Instead of placing the complex variable on a basis logically irreproachable, it only serves to make it ridiculous in the eyes of the average freshman. The teacher, on the defensive, attempts to point out that in the higher analysis, and in theoretical physics, the complex variable is an invaluable instrument. Unfortunately, the freshman is not prepared to investigate such matters as conformal mapping and its applications to the theory of electricity, hydrodynamics, aeronautics, and the like.

It is perhaps worth while, therefore, to point out ways in which the complex variable may be made a useful tool in the solution of problems in elementary plane analytics. No further knowledge of complex numbers is required than that of the laws governing the fundamental operations, and DeMoivre's theorem. With no further acquaintance with complex numbers than is acquired in a freshman course it is frequently possible to solve problems in plane analytics more simply and more elegantly than by cartesian coordinates. The greater simplicity is in many cases so striking that one wonders that the texts on elementary analytic geometry have not called attention to this method long ago. There is undoubtedly a lamentable conservatism and inertia in the handling of the material of the elementary courses in mathematics. Of the many new texts on elementary analytic geometry appearing yearly scarcely one dares venture beyond certain well-trodden fields. It may be that the teachers share with the students a certain timidity toward imaginaries, hesitating to call them to their service until driven to it. Professor Oswald Veblen expresses it very well (Veblen & Young, *Projective Geometry*, vol. 2, p. 169) when he says, "Another criticism on current books is that they employ imaginary points in a rather shy and awkward manner. This is doubtless due to the fact that, ... the geometry of reals was regarded as having, somehow, a higher degree of validity than the complex geometry. The reader will often find it easy to abbreviate the proofs of theorems in the literature by a free use of imaginary elements."

This paper is not attempting an exposition of the interesting field of inversive geometry but applies only the most elementary properties of imaginaries to the solution of problems. Illustrations of the methods of inversive geometry as applied to analytic problems of a more advanced character, have appeared in this MONTHLY, as, for instance, an article by Professors Frank Morley and F. D. Murnaghan, "Note on Neuberg's Cubic," in the issue of October, 1925.

Instead of the cartesian coordinates of a point in a plane referred to perpendicular axes, we shall associate with each point a complex number as in the Argand diagram. The conjugate of a point a we shall designate as $\bar{a}$. If

$a=\rho(\cos\theta+i\sin\theta)=\rho e^{i\theta}$, then $\bar{a}=\rho(\cos\theta-i\sin\theta)=\rho e^{-i\theta}$ and $(a/\bar{a})=(\cos\theta+i\sin\theta)^2$. Giving the name t (a turn) to $\cos\theta+i\sin\theta$, we have

$$(a/\bar{a}) = t^2. \tag{1}$$

If a lies on the axis of reals, $a=\bar{a}$. Given two complex numbers a and b, then $a-b$ is represented by the line segment joining the points a and b. We then have

$$\frac{a-b}{\bar{a}-\bar{b}} = t^2. \tag{2}$$

That is, the quotient t^2 is a complex number of modulus unity (a turn) whose θ is twice the angle of inclination of the line segment ab. We shall call t^2 the *clinant* of ab. Its analogy to the "slope" of a line in cartesian geometry is obvious. If the clinants of two line segments are given, the quotient of the first by the second is called the *relative clinant* of the two line segments. The analogy to the cartesian formula for the angle between two lines in terms of their slopes is again obvious. With nothing more than this material it is possible to solve a great number of problems.

The problem which has been chosen to illustrate the method is one which has been going the rounds of naval circles for some years. Several officers in the construction corps had been attempting a solution, and finally sent the problem on to a member of the department of mathematics of the Naval Academy. Cartesian geometry applied by main strength will of course give a solution, but it was felt that there ought to be a simpler method of attack and that the solution ought to suggest a ready method of locating the points sought. The use of imaginaries suggested itself to me, and a complete and practicable solution was found. The problem is as follows:

Given any two complex points x and y lying within a circle whose center is o; required the points z on the circle, so situated that $\angle xzo=\angle ozy$. The restatement of the problem in terms of light reflection is obvious.

We shall consider the locus of points z, without at first introducing the condition that z lies on the circle. The clinants of zy, zo, zx are respectively: $(z-y)/(\bar{z}-\bar{y})$, $z/\bar{z}$, $(z-x)/(\bar{z}-\bar{x})$. Now if we find the relative clinant of zy and zo and equate it to the relative clinant of zo and zx, we have the equation:

$$z\bar{z}^2(x+y)-\bar{z}z^2(\bar{x}+\bar{y})-xy\bar{z}^2+\bar{x}\bar{y}z^2=0. \tag{3}$$

This is a cubic since it is of the third degree in z and $\bar{z}$, which are the coordinates. Further, it is a circular cubic since $z\bar{z}$ is a factor of the leading term. The expression $z\bar{z}$ is equivalent to the cartesian expression X^2+Y^2, as is seen from the substitution $z=X+iY$; $\bar{z}=X-iY$. Taking our circle as the unit circle we have as the equation of the circle $z\bar{z}=1$. It is interesting to note by way of digression that this equation regards the circle as the locus of double points of the involution of points inverse to each other with respect to the circle. Setting $z\bar{z}=1$ in equation (3) we have

$$\bar{z}(x+y)-z(\bar{x}+\bar{y})-xy\bar{z}^2+\bar{x}\bar{y}z^2=0. \tag{4}$$

Without loss of generality it is possible so to choose the axis of reals that xy is real, that is, by making the axis of reals bisect the angle xoy. Since xy is real, $xy=\bar{x}\bar{y}$ and (4) becomes

$$z^2-\bar{z}^2-z(\bar{x}^{-1}+\bar{y}^{-1})+\bar{z}(x^{-1}+y^{-1})=0. \tag{5}$$

This is a conic and passes through the center of the unit circle. If we transform this equation to cartesian coordinates by setting

$$z=X+iY, \qquad \bar{z}=X-iY; \qquad x^{-1}=r_1e^{i\theta_1}, \qquad y^{-1}=r_2e^{-i\theta_1}, \tag{6}$$

and making use of well-known relations between the sine, cosine, and the exponential function, we have[1]

$$2XY+X(r_1-r_2)\sin\theta_1-Y(r_1+r_2)\cos\theta_1=0. \tag{7}$$

This is seen to be a rectangular hyperbola whose center is the point (h,k), where

$$2h=(r_1+r_2)\cos\theta_1, \quad 2k=(r_2-r_1)\sin\theta_1. \tag{8}$$

It is more convenient to have the center expressed as the complex point:

$$(\bar{x}^{-1}+\bar{y}^{-1})/2. \tag{9}$$

That this is the point given by (8) the reader can easily verify.

The complex point given by (9) is found by constructing the points inverse to x and y, with respect to the unit circle and laying off half the vector sum (parallelogram law). These constructions are of a most elementary nature.

In brief, then, with (9) as origin, draw axes parallel to the axes of reals and imaginaries, the axis of reals bisecting the angle xoy; construct a rectangular hyperbola with these new axes as asymptotes and passing through the center of the unit circle. The real intersections of the hyperbola with the unit circle give the required points.

The analysis of inversive geometry lends itself admirably to the study of the Apollonian circles, the Euler line, the nine-point circle, the Simson line and other topics usually not taken up in a first course in analytic geometry. Inversive geometry seems the simplest method of approach to these topics and might very advantageously be taught concurrently with the synthetic treatment of the same topics. The student and teacher alike should see that imaginaries may be employed as analytic aids at a much earlier stage of mathematical study than is usually the case.

II. An Application of Fourier Series and a Theorem on Definite Integrals

By E. J. McShane, Tulane University

1. The results of this paper are of interest to an analyst only in that they contain a theorem on definite integrals. The primary intention was, however, to explain a certain physical phenomenon—the decrease of the power factor of the tungar rectifier under load.

Let us assume that we have two functions of the time, $e=f_1(t)$, $i=f_2(t)$, each periodic with the same period, and each of which is continuous and possesses a bounded first derivative. Choose the unit of time so that the period will be 2π.

Define

$$E=\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}e^2dt}, \quad I=\sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}i^2dt}, \quad P=\frac{1}{2\pi}\int_{-\pi}^{\pi}eidt, \quad \varphi=\frac{P}{EI}$$

[1] That the points sought are the intersection of a rectangular hyperbola with the unit circle was first communicated to me by Dr. F. D. Murnaghan of the Johns Hopkins University. I am also indebted to Dr. Murnaghan for equation (9).

The physicist will recognize in e the expression for an alternating E.M.F., in i that for an alternating current, and in E, I, P, and φ, the effective voltage, effective current, power, and power factor respectively.

Let us develop e and i into Fourier's series:

$$e = \frac{a_1}{\sqrt{2}} + a_2 \sin t + a_3 \cos t + a_4 \sin 2t + a_5 \cos 2t + \cdots$$
$$= \frac{a_1}{\sqrt{2}} + \sum_{j=1}^{\infty} (a_{2j} \sin jt + a_{2j+1} \cos jt), \tag{1}$$
$$i = \frac{b_1}{\sqrt{2}} + \sum_{j=1}^{\infty} (b_{2j} \sin jt + b_{2j+1} \cos jt) \cdot \tag{2}$$

Since $e=f_1(t)$ is everywhere continuous and possesses a bounded derivative, and $f_1(-\pi)=f_1(\pi)$, the series $\sum_{j=1}^{\infty} a_j$ is absolutely convergent; and since $|\cos jt| \leqq 1$, $|\sin jt| \leqq 1$, it follows that (1) converges absolutely and uniformly.[1]

The same remarks apply to series (2).

We may now multiply (1) by (2), term by term:

$$\begin{aligned} ei = \frac{a_1 b_1}{2} &+ \frac{a_1}{\sqrt{2}} \sum_{j=1}^{\infty} (b_{2j} \sin jt + b_{2j+1} \cos jt) \\ &+ \frac{b_1}{\sqrt{2}} \sum_{j=1}^{\infty} (a_{2j} \sin jt + a_{2j+1} \cos jt) \\ &+ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (a_{2j} b_{2k} \sin jt \sin kt + a_{2j}\, b_{2k+1} \sin jt \cos kt \\ &+ a_{2j+1} b_{2k} \cos jt \sin kt + a_{2j+1}\, b_{2k+1} \cos jt \cos k) \cdot \end{aligned} \tag{3}$$

The above series of coefficients is absolutely convergent and the series is therefore uniformly convergent[2] everywhere. Let us integrate it term by term between the limits $-\pi$ and π. Since all the integrals vanish except

$$\int_{-\pi}^{\pi} \tfrac{1}{2} dt = \int_{-\pi}^{\pi} \cos^2 jt dt = \int_{-\pi}^{\pi} \sin^2 jt dt = \pi,$$

we have

$$P = \frac{1}{2\pi} \int_{-\pi}^{\pi} ei dt = \tfrac{1}{2} \sum_{j=1}^{\infty} a_j b_j \cdot \tag{4}$$

Similarly

$$E^2 = \tfrac{1}{2} \sum_{j=1}^{\infty} a_j^2, \tag{5}$$
$$I^2 = \tfrac{1}{2} \sum_{j=1}^{\infty} b_j^2 \cdot \tag{6}$$

Since $\sum_{j=1}^{\infty} |a_j|$ and $\sum_{j=1}^{\infty} |b_j|$ converge, the three series above converge absolutely.

We have then

$$P^2 = \tfrac{1}{4} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (a_j b_j a_k b_k), \quad E^2 I^2 = \tfrac{1}{4} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j^2 b_k^2 \cdot$$

Since the terms may be rearranged in any desired order,

$$\begin{aligned} 8(E^2 I^2 - P^2) &= 2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (a_j^2 b_k^2 - a_j b_j a_k b_k) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \{ a_j^2 b_k^2 - a_j b_j a_k b_k) + (a_k^2 b_j^2 - a_j b_j a_k b_k) \} \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (a_j b_k - a_k b_j)^2 \cdot \end{aligned} \tag{7}$$

[1] Carslaw's *Fourier's Series and Integrals*, page 249.

[2] *Mathematical Analysis*, Goursat-Hedrick, page 351; *Crelle's Journal*, vol. LXXIX.

This leads immediately to the corollary $E^2I^2-P^2\geqq 0$, since every term on the right is zero or positive; i.e., for no physically possible wave forms of voltage or current can the power factor φ exceed unity. The special case $\varphi=1$ implies $e=ri$, where r is a constant. For in that case $a_jb_k-a_kb_j=0$ for every j and k; whence $a_j=rb_j$, or $e=ri$.

2. In order to apply the preceding results to the case of the tungar rectifier, we might mention first that the tungar rectifier is a device that, having impressed upon it an alternating voltage, allows the current to pass in one direction but reduces it almost to zero in the other. The following assumptions fit the physical data to a fairly high degree of approximation.

(*a*) The voltage is a pure sine wave ($a_j=0, j\neq 2$),

(*b*) At zero load, the current is also a pure sine wave in phase with the voltage ($b_2=B_2; b_j=0, j\neq 2$),

(*c*) If the load on the rectifier increase, then each b_j will be given by

$$b_2 = B_2 + \lambda\beta_2\ ; \quad b_j = \lambda\beta_j, \quad j \neq 2,$$

where each β_j is a constant and the factor λ is a monotonically increasing function of the load.

Referring to (7), we have

$$(a_jb_k-a_kb_j)^2=0 \text{ if } j=k=2 \text{ or if neither } j \text{ nor } k=2;$$
$$(a_jb_k-a_kb_j)^2=(a_2b_k)^2, \text{ if } j=2, k\neq 2;$$
$$=(a_2b_j)^2, \text{ if } j\neq 2, k=2.$$

Therefore

$$E^2I^2 - P^2 = \tfrac{1}{4}\sum_{j=1}^{\infty}(a_2b_j)^2 = \tfrac{1}{4}\,a_2^2\sum_{j=1}^{\infty}\lambda^2\beta_j^2 = \tfrac{1}{4}K^2\lambda^2\,. \qquad (j\neq 2)$$

If now we divide by P^2, replace (EI/P) by $(1/\varphi)$, and apply (4), we find

$$\varphi^{-2}-1 = \left(\frac{\lambda K}{a_2b_2}\right)^2 = \frac{K^2}{\dfrac{a_2^2B_2^2}{\lambda^2}+\dfrac{2a_2^2B_2\beta_2}{\lambda}+a_2^2\beta_2^2}\,.$$

Now φ, B_2 and β_2 must all be positive, since the contrary would lead to the result that the rectifier was returning energy to the source. We therefore see that as the load increases (*i.e.*, as λ increases), φ must decrease.

3. It is easy to prove, by use of equation (7), the following theorem[1] known as "Schwarz's inequality"

THEOREM: *If $y=f_1(x)$, $z=f_2(x)$, are continuous and possess bounded first derivatives in the interval (a,b), then*

$$\left[\int_a^b y^2dx\right]\left[\int_a^b z^2dx\right] \geqq \left[\int_a^b yzdx\right]^2$$

For, by applying the transformation $x=b+t(b-a)/\pi$, we find $y=F_1(t)$, $z=F_2(t)$, continuous and possessing bounded first derivatives in the interval

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[1] *Theory of Functions of a Real Variable*, Hobson, p. 534.

$(-\pi,0)$. Define $F_1(-t)=F_1(t)$, $F_2(-t)=F_2(t)$. Then $F_1(t)$ and $F_2(t)$ satisfy the conditions set upon the functions e and i above, whence

$$\left\{\int_{-\pi}^{0}\overline{F_1(t)}^2dt+\int_0^{\pi}\overline{F_1(t)}^2dt\right\}\left\{\int_{-\pi}^{0}\overline{F_2(t)}^2dt+\int_0^{\pi}\overline{F_2(t)}^2dt\right\}$$
$$\geqq\left\{\int_{-\pi}^{0}F_1(t)F_2(t)dt+\int_0^{\pi}F_1(t)F_2(t)dt\right\}^2.$$

Using the transformation $t=-r$ on each integral which has limits 0 and π, and then returning to x we have the theorem.

III. Concerning the Remainder Term in Taylor's Formula

By L. M. Blumenthal, University of Chicago

1. In Schlömilch's *Kompendium der Höheren Analysis* the following expression is obtained for the remainder term, after n terms, in Taylor's formula:

$$R_n=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)\,!}\cdot\frac{\varphi(b)-\varphi(a)}{\varphi'(a+\theta h)}F^{(n)}(a+\theta h)\,,$$

where $F(x)$ is the function, continuous with its first n derivatives on the interval (a,b) to which Taylor's formula is applied; $\varphi(x)$ is an arbitrary function, continuous and differentiable in the interval (a,b); $h=b-a$ and θ is a positive number less than unity.

In this note I propose to find by a sort of generalization of the method of Schlömilch, a more general expression for R_n which, for special cases, reduces to the well-known expressions for the remainder term, after n terms, given by Lagrange, Cauchy, and Roche, as well as to the form given by Schlömilch. A further extension of the results obtained will be indicated.

2. Let $f(x)$, $\varphi(x)$, $\psi(x)$ be three functions continuous and differentiable in the interval (a,b).

Then there exists[1] a value $x=\xi$ in the interval such that

$$\begin{vmatrix} f'(\xi) & \varphi'(\xi) & \psi'(\xi)\\ f(a) & \varphi(a) & \psi(a)\\ f(b) & \varphi(b) & \psi(b)\end{vmatrix}=0. \qquad (A)$$

Now we may substitute[2]

$$f(x)=F(b)-F(x)-(b-x)F'(x)-\frac{(b-x)^2}{2!}F''(x)-\cdots-\frac{(b-x)^{n-1}}{(n-1)\,!}F^{(n-1)}(x)-\cdots.$$

[1] *E. Pascal*: *Repertorium der Höheren Mathematik*, *Vol.* 1, p. 117.

[2] Schlömilch: *Kompendium der Höheren Analysis*, p. 237.

Differentiating, we find

$$f'(\xi) = \frac{-(b-\xi)^{n-1}}{(n-1)\,!}F^{(n)}(\xi).$$

Also

$$f(a) = F(b) - F(a) - (b-a)F'(a) - \frac{(b-a)^2}{2!}F''(a) - \cdots = R_n.$$

$$f(b) = 0.$$

Putting these values in (1),

$$\begin{vmatrix} \dfrac{-(b-\xi)^{n-1}}{(n-1)\,!}F^{(n)}(\xi) & \varphi'(\xi) & \psi'(\xi) \\ R_n & \varphi(a) & \psi(a) \\ 0 & \varphi(b) & \psi(b) \end{vmatrix} = 0.$$

Developing according to elements of the first column, and solving for R_n we obtain

$$R_n = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)\,!}\;\frac{\begin{vmatrix} \psi(a) & \varphi(a) \\ \psi(b) & \varphi(b) \end{vmatrix}}{\begin{vmatrix} \varphi'(\xi) & \psi'(\xi) \\ \varphi(b) & \psi(b) \end{vmatrix}}F^{(n)}(a+\theta h),$$

where we assume that

$$\begin{vmatrix} \varphi'(\xi) & \psi'(\xi) \\ \varphi(b) & \psi(b) \end{vmatrix} \neq 0$$

and where we have substituted $a+\theta(b-a)=a+\theta h$ for ξ, θ being a positive number less than unity.

3. Special cases.

(1) Let $\psi(x) \equiv$ a constant (not zero), then

$$R_n = \frac{h^{n-1}(1-\theta_1)^{n-1}}{(n-1)!}\cdot\frac{\varphi(b)-\varphi(a)}{\varphi'(a+\theta_1 h)}F^{(n)}(a+\theta_1 h),\quad 0<\theta_1<1,$$

which is Schlömilch's form.

(2) Let $\psi(x) \equiv$ a constant (not zero) and $\varphi(x) \equiv (b-x)^r$, then

$$R_n = \frac{h^n(1-\theta_2)^{n-r}}{r\times(n-1)\,!}F^{(n)}(a+\theta_2 h),\quad 0<\theta_2<1,$$

which is due to Roche.[1]

[1] *Mem. de L'Acad. de Montpellier* (1858).

(3) If, in (2), we let $r \equiv n$ we obtain

$$R_n = \frac{h^n}{n\,!} F^{(n)}(a+\theta_3 h), \quad 0<\theta_3<1,$$

due to Lagrange.

(4) If we let $r=1$ in (2), we obtain

$$R_n = \frac{h^n(1-\theta_4)^{n-1}}{(n-1)\,!} F^{(n)}(a+\theta_4 h), \quad 0<\theta_4<1,$$

due to Cauchy.

By generalizing (A) a remainder term containing k arbitrary functions may be obtained.

So far as the writer knows, the formula given in this paper is not to be found in the literature.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Elements of Physical Biology. By A. J. LOTKA, Baltimore, Williams and Wilkins Co., 1925. xxx+460 pages. Price $6.00.

Let x be the total mass contained in the bodies of a given species of animals growing in an environment which is practically unchanging except for the change in x itself. Then the rate dx/dt of change of x with respect to the time will be some function $f(x)$ of x, so that we have the differential equation

$$\frac{dx}{dt} = f(x).$$

We shall suppose that $f(x)$ may be expanded in a power series in x. In a state of equilibrium we have $dx/dt=0$, and hence $f(x)=0$. Now the species is in equilibrium when $x=0$ since at least one female is required to start the growth of a population. Hence $f(0)=0$, so that $f(x)$ can be expressed in the form $f(x)=\alpha x+\beta x^2+\cdots$. Now the mass x can not increase indefinitely. Hence we may assume a second equilibrium state so that $f(x)$ shall vanish for another value of x besides $x=0$. Let μ be such a value. Then we naturally write $f(x)=ax(\mu-x)g(x)$. As the simplest example of such a function $f(x)$ Lotka (*p*. 65) takes the function $f(x)=ax\,(\mu-x)$. Then the differential equation for the growth of the species becomes

$$\frac{dx}{dt} = ax(\mu - x).$$

The general solution of this equation is $x=\mu/(1-Ce^{-a\mu t})$, where C is the constant of integration. This affords a formula for the law of growth of a species under the named conditions, provided that the hypotheses employed in its derivation are legitimate. It contains the three constants a, μ, C to be determined empirically.

This treatment affords one of the simplest instances of the applications of differential equations to the problems of biology as the theory is developed in the book under review. In the seventh chapter the author shows by a number of examples how this formula is applicable to actual biological conditions, determining the constants for each case. The first example is that of the growth of population in the United States, the number of persons in the population being taken as a measure of the mass x.

When there are two species living in mutual interdependence, as in the case of an animal species and one of its parasites for instance, and the conditions are practically uniform except for the change in mass of these two species themselves, one obtains the following system of equations:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots, \quad \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots$$

A partial analysis of these equations is made by Lotka in his eighth chapter and the results are applied in the study of quantitative epidemiology. Various types of cases arise and the problem on the whole is a difficult one. The analysis of some cases is carried far enough to show that this method leads to equations in fairly satisfactory agreement with such data as are available for checking them.

These examples will serve to indicate the method of attack upon biological problems here employed by Lotka. A considerable part of the work is based on rather general differential equations; and the author has "striven to infuse the mathematical spirit also into those pages on which symbols do not present themselves to the eye." His book is the outcome of a systematic attempt to carry into biology a method of treatment like that which has been so successful in theoretical mechanics.

In physical chemistry we have a development of the laws governing the sequence of events which we speak of as chemical reactions; we may call them the laws of evolution of chemical systems. "This sequence of events may be described as the progressive redistribution of matter among competing chemical species, elements and compounds." A systematic study of this sequence results in physical chemistry. "Can biological evolution, which is the progressive change in the distribution of matter among competing species of organisms be $\cdots$ brought under the scope of physical law?" Lotka answers this question

in the affirmative; and his detailed reasons for the answer result in what appears to be a new chapter of science. To it he gives the name of physical biology. Throughout it is infused by the mathematical spirit; and the guiding ideal is the general form of classical dynamics. Such an injection of the mathematical method into biology must be of considerable interest to mathematicians. How far the partial successes already attained can be extended remains to be seen. The method is at least a promising one, and it has been launched with a skill which rests on the requisite knowledge of mathematics.

The fundamental equations of the kinetics of evolving systems (including the evolution of living species) are written in the form

$$\frac{dx_i}{dt} = F_i(X_1,X_2, \cdots , X_n ; P_1,P_2, \cdots , P_j ; Q_1,Q_2, \cdots , Q_k) ; i = 1,2, \cdots , n.$$

Here the x_i denote the masses of certain components (biological species, for instance) denoted by $S_1, S_2, \cdots, S_n$; the $P_1, P_2, \cdots, P_j$ relate to measurable elements of topography, climate, and other general environmental conditions; the $Q_1, Q_2, \cdots, Q_k$ are parameters defining the characters of the several components S, consisting in part of parameters relating to frequency distribution. The author adds: "To read these equations in their broadest interpretation we must be prepared to consider cases in which the phenomenon of lag or lead enters." It is in connection with these phenomena that his system finds a possible place for memory and will and other features of consciousness.

In the present state of empirical knowledge the author can not get far with these equations in their general form; but he does draw some significant conclusions from them (see pages 57–63). Their principal value lies in the general method of attack involved in them and in the special cases which are amenable to a more detailed analysis. The partial successes already attained in this volume afford grounds for the hope that this general point of view will lead to further developments of an important character. We have here not only a contribution to biology but also *the foundations of a new chapter in applied mathematics.*

R. D. Carmichael.

A Correction—In the review (*1926*, 332) of Brink's *Analytic Geometry* there appeared the comment, "It seems strange that the existence of a second focus and directrix in the ellipse and hyperbola should not be mentioned". Article 89 of the book, page 177, which is entitled "The second focus of a central conic", mentions and explains the existence of the second focus and directrix of the ellipse and hyperbola, and, for curves in standard position, gives the coordinates of the foci and the equations of the directrices.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3212. Proposed by A. A. Bennett, Lehigh University.

The following scheme portrays an example in ordinary "exact" long division. The symbol, x, merely indicates the presence of a digit. The divisor appears upon the left and the quotient on the right. A letter, such as a, b, c, etc. denotes a digit, and when one of the letters, (not x) is used in two places, this signifies that the same digit is to be used in both places. However, distinct letters do not necessarily denote distinct digits.

Find the divisor and quotient and prove that your solution is the only one possible.

```
x x x x x a b x x x x ) x x x x x x x x x x x x x c c f x ( x c x x x x x
                        b x x x x x x x x x x
                        ----------------------
                          x x x x x x x x x x x x
                          x x x x b d a x e f x x
                          ------------------------
                            x x x x x x x x x x x c
                            x x x x x x x x x x x e
                            ------------------------
                              x x x x x x x x x x b c
                              x g g g g g g g g g g d
                              ------------------------
                                x x x x x x x x x x a f
                                x x x x x x x x x x x d
                                ------------------------
                                  x x x x x x x x x x b x
                                  x x x x x x x x x x b x
                                  ------------------------
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3213. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Prove the proposition: If AB, CD are two harmonic segments, the harmonic conjugate of the middle point of AB with respect to the couple C, D is identical with the harmonic conjugate of the middle point of CD with respect to the couple A, B. Generalize.

3214. Proposed by the late Laenas G. Weld.

A block sliding without friction and a ball rolling without friction start together down an inclined plane with the same initial velocity: Determine their subsequent relative motion.

3215. Proposed by R. M. Mathews, University of Illinois.

When a quadrangle is inscribed in a central conic so that two of its opposite sides pass through the foci, then the tangent pairs at points one on each of these sides meet on the bisectors of the angles formed by the sides.

When a quadrangle is inscribed in a central conic so that two of its opposite sides are symmetric with respect to the bisectors of the angle subtended at their intersection by the foci, then the tangents at point pairs, one on each of those sides, meet on the said bisectors.

Dualize. Also modify for parabola.

3216. Proposed by S. A. Corey, Des Moines, Iowa.

Let L, M, and N be any three unit vector space co-ordinates. Also let X, Y, and Z be three other vector space co-ordinates such that $X=a^2pL+acmM+c^2nN$, $Y=b^2pL+bdmM+d^2nN$, and $z=2abpL+(ad+bc)mM+2cdnN$, a, b, c, d, p, m, and n being ordinary scalars. Then prove that 4(tensor of X) (tensor of Y) $(\cos \widehat{XY})-(\text{tensor of } Z)^2=(ad-bc)^2(4\,pn\cos\widehat{LN}-m^2)$.

CORRIGENDUM: To rectify two errors, contributors to this department are asked to re-number the problems proposed since October, 1925. All numbers from **3144**[**1925, 433**] to **3180**[**1926, 159**] should be increased by four. All numbers from **3173**[**1926, 228**] to **3193**[**1926, 338**] should be increased by twelve. The problems proposed in the August-September number, 1926, begin with **3206** as they should.

EDITOR.

SOLUTIONS

2841 [1920, 274; 1925, 482]. Proposed by William Hoover, Columbus, Ohio.

The square number

$$9\frac{49}{64}=9+\frac{49}{64}=3^2+\frac{7^2}{8^2}\text{ is of the type form } k^2+\frac{(2k+1)^2}{(2k+2)^2};$$

how may the forms of the terms of the fractional part be determined deductively?

Generally, required that

$$k^2+\frac{\{\phi_1(k)\}^2}{\{\phi_2(k)\}^2} \tag{1}$$

be a perfect square, show how $\phi_1(k)$ and $\phi_2(k)$ may be found.

SOLUTION BY W. E. ROTH, West Allis, Wisconsin.

The problem does not imply that k shall be restricted to integral values so in the following we shall assume that k is a rational number.

Since $\phi_1^2(k)$ and $\phi_2^2(k)$ in (1) shall give square numbers for all rational values of k, it is evident that $\phi_1(k)$ and $\phi_2(k)$ must be polynomials in k with rational coefficients and that $\phi_2(k)$ must not be identically zero. The expression (1) takes on the form

$$\frac{k^2\phi_2^2(k)+\phi_1^2(k)}{\phi_2^2(k)}, \tag{2}$$

where the denominator, $\phi_2^2(k)$, is a perfect square for any arbitrary choice of the rational polynomial, $\phi_2(k)$, not identically zero; it remains then to determine what restrictions must be imposed upon $\phi_1(k)$ and $\phi_2(k)$ to make the numerator a perfect square also. Asssume that

$$k^2\phi_2^2(k)+\phi_1^2(k)\equiv M^2(k), \tag{3}$$

where $M(k)$ must be a rational polynomial in k; in other words, the numerator in (2) must be expressible in $M(k)$ as given above. From (3) we have

$$\begin{aligned}\phi_1^2(k) &\equiv M^2(k)-k^2\phi_2^2(k)\\ &\equiv [M(k)+k\phi_2(k)]\,[M(k)-k\phi_2(k)]\end{aligned} \tag{4}$$

Now since $\phi_1(k)$ must be a rational polynomial in k, it is evident that the factors of the right member of the identity (4) must be reducible as follows

$$M(k)+k\phi_2(k)\equiv R^2S,\quad M(k)-k\phi_2(k)\equiv ST^2, \tag{5}$$

where R, S, and T are entirely arbitrary polynomials save that no one of them is identically zero and that R and T are not identically equal. The restriction upon $\phi_2(k)$ and $M(k)$ as given in (5) is a necessary one in order that $\phi_1(k)$ be a rational polynomial. From (5) we obtain

$$M(k) \equiv \frac{S(R^2+T^2)}{2}\,; \quad \phi_2(k) \equiv \frac{S(R^2-T^2)}{2k} \tag{6}$$

and from (4) that

$$\phi_1(k) \equiv \pm RST, \tag{7}$$

where the double sign in the latter may be dropped since R, S, and T are arbitrary. For symmetry in the final result and to eliminate the k from the denominator of $\phi_2(k)$ in (6), we may let $R \equiv 2kU+T$, where U is now an arbitrary rational polynomial in k and thus obtain

$$\phi_1(k) \equiv ST(2kU+T), \quad \phi_2(k) \equiv 2SU(kU+T), \tag{8}$$

where S, T, and U are arbitrary rational polynomials in k no one of which is identically zero and where kU is not identically equal to $-T$. Thus the conditions given above are necessary and sufficient; that the latter is the case may be readily seen by substituting the values in (8) into the expression (2).

If the terms of the fractional part in (1) shall have no common factor in k, then we put $S \equiv 1$ and require that the polynomials T and U have no common factor in k and that T shall not have k as a factor in the identities as given in (8).

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor W. C. GRAUSTEIN has presented to the Société Mathématique de Belgique the amount of the prize that he received from the Royal Academy of Belgium for his memoir entitled "Methodes invariantes dans la géométrie infinitesimale des surfaces" with the request that it be used to found a prize. This prize, which will be known as the Prix Graustein, will be awarded for an important contribution to infinitesimal geometry; competition will be restricted to the United States and Belgium. Competing memoirs should be sent to the Société Mathématique de Belgique, rue d'Egmont 11, Brussels, before October 15, 1927.

Professor HERMANN WEYL of Zurich, a delegate from Switzerland to the International Congress of Philosophy at Cambridge, Mass., recently visited several of the universities of the east and middle west. While at the University of Michigan he gave lectures on the "Rôle of infinity" and "Modern concepts of gravitation."

It is announced that *The Reorganization of Mathematics in Secondary Education*, the Report of the National Committee on Mathematical Requirements, is out of print and is therefore no longer available for distribution. Two large editions, aggregating 25,000 copies, of this report have been distributed since its publication in 1923. Since the supply became exhausted last March requests for this Report have continued in considerable numbers. The trustees of the association have accordingly appointed a committee to investigate the possibility of issuing a new edition.

TWO NOTABLE GIFTS TO THE ASSOCIATION

I. THE CARUS MATHEMATICAL MONOGRAPHS.

The entire expense for publishing and distributing these MONOGRAPHS is provided by MRS. MARY HEGELER CARUS as a gift to the ASSOCIATION. The sale of these books at cost to its members by the ASSOCIATION is thus made possible, and the receipts from such sales are used to build up an endowment fund of the ASSOCIATION to be known as the CARUS PUBLICATION FUND. Hence, when a member buys a CARUS MONOGRAPH he not only gets full value at minimum cost but he also contributes to a fund which will ultimately be of the utmost value to the ASSOCIATION. Can any member show good reason for not rendering this service to the ASSOCIATION? The first and second Monographs are still available to members at the cost price.

II. THE RHIND MATHEMATICAL PAPYRUS.

CHANCELLOR ARNOLD BUFFUM CHASE, of Brown University, who has repeatedly shown his vital interest in the Association by cash contributions to its depleted budget, has now made a notable gift which was fully explained in the last issue of the MONTHLY. He has done the ASSOCIATION signal honor by publishing at great expense his RHIND MATHEMATICAL PAPYRUS under its auspices. The entire receipts from the sale of this work will be devoted to an endowment fund of the ASSOCIATION to be known as the ARNOLD BUFFUM CHACE FUND. Individuals and institutions not now members of the ASSOCIATION may secure the special rate to members by making application for membership before the sale begins, on or about January 1, 1927.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7–8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Annapolis, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

THE AMERICAN MATHEMATICAL MONTHLY

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MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The nineteenth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held on Saturday, May 8, 1926, at the Johns Hopkins University, Baltimore, Md., the morning session opening at 11 A.M. and the afternoon session at 2 P.M. Those attending the meeting were guests of the University at luncheon. Chairman W. D. Lambert presided at both sessions.

There were 64 present, including the following 51 members of the Association: O. S. Adams, Katherine S. Arnold, R. N. Ashmun, H. G. Avers, Clara L. Bacon, Helen Barton, W. W. Bigelow, G. A. Bingley, R. F. Borden, C. C. Bramble, J. A. Bullard, P. Capron, G. R. Clements, A. Cohen, C. W. R. Crum, A. Dillingham, J. A. Duerksen, H. English, H. Gwinner, J. Hall, W. M. Hamilton, S. C. Harry, Bertha I. Hart, L. S. Hulburt, W. D. Lambert, A. E. Landry, Florence P. Lewis, F. Morley, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, B. C. Patterson, E. C. Phillips, G. Y. Rainich, O. Ramler, J. N. Rice, A. W. Richeson, H. M. Robert, H. A. Robinson, R. E. Root, G. A. Ross, J. T. Spann, T. H. Taliaferro, A. A. Tomelden, M. M. Torrey, J. Tyler, P. S. Wagner, W. J. Wallis, Elizabeth W. Wilson, F. I. Winant, E. W. Woolard.

The program follows, accompanied by abstracts of the papers:

1. "Percircular rational curves," by Professor E. C. PHILLIPS, Georgetown College Observatory.
2. "Inversive coordinates," by Professor FRANK MORLEY, Johns Hopkins University.
3. "Dividing a circle into any number of equal parts graphically," by Professor HARRY GWINNER, University of Maryland.
4. "A problem connected with the ellipse," by Professor F. D. MURNAGHAN, Johns Hopkins University.
5. "The complex variable in the solution of problems in elementary geometry," by Professor G. A. BINGLEY, St. Johns College.
6. "Families of circles representing moving points," by Dr. G. Y. RAINICH, Johns Hopkins University.
7. "Statistics as an aid in secondary school administration," by Miss ELIZABETH W. WILSON, Central High School, Washington, D. C.
8. "A special integrator," by Mr. H. A. ROBINSON, Johns Hopkins University.

9. "A substitution operator," by Professor JOHN TYLER, U. S. Naval Academy.

10. 'Three transcendental functions derived from difference equations and related to the gamma function," by Dr. O. S. ADAMS, U. S. Goast and Geodetic Survey.

11. "A course in analytic geometry, special methods used," by Professor KATHERINE S. ARNOLD, Hood College.

12. "A problem in engine vibrations," by Professor R. E. ROOT, U. S. Naval Academy.

1. The name "percircular curve" is used to designate a curve which goes through the circular points at infinity as often as possible, that is, k times for a curve of order $2k$ or $2k+1$; familiar examples beside the circle itself are the circular cubic, the limaçon, and cardioid. The case of rational curves was considered; the general equation of such curves was expressed in parametric form and a study made of the singularities of such curves in the finite part of the plane. A consideration of these singularities enables us to establish a convenient relationship between proper rational percircular curves and systems of k circles or of k circles and a straight line according as the rational curve is of order $2k$ or $2k+1$ respectively. This relationship further enables us to give the general appearance of the various percircular curves of any chosen order by deforming into a continuous curve the system of circles or of circles and a straight line just referred to. (Preliminary report.)

2. For four points x_1, x_2, x_3, x_4 of a plane we consider the difference products $\pi_1=(x_2-x_3)(x_1-x_4)$, $\pi_2=(x_3-x_1)\ (x_2-x_4)$, $\pi_3=(x_1-x_2)\ (x_3-x_4)$. These we regard as 3 strokes forming a representative triangle, with angles θ_i and sides r_i. These angles, or their cotangents, as may be convenient, are called the homographic coordinates of any one of the points (say x_4) with regard to the other three. When the representative triangle is flat, so that its angles are in some order 0, 0, π, we take instead the sides r_i as homographic coordinates of x_4 which is then a point of the circle x_1, x_2, x_3. We may take the images of x_4 in the sides of $x_1x_2x_3$ as the representative triangle. The connection of the affine coordinates r_i of a point x_4 and its homographic coordinates is given by 3 equations such as $kr_1=(c_2+c_3)(\cot\theta_1+c_1)$ where c_i is the cotangent of an angle of the given triangle $x_1x_2x_3$.

3. An investigation of the accuracy of the following construction used in mechanical drawing and in machine design when laying out gear teeth would be very interesting. In the given circle divide the diameter AB into the same number of equal parts as that into which it is desired to divide the circle. With A and B as centers and a radius equal to AB, describe intersecting arcs. Let the point of intersection be D. Through D and the second

point of division on the diameter pass a line, as DE (E being the further intersection of the line and the circle). The chord AE will give one of the divisions desired.

4. Dr. Murnaghan considered the problem of finding the limiting point reached when one starts at any point on an ellipse and joins this point to a focus, and joins the second intersection of this chord and the ellipse to the other focus, and so on, *cf.* Problem **3167** of this MONTHLY. The mapping $z=t+(a^2/t)$ sends the unit circle $|t|=1$ into an ellipse and if the various points on the unit circle corresponding to the points on the ellipse are, in order, t_1, t_1', t_2, t_2', etc. it is readily found that

$$\frac{t_2+1}{t_2-1}=q^2\frac{t_1+1}{t_1-1}\text{ where } |q|<1.$$

Hence

$$\frac{t_n+1}{t_n-1}=q^{2n}\frac{t_1+1}{t_1-1}\text{ so that }\operatorname*{lt}_{n\to\infty} t_n=-1.$$

The limiting point on the ellipse is an end of the major axis.

5. Professor Bingley's paper appears in this MONTHLY (*1926*, 418).

6. The representation of point events (an event being given by two coordinates x, y and the time t) with the aid of directed circles in the plane was discussed. A continuous series of events, *i. e.*, a particle in motion, is represented by a family of such circles; collision between two particles—by two families which have one circle in common. The "distance" or "interval" between two events is represented by the distance between the corresponding circles measured along the common tangent. This leads in a natural way to the formula for the interval with one negative and two positive squares.

7. Distribution and nativity maps aid in location of school sites and Americanization program respectively. Comparison of statistics showing percent of each mark given by each teacher aids both teacher and rating officials. Non-conformity to probability curve shown in (1) percents of marks in different subjects, (2) in averages of graduates (computed for college certification), (3) in averages of undergraduates (computed for admission to the honor society) and (4) in the IQ's of entrants, makes marking on basis of normal curve unjust. Insignificant correlation is found between IQ's and achievement ($r=.38$) and between absence and failure.

8. Mr. Robinson demonstrated an instrument that would draw a certain class of curves, namely those which have the property that when a circle rolls upon them a point connected with the circle will trace out a circle in the plane over which the rolling circle moves. These curves are in general given analytically by elliptic integrals. Adjustments can be made on the instru-

ment so that the graph of any particular elliptic integral of the first or third kind can be found.

9. Professor Tyler discussed the properties and applications of the operator e^{PD_x}, where P is a function of x and D_x is the differential operator d/dx. This operator e^{PD_x} has the property of producing a substitution. Thus $e^{PD_x} \cdot f(x) = f(\lambda)$, where λ is a function of x, for example, $e^{\log y\, x D_x} \cdot f(x) = f(xy)$. The relation between P and λ is given by the functional equation

$$\int \frac{d\lambda}{P(\lambda)} - \int \frac{dx}{P(x)} = 1.$$

Various illustrations were given as to how this operator could be used to solve functional equations.

10. The three difference equations

$$F(x + 1) = x^x F(x), \; G(x + 1) = \Gamma(x)G(x), \; f(x + 1) = \sin x f(x),$$

can be solved very elegantly by passing to a sufficiently high derivative of the logarithms of the equations. This device is also sometimes used in the derivation of the gamma function itself. The F function was first derived by Kinkelin in *Crelle's Journal*, vol. 57; the G function was first derived by Alexeiewsky and later by Barnes in the *Quarterly Journal*, vol. 31. Kinkelin used the sumbol G for his function but the symbol F was used in this paper to distinguish it from the Barnes function. Besides these three functions, the paper included a fourth function defined by the difference equation $Z(x+1) = F(x)Z(x)$, in which the F function is the Kinkelin function.

11. The use of lists of extra credit problems in connection with a course in analytic geometry was discussed. Miss Arnold has used such lists of problems very successfully and has not sectionized her classes.

12. Professor Root discussed the problem of finding the frequency of free torsional vibrations of a shaft with a reciprocating engine at one end and a rotating mass at the other end. The vibrations of a shaft with two rotating bodies rigidly attached are given by linear differential equations, but the reciprocating parts of the engine introduce the square of the first derivative and bring into the coefficients periodic functions of the dependent variable. For certain multicylinder engines the lower harmonics of the periodic coefficients balance out, and the problem is solved approximately by linear differential equations with constant coefficients.

The following officers were elected for the coming year: Chairman, J. A. Bullard, U. S. Naval Academy; Secretary-Treasurer, J. R. Musselman, Johns Hopkins University; Members of Executive Committee, E. C. Phillips, Georgetown University, and T. H. Taliaferro, University of Maryland.

J. A. Bullard, *Secretary-Treasurer.*

THE ORIGIN OF THE TERM "ALGEBRA"

By SOLOMON GANDZ, Rabbi Isaac Elchanan Theological Seminary, New York City

1. A small volume might be filled with an enumeration of the various attempts to translate and explain the meaning of the word "algebra". Only a few examples, taken at random from the current reference books, need be given in order to illustrate the uncertainty and lack of clearness in the use of this mathematical term.

The word first occurs in the phrase *"al-jebr w'al muqâbalah,"* this being the title of the first book of its type, written by Mohammed ibn Mûsâ al-Khowârizmî (*c.* 825). Because of this double title, all the explanations that have been advanced contain a comment upon the second word, "al-muqâbalah," as well as the first one. For example, consider the following instances:

(1) Murray's *English Dictionary*, under "Algebra":

"The redintegration or reunion of broken parts; hence the surgical treatment of fractures, bone-setting." 'Ilm al-jebr wa'l-muqâbalah" = science of redintegration and equation (opposition, comparison, collation)."

(2) *Encyclopaedia Britannica:* "Transposition and removal."

(3) Lane, *Arabic English Dictionary*, (I, 374): "Perfective addition and compensative subtraction," or "restoration and compensation."

(4) Rosen, in his edition of the *Algebra* of Mohammed al-Khowârizmî, English text, p. 3, translates: "Completion and reduction." In his notes, pp. 177-186. there is an extended discussion of the subject, based upon certain old Arabic authorities, and reading about as follows: "Al-jabr = restoration of a broken bone; in mathematical language: removal of the negative quantity. Al-muqâbalah = to put two things face to face, to confront or compare; in mathematical language, to reduce the equation by removal of two positive quantities which are equal on both sides." This is also the way in which al-Khowârizmî uses the two expressions. For the restoration of a fraction by means of a multiplication al-Khowârizmî uses the verb "ikmâl," and for the reduction by means of a division he uses the verb "radd."

(5) Cantor, *Vorlesungen zur Geschichte der Mathematik* (I (1), 619, 620), quotes the usual Latin translation: "restauratio et oppositio," which he renders in the German: "Herstellung und Gegenüberstellung." From the fact that al-Khowârizmî used these words without any explanation, this eminent historian of mathematics draws the conclusion that they must have been traditional terms, well known a long time before they appeared in the book itself.[1]

[1] This conclusion is reasonable as well as logical and will serve to support the writer's own suggestion as given later in this paper.

(6) The discussion given by Professor Smith in his *History of Mathematics* (II. p. 386–392) is also very instructive. He quotes the explanation given by Behâ Eddîn (1600) in his *Kholâsat al-Hisâb*[1], and interprets it very clearly as follows: "*al jabr* has as the fundamental idea the transposition of a negative quantity, and *muqâbalah* the transposition of a positive quantity and the simplification of each number." Smith also gives an interesting list of the distortions of the terms *al-jabr w'al-muqâbalah* by the Latin translators (p. 390), and is the first to suggest[2] the comparative use of the other Semitic languages for the explanation of *al jabr* (p. 388, note 2).

(7) Among the Arabic authorities, al-Karkhî (*c.* 1020), offers an explanation which differs from the traditional one, but his idea has had two quite different interpretations. Woepcke[3], referring to his *Fakhrî*, states that *al-jabr* is the removal of the negative and positive members of the equation, while *al-muqâbalah* simply means: to set up the equation. Ruska,[4] however, referring to the definition given in the book *Al Kâfî*, relates that the operations of removing the negative quantities and the fractions is meant by *al-jabr*, while the removal of the equal positive quantities is indicated by *al-muqâbalah*. Since the Arabic texts of *Fakhrî* and *Al-Kâfî* are not yet edited and available, the writer is not able to decide which relation is the correct one.

(8) Later Arabic authorities, of the 14th and 15th centuries, introduce the terms *al-jabr w'al-ḥaṭṭ* instead of *al-jabr w'al-muqâbalah. Al-jabr* means completion, *i.e.*, the removal of the negative quantities and fractions by the two operations $a+x=b$, $a \cdot x=b$; while *al-ḥaṭṭ* means the reduction, *i.e.*, the removal of the equal positive members, and of the multiplying factors by the two operations[5] $a-x=b$, $(a/x)=b$.

(9) Finally Ruska, in his *Zur ältesten arabischen Algebra*, pp. 5–14, devotes a whole chapter to this question. He realizes a great many of the difficulties, gives a full account of the literature on the subject, and also sets forth the usage of those terms by the old Arabic mathematicians. His own translation of the expression,—*Ergänzung und Ausgleichung* (completion and compensation) does not, however, bring us any nearer to the solution of the problem.

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[1] Arabic text, ed. Nesselmann, p. 41.

[2] This paper owes its inception to that suggestion.

[3] See *Extrait du Fakhrî*, pp. 7, 63-64; compare also Smith, *History*, I, 283–284.

[4] *Zur ältesten arabischen Algebra*, Heidelberg 1917, pp. 13–14; compare also Hochheim, *Al-Kâfî fîl Ḥisab*, III, p. 10.

[5] See Carra de Vaux in *Bibliotheca Mathematica*, 1897, pp. 1–2; Ruska, *l.c.*, pp. 11–12. A similar explanation of *al-jabr* is also given by Ibn Khaldûn (1332–1406), in his *Muqaddamah*, ed. Beirut, 1886, p. 422; see Ruska, *l.c.*, p. 14.

2. Before proposing his own solution the writer wishes to set forth all of the difficulties, as follows:

(1) *Jabara* apparently has no etymology in the Arabic language; "to set a broken bone" is a derived meaning. We are at once confronted by the question, Why should we use an artificial surgical term for a mathematical operation, when there are such good plain words as *zâda* and *tamma* for the operation of addition and completion?

(2) *Muqâbalah* means "putting face to face, confronting, equation," and the question arises as to the reason for giving to it the meaning of the special operation of removing the equal positive members.

(3) Why should the names of these two operations give the name for the whole science of equations?

(4) There are still remnants in the mathematical literature suggesting that in olden times the term *al-jabr* alone was used for the science of equations, and the term *al-jabriyyun* was taken for the masters of algebra.[1] On the other hand the term *al-muqâbalah* alone, according to its real meaning of "putting face to face, confronting, equation," seems to be the most appropriate name for equations in general. With these difficulties in mind, the writer undertook to search out the real meaning of *jabara* in the related Semitic languages. Now the Assyrian name *gabrû-maḫâru* means to be equal, to correspond, to confront, or to put two things face to face; see Delitzsch, *Assyrisches Handwörterbuch*, under *gabru* and *maḫâru*, pp. 193, 401, and Muss-Arnolt, *Assyrian Dictionary*, under *gabru and maxaru*,[2] pp. 210, 525. From the first of these we have the etymology of the Hebrew *geber* and *gibbôr*. *Geber* is the mature man leaving the state of boyhood and being *equal in rank and value* to the other men of the assembly or army. *Gibbôr* is the hero who is strong enough *to fight and overcome his equals and rivals* in the hostile army. *Gabara* =*jabara*, in its original Assyrian meaning, is therefore the corresponding name for the Arabic *qâbala* (verbal noun *muqâbalah*), and an appropriate name for equations in general.

3. The Egyptians[3] knew and wrote books on algebra as early as 1600 B.C., and it would be very strange if the Assyrians, having the same level of culture as the Egyptians and having close political and economic relations with them, were quite ignorant of this art. *Gabr* must have been the original Assyrian form of the word. The Arabs received this ancient science, with its original Assyrian name (in Arabic pronunciation *al-jabr*) from the Aramaeans

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[1] See Omar al-Khayyâm, ed. Woepcke, Arabic text, pp. 2, line 7; p. 4, l. 14; p. 5, ll. 6, 17; p. 6, l. 8; p. 7, l. 7, and Ruska *l.c.*, p. 13, seq.; p. 29, note 1.

[2] The latter transcribes *maxaru* instead of *maḫaru*.

[3] Smith, *History of Mathematics*, I, 47 seq.; II, 370.

and Syrians, who lived on Assyrian territory, and added the Arabic name *al-muqâbalah,* which is nothing else than the literal Arabic translation of *al-jabr.* This took place many hundreds of years before Mohammed ibn Mûsâ al-Kowârizmî. Later on the real meaning of the word was forgotten, and the simple meaning seems not to have been regarded as scholarly enough for good usage. The scholastic method at that time was common in both the philosophical and the theological schools. The scholars tried to find in the Bible, the Koran, and the old philosophical texts everything but the simple, plain meaning. The same method was followed towards these two mathematical terms. The masters simply declared them to signify the first two operations of algebra, namely the removal of the negative and positive quantities, without worrying much about philological reasons. This conception of the two words was already well known at the time of Mohammed ibn Mûsâ al Khowârizmî, and the latter used the terms in the traditional way without any further explanation, as shown by both Rosen and Ruska.

For more than a thousand years this scholastic interpretation prevailed in the Arabic and European worlds. In reality, however, it would seem that the expression *'Ilm al-jabr w'al-muqâbalah* ought to be rendered simply as *Science of equations, al-jabr* being the Assyrian and *al-muqâbalah* the Arabic name for equation.

ON THE CORRELATION BETWEEN TWO FUNCTIONS[1]

By F. M. WEIDA, Lehigh University.

1. **Introduction.** The recent papers of P. R. Rider (1924, 227–231), Karl Pearson (1925, 70–73), and H. L. Rietz[2] suggested to me the investigation of the correlation between any two rational integral functions of a given set of uniformly distributed values on a given range.

It is the purpose of the present paper to consider the measurement of the degree of correlation between two variates X and Y when

$$X = \sum_{i=0}^{n} a_i t_i, \qquad (i = 0,1,2,\cdots,n) \tag{1}$$

and

$$Y = \sum_{j=0}^{m} b_j t_j, \qquad (j = 0,1,2,\cdots,m) \tag{2}$$

where the $a_i's$ and the $b_j's$ are real numbers, and the variate t is defined for all values t_k of t on the range of real numbers $c \leqq t_k \leqq d$.

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[1] Presented before the American Mathematical Society, New York, Jan. 1, 1926.

[2] *The Quarterly Publications of the American Statistical Association*, Sept. 1919, pp. 472–476.

In the conventional notation, let $\bar{X}$ and σ_x be the arithmetic mean and standard deviation of the values of X. Similarly, let $\bar{Y}$ and σ_y be the arithmetic mean and standard deviation of the values of Y. It will be found convenient to let

$$A_0 = \frac{a_0 - \bar{X}}{\sigma_x}, \quad A_i = \frac{a_i}{\sigma_x}, \quad B_0 = \frac{b_0 - \bar{Y}}{\sigma_y}, \quad B_j = \frac{b_j}{\sigma_y}. \tag{3}$$

Then we shall prove the following

2. **Theorem.** *The correlation coefficient between corresponding values of X and Y is given by*

$$r = \frac{1}{d-c} \sum_{s=0}^{n+m} \frac{L_s}{s+1} (d^{s+1} - c^{s+1}), \tag{4}$$

where $L_s = A_s B_0 + A_{s-1} B_1 + A_{s-2} B_2 + \cdots + A_0 B_s$, when $0 \leqq s \leqq n$; and $L_s = A_{s-k} B_k + A_{s-k-1} B_{k+1} + \cdots + A_{k-1} B_m$, when $s = n + k$; $0 < k \leqq m$.

To prove the theorem, we have almost directly from the definitions[1] of arithmetic mean and standard deviation

$$\bar{X} = \frac{1}{d-c} \int_c^d X dt, \quad (5), \qquad \bar{Y} = \frac{1}{d-c} \int_c^d Y dt, \tag{6}$$

$$\sigma_x^2 = \frac{1}{d-c} \int X^2 dt - \bar{X}^2, \quad (7) \qquad \sigma_y^2 = \frac{1}{d-c} \int_c^d Y^2 dt - \bar{Y}^2. \tag{8}$$

By substituting the values of X and Y from (1) and (2) in (5), (6), (7), (8) and integrating, we obtain

$$\bar{X} = \frac{1}{d-c} \sum_{i=0}^{n} \frac{a_i}{i+1} (d^{i+1} - c^{i+1}), \tag{9}$$

$$\bar{Y} = \frac{1}{d-c} \sum_{j=0}^{m} \frac{b_j}{j+1} (d^{j+1} - c^{j+1}), \tag{10}$$

$$\sigma_x = \left[\frac{1}{d-c} \sum_{i=0}^{2n} \frac{m_i}{i+1} (d^{i+1} - c^{i+1}) - \bar{X}^2 \right]^{\frac{1}{2}}, \tag{11}$$

$$\sigma_y = \left[\frac{1}{d-c} \sum_{j=0}^{2m} \frac{n_j}{j+1} (d^{j+1} - c^{j+1}) - \bar{Y}^2 \right]^{\frac{1}{2}}, \tag{12}$$

[1] E. V. Huntington, Monthly, (1919, 426).

where m_i is the sum $\sum_{r=1}^{i+1} a_{r-1}a_{i-r+1}$ when $0 \leqq i \leqq n$. When $(n+1) \leqq i \leqq 2n$, m_i is the sum $\sum_{r=1}^{2n-i+1} a_{i-n+r-1}a_{n-r+1}$. Similarly n_j is the sum $\sum_{r=1}^{j+1} b_{r-1}b_{j-r+1}$ when $0 \leqq j \leqq m$. When $(m+1) \leqq j \leqq 2m$, n_j is the sum $\sum_{r=1}^{2m-j+1} b_{j-m+r-1}-b_{m-r+1}$.

If now we introduce the notation $x=(X-\overline{X})/\sigma_x$, $y=(Y-\overline{Y})/\sigma_y$, we may write

$$x = \sum_{i=0}^{n} A_i t^i, \qquad y = \sum_{j=0}^{m} B_j t^j. \tag{13}$$

Then applying the definition that the correlation coefficient is the arithmetic mean of the products xy over the range on t from c to d, we have

$$r = \frac{1}{d-c}\int_c^d \left(\sum_{i=0}^{n} A_i t^i \sum_{j=0}^{m} B_j t^j\right)dt = \frac{1}{d-c}\int_c^d \sum_{s=0}^{s=n+m} L_s t^s dt \tag{14}$$

$$= \frac{1}{d-c}\sum_{s=0}^{n+m} \frac{L_s}{s+1}(d^{s+1}-c^{s+1}), \tag{15}$$

which was to be proved.

This form of r is useful for numerical calculation when application is to be made to particular cases.

3. **Special cases.** Let us now consider the following simple special cases, namely,

(1) $X = a_0 + a_1 t,\qquad Y = b_0 + b_1 t$;

(2) $X = a_0 + a_1 t,\qquad Y = b_0 + b_1 t + b_2 t^2$;

(3) $X = a_0 + a_1 t + a_2 t^2,\qquad Y = b_0 + b_1 t + b_2 t^2$;

and attempt a simplification of our formula (15) for r.

(1) When $X=a_0+a_1t$, $Y=b_0+b_1t$. Here $m=n=1$, and we write for (15)

$$r = \frac{1}{d-c}\left\{\frac{L_0}{1}(d-c) + \frac{L_1}{2}(d^2-c^2) + \frac{L_2}{3}(d^3-c^3)\right\}, \tag{16}$$

where $L_0 = A_0B_0$, $L_1 = A_1B_0 + A_0B_1$, and $L_2 = A_1B_1$.
From (3), (5), (6), (7), (8), we find that

$$A_0 = \frac{\mp (d+c)\sqrt{3}}{d-c};\ A_1 = \frac{\pm 2\sqrt{3}}{d-c};\ B_0 = \frac{\mp (d+c)\sqrt{3}}{d-c};\ B_1 = \frac{\pm 2\sqrt{3}}{d-c}.$$

Hence,

$$L_0 = \frac{\pm 3(d+c)^2}{(d-c)^2};\quad L_1 = \frac{\mp 12(d+c)}{(d-c)^2};\quad L_2 = \frac{\pm 12}{(d-c)^2},$$

Substituting the values of L_0, L_1, L_2, just found, in equation (16) and simplifying, we find that

$$r = \pm 1. \tag{17}$$

(2) When $X = a_0 + a_1t$, $Y = b_0 + b_1t + b_2t^2$. In this case $n = 1$ and $m = 2$, and we write for (15)

$$r = \frac{1}{d-c}\left\{\frac{L_0}{1}(d-c) + \frac{L_1}{2}(d^2-c^2) + \frac{L_2}{3}(d^3-c^3) + \frac{L_3}{4}(d^4-c^4)\right\}, \tag{18}$$

where $L_0 = A_0B_0$, $L_1 = A_1B_0 + A_0B_1$, $L_2 = A_1B_1 + A_0B_2$, and $L_3 = A_1B_2$.

If now we find the values of the L's by making use of (3), (5), (6), (7), (8); and if we substitute these values repectively in (18) and simplify, we find that

$$r = \sqrt{15}\,\{b_1 + b_2(d+c)\}/\{15b_1^2 + 30b_1b_2(d+c) + 4b_2^2(4d^2+7dc+4c^2)\}^{\frac{1}{2}}. \tag{19}$$

From this result, let us investigate the conditions under which $r^2 = 1$. If we square both members of equation (19) and impose the condition that $r^2 = 1$ and solve the resulting equation, we find that when $b_2 \neq 0$, $r^2 = 1$ only in the trivial case when $d = c$.

(3) When $X = a_0 + a_1t + a_2t^2$, $Y = b_0 + b_1t + b_2t^2$. Here $m = n = 2$, and we write for (15) that

$$r = \frac{1}{d-c}\left\{\frac{L_0}{1}(d-c) + \frac{L_1}{2}(d^2-c^2) + \frac{L_2}{3}(d^3-c^3) + \frac{L_3}{4}(d^4-c^4) + \frac{L_4}{5}(d^5-c^5)\right\}, \tag{20}$$

where $L_0 = A_0B_0$, $L_1 = A_1B_0 + A_0B_1$, $L_2 = A_2B_0 + A_1B_1 + A_0B_2$, $L_3 = A_2B_1 + A_1B_2$, and $L_4 = A_2B_2$.

If now we find the values of the L's by making use of (3), (5), (6), (7), (8); and if we substitute these values respectively in (20) and simplify, we find that

$$r = \frac{15a_1b_1 + 15(a_2b_1 + a_1b_2)(d+c)}{\{15a_1^2 + 30a_1a_2(d+c) + 4a_2^2(4d^2+7dc+4c^2)\}^{1/2}} \frac{+\,4a_2b_2(4d^2+7dc+4c^2)}{\{15b_1^2 + 30b_1b_2(d+c) + 4b_2^2(4d^2+7dc+4c^2)\}^{1/2}}. \tag{21}$$

In particular, if $a_1/a_2 = b_1/b_2$, and if a_2 and b_2 have the same sign, the vertices of our (X, t) and (Y, t) parabolas have the same t-coordinate and the para-

bolas extend in the same direction. If now we impose these conditions upon the a's and b's in (21) we readily find that

$$r = 1. \tag{22}$$

Again if $a_1/a_2 = b_1/b_2$, and if a_2 and b_2 have opposite signs, the vertices of our (X, t) and (Y, t) parabolas have the same t-coordinate and the parabolas extend in opposite directions. If we impose these conditions upon the a's and the b's in (21) we find that

$$r = -1. \tag{23}$$

The results $r = \pm 1$ from $a_1/a_2 = b_1/b_2$ are of analytic interest because $r = \pm 1$ without regard to the distances from the vertices to the foci of the parabolas.

4. **Concluding remarks.** It is fairly obvious that the correlation we have studied may be regarded geometrically as the correlation between the ordinates of two curves given by (1) and (2) on the range $c \leqq t \leqq d$.

While the substitution of special values in (15) shows that the correlation coefficient between these ordinates may take a wide range of values varying from -1 to 1, it should perhaps be remarked that the correlation ratio[1] η_{yx} of y on x would be equal to unity when the elimination of t from (1) and (2) gives y as a single valued function of x.

In the preceding discussion, we have placed no restrictions on c or d except that c and d are real numbers. A case of theoretic interest arises when the difference between c and d is very small. We then write $d = c + \Delta c$ where Δc is very small. As $\Delta c \to 0$, the segments of the two continuous curves may be taken as strictly linear and the correlation coefficient will approach either 1, -1, or 0. An examination of (19) and (21) when $d = c + \Delta c$ and $\Delta c \to 0$ shows that $r = \pm 1$.

CONCERNING THE PROBABILITY CURVES OF N POINTS TAKEN AT RANDOM ON A STRAIGHT LINE SEGMENT OF CONSTANT LENGTH

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In the theory of probability, when a point P is taken at random upon a segment of a straight line and the probability that P shall fall at a given point is constant for all points of the line, the probability that P shall fall in a segment of length Δx is said to be $\Delta x/a$, where a is the length of the original segment upon which it is certain the point must fall. If the probability is not constant for all points but is a continuous function of x, the abscissa of the corresponding

[1] *Handbook of Mathematical Statistics* by H. L. Rietz and others, (1924), pp. 129–131.

point, $y=f(x)$ is said to be the probability curve of the point P and we have the condition $\int_0^a f(x)dx=1$. In the case of one point taken at random dx/a is the probability that P will fall between x and $x+dx$ and $\int_0^a(1/a)dx=1$, so that $y=1/a$ is the probability curve of the point. When two points X, Y are taken at random upon the line AB, whose length is a, X being the point nearer A, dx/a is the probability that X is between x and $x+dx$ and dy/a that Y is between y and $y+dy$, so that $dxdy/a^2$ is the probability that X is between x and $x+dx$ and Y is between y and $y+dy$. Then $(dx/a^2)\int_x^a dy$ sums the probabilities when Y is to the right of X, so that we have $y=c(a-x)/a^2$ as the probability curve for X, where c must be determined subject to the condition that $(c/a^2)\int_0^a (a - x)dx = 1$, for although $\int_0^a\int_0^a dxdy/a^2 = 1$, by imposing the condition that X is to the left of Y, we have restricted the range of values in determining the probability curve as a function of x so that there is no reason to expect that this function integrated over the entire range will give unity as a result. Accordingly, $y=2(a-x)/a^2$ is the probability curve of X, and similarly $y=2x/a^2$ is the probability curve of Y. We are now ready to turn our attention to the general case.

LEMMA ONE. If n points are taken at random upon a straight line segment of constant length a, and P_k be the kth point from the left, then the probability curve of P_k has a zero of the $(k-1)$ th order at the origin and a zero of the $(n-k)$th order at the point $(a, 0)$, where the segment a is that portion of the x-axis between the origin and $(a, 0)$. For, summing up the probabilities of the positions of the $(n-1)$ other points, one possibility is that $(k-1)$ of them may be adjacent at the origin and that $(n-k)$ may be adjacent at the point $(a, 0)$. Hence the ordinate of the probability curve for the point P_k is zero for the $(k-1)$ adjacent points at the origin and the $(n-k)$ adjacent points at $(a, 0)$.

LEMMA TWO. The probability curve of P_k any one of the n points taken under the above hypothesis is of degree $(n-1)$. For we have:

$$\frac{dp_k}{a^n}\int_{p_{n-1}} \cdots \int_{p_{k+1}}\int_{p_{k-1}} \cdots \int_{p_1} dp_1 \cdots dp_{k-1}dp_{k+1} \cdots dp_{n-1}$$

upon summing up the various probabilities, and each integration is performed with respect to at least one variable.

From these lemmas we infer that $y=c(a-x)^{n-k}\, x^{k-1}$ is the probability curve of the kth point. To determine c set $c\int_0^a(a-x)^{n-k}\, x^{k-1}\, dx=1$. By means of the Beta and Gamma functions we obtain:

$$c = \frac{n}{a^n}\,\frac{(n - 1)(n - 2) \cdots (n - k + 1)}{(k - 1)\,!},$$

so that the probability curve of the kth point from the left is: $y=(n/a^n)$ multiplied by the kth term in the expansion of $[(a-x)+x]^{n-1}$. The binomial expansion thus plays an important part here as in other types of probability problems.

To find the maximum point of this curve we set (dy/dx) equal to zero and find

$$x=a(k-1)/(n-1). \tag{1}$$

Note that when $k=1$ or n we still have the maximum of the first or the last curve with respect to that portion of the plane under consideration, namely, that portion within the rectangle three of whose sides are the x-axis, the y-axis, and the line $x=a$, although these are not maximum points of the curves themselves. This shows that the abscissas of the n points divide the x-axis from the origin to $(a, 0)$ into $(n-1)$ equal parts.

By substitution the corresponding y-coordinate of the maximum for the curve is found to be:

$$y = \frac{n(n-1)(n-2)\cdots(n-k+1)}{a(k-1)!\,(n-1)^{n-1}}(n-k)^{n-k}(k-1)^{k-1} = A\,. \tag{2}$$

When $k=1$ or n we have a term of the form 0^0 but since the limit of $(k-1)^{k-1}$ or $(n-k)^{n-k}$ as k approaches 1 or n respectively, is unity, our formula will still apply in these cases.

When k is less than $\frac{1}{2}n$, the ordinate of the maximum point for the $(k+1)$th curve is less than the corresponding ordinate for the kth curve. For if in (2) we let k be replaced by $k+1$, we get $y=B$, where

$$B = A\left(\frac{n-k-1}{n-k}\right)^{n-k-1}\left(\frac{k}{k-1}\right)^{k-1}.$$

Consider

$$\left(\frac{n-k-1}{n-k}\right)^{n-k-1}\cdot\left(\frac{k}{k-1}\right)^{k-1} \tag{3}$$

as a function of n and k.

When $n=2k$, we have

$$\left(\frac{k-1}{k}\right)^{k-1}\left(\frac{k}{k-1}\right)^{k-1} = 1.$$

When $n>2k$, or $k<\frac{1}{2}n$, let $n=2k+\Delta n$.

Then (3) becomes

$$\left(\frac{k+\Delta n-1}{k+\Delta n}\right)^{k+\Delta n-1}\left(\frac{k}{k-1}\right)^{k-1} = \frac{1}{\left(1+\dfrac{1}{k+\Delta n-1}\right)^{k+\Delta n-1}}\left(\frac{k}{k-1}\right)^{k-1}$$

and

$$\frac{1}{\left(1+\dfrac{1}{k+\Delta n-1}\right)^{k+\Delta n-1}}$$

represents terms in the expansion of

$$\frac{1}{\left(1+\dfrac{1}{x}\right)^{x}},$$

where $(k+\Delta n-1)=x$ and x becomes infinite. This, however, is a decreasing function whose limit is $1/e$. Hence $B=A$ multiplied by a factor less than one when $k<\frac{1}{2}n$. Because, for a given value of n, the probability curves are evidently symmetrical with respect to the line $x=\frac{1}{2}a$, since the theory might have been developed starting from the right rather than the left, with no change, we infer that when $k>\frac{1}{2}n$, $B>A$. When n is odd, the lowest maximum point for any one of the n curves is for the kth curve when $k=\frac{1}{2}(n+1)$; if n is even the lowest maximum points are the two having equal ordinates where $k=\frac{1}{2}n$ and $k=\frac{1}{2}n+1$. Using the parametric equations (1) and (2) and eliminating the parameter k, we find that the n maximum points lie on the curve:

$$y=\frac{(a-x)^{\left(\frac{n-1}{a}\right)(a-x)}x^{\frac{(n-1)x}{a}}}{\int_0^a (a-z)^{\left(\frac{n-1}{a}\right)(a-x)}z^{\frac{(n-1)}{a}x}dz}$$

which is continuous when $-a/n-1<x<na/n-1$. This curve may easily be proved symmetric with respect to the line $x=\frac{1}{2}a$.

We shall now show that each one of the n curves intersects every other one once and only once in the region we are considering, except for intersections at the origin and at $(a, 0)$.

If we take the equations of any two curves and solve them simultaneously we shall have, after factoring out all factors of the form $(a-x)$ and x, $(a-x)^p=cx^p$, $p<n$, $c>0$. All the pth roots of this equation will be found in the determinations of the pth roots of c. When p is odd, c has only one real root and x is between 0 and a. When p is even, there are three cases:

(1) $c=1$. Here $(a-x)=\pm x$, gives the real values of x, but only the positive sign can be used.

(2) $c>1$. Here we have $(a-x)=\pm\sqrt[p]{c}x$ and the positive sign gives the only real positive value for x.

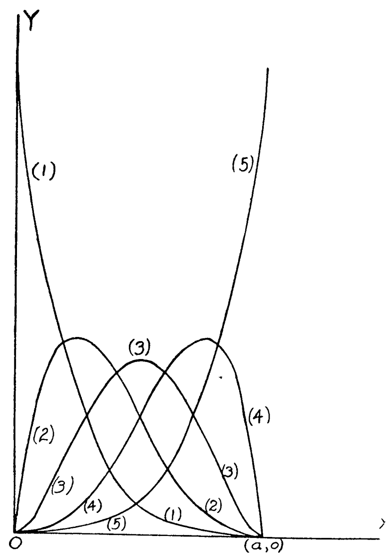
(3) $c<1$. Let $cc'=1$; then

$$(a-x)=\pm\frac{x}{\sqrt[p]{c'}}\,;\quad x=\frac{a\sqrt[p]{c'}}{1+\sqrt[p]{c'}}\ \text{or}\ \frac{a\sqrt[p]{c'}}{\sqrt[p]{c'}-1}\,;$$

the second of these values for x is greater than a and hence is outside the region under consideration. The ordinates of the points of intersection are, of course, positive when $0<x<a$.

As an example of the preceding consider the case $n=5$. Here the probability curves are:

$k=1$	$y=5(a-x)^4/a^5$	Max. $(0, 5/a)$
$k=2$	$y=20(a-x)^3x/a^5$	Max. $(.25a, 2.11/a)$
$k=3$	$y=30(a-x)^2x^2/a^5$	Max. $(.5a, 1.88/a)$
$k=4$	$y=20(a-x)x^3/a^5$	Max. $(.75a, 2.11/a)$
$k=5$	$y=5x^4/a^5$	Max. $(a, 5/a)$.



The curve upon which the maximum points lie is:

$$y=\frac{(a-x)^{4(a-x)/a}\,x^{4x/a}}{\int_0^a (a-z)^{4(a-x)/a}\,z^{4x/a}\,dz}\,;\quad \frac{-a}{4}<x<\frac{5a}{4}.$$

The points of intersection of the curves are:
Curves 1 and 2—$(.2a, 2.05/a)$
Curves 1 and 3—$(.29a, 1.27/a)$
Curves 1 and 4—$(.39a, .71/a)$
Curves 1 and 5—$(.5a, .31/a)$
Curves 2 and 3—$(.4a, 1.73/a)$
Curves 2 and 4—$(.5a, 1.25/a)$.

Corresponding curves counting from the right
Curves 4 and 5—$(.8a, 2.05/a)$
Curves 3 and 5—$(.71a, 1.27/a)$
Curves 2 and 5—$(.61a, .71/a)$
Curves 1 and 5—$(.5a, .31/a)$
Curves 3 and 4—$(.6a, 1.73/a)$
Curves 2 and 4—$(.5a, 1.25/a)$.

The figure illustrates clearly the symmetry of the curves and of their points of intersection. The area under each curve, of course, is unity.

Finally it may be remarked that if, to find the probability curve of a certain point we sum the probabilities by means of $(n-1)$ integrals, allowing the position of each point to vary between that of the point preceding and that of the point following, the constant determined so that the area under the curve shall be unity, is $n!$, corresponding to the $n!$ permutations of the n points. As an example let $n=7$, $k=3$.

$$\frac{(0,0)\qquad\qquad (a,0)}{X\ Y\ Z\ T\ U\ V\ W}$$

$$\frac{dz}{a^7}\int_z^a\!\!{}_{w}\int_z^w\!\!{}_{v}\int_z^v\!\!{}_{u}\int_z^u\!\!{}_{t}\int_0^z\!\!{}_{y}\int_0^y\!\!{}_{x}\,dxdydtdudvdw=(a-z)^4z^2\frac{dz}{48a^7},$$

and upon multiplying by $7! = 5040$, we obtain $(105/a^7)\ (a-z)^4\ z^2\ dz$, as we may get also by our formula.

Similarly

$$\frac{dz}{a^7}\int_z^a\int_z^a\int_z^a\int_z^a\int_0^z\int_0^z dxdydtdudvdw,$$

which sums the probabilities when X and Y are allowed to take any position between the origin and Z, and T, U, V, W, between Z and $(a, 0)$, gives: $(a-z)^4$ $z^2\ dz/a^7$ and the required constant $c=105$ may be obtained by multiplying by $15={}_{n-1}C_{k-1}$ the number of ways in which the points may occupy the $(n-1)$ positions by crossing from left to right of Z or vice-versa, and by 7, the number of points. This last method clearly shows the existence and order of the zeros at the origin and at $(a, 0)$.

EARLY LITERARY EVIDENCE OF THE USE OF THE ZERO IN INDIA

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During the last few decades there have been many earnest attempts to find the time and country of invention of the zero and the place-value for the decimal system of numeration. Though some historians of mathematics have suggested the name of one or more of the nations that once flourished on the rich soil of Mesopotamia, the world is gradually adopting the view that the credit of the invention is entirely due to the Hindus.[1] Such was also the belief of the mediaeval and Renaissance writers of Europe. It is noteworthy that though the modern numerals which are in use all over the civilized world are commonly called "Arabic," and though some influential ancient writers pleaded with fervor in favor of Arabia, the Arabs themselves were of opinion that the invention was of Hindu origin. About the exact time of the invention there have been put forth different opinions. Most of the epigraphical instances so far found in India have been considered more or less as forgeries by some writers. Kaye[2] has been the most eloquent exponent of this group. His attitude will be known from the following quotation from this article:

> On palaeographic grounds we are forced to fix the 9th century A.D. as the earliest period in which the modern place-value system of notation may have been in use in India. This earliest period depends upon one inscription only. If this inscription, on further light being thrown upon it, prove unreliable

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[1] *Vide* Smith's introductory note to Ginsburg's article, "New Light on our numerals," *Bull. Amer. Math. Soc.*, vol 23(1917); also compare Smith and Karpinski, *The Hindu-Arabic Numerals* (Boston, 1911), Smith, *History of Mathematics*, ii, p. 64.

[2] Kaye, "Notes on Indian Mathematics—Arithmetical Notation," *Jour. Asiat. Soc. Bengal, vol. 3* (1907) p. 487.

(as it possibly will), then we shall have to fix the tenth century as the earliest period. Even for the tenth century there is not an excessive amount of good evidence, and it is within the bounds of possibility that we may have finally to turn to the eleventh century for evidence of the use of our modern system in India.

On the other hand authorities on Indian epigraphy like Bühler, V. A. Smith, Bhandarkar, and others bear testimony to the genuineness of some of the inscriptions at least and the earliest of them recording numerals with place-value is dated 595 A. D.[1] Among the writers on literary evidence, Haraprasad Shastri is of opinion that the Indians knew the place-value system of decimal numeration at least in the early centuries of the Christian era.[2] Thibaut puts the date at 590 A.D.[3] In a note recently contributed to the MONTHLY, (*1926* 220–221), I have quoted two almost identical passages from philosophical writings of the sixth and seventh centuries which expressly refer to the place value as an illustration of a form of philosophical category. I have there suggested that the higher limit of the time of invention of the place-value system of decimal numeration in India must be fixed at the fifth century. This, however, does not settle as a matter of course the date of invention of the zero as a numeral. Strictly speaking these illustrations do not mention whether the place value was indicated in writing numbers or was being indicated on the abacus. In other words it may be asked whether it will be safe to conclude that the zero was known in that early age.[4] No doubt there is the very significant fact that until now no mention of the existence of the abacus, direct or indirect, has been traced in any Indian literature so that it may be taken without any fear of contradiction that if an abacus was ever in use among the learned men of India, it was discarded long ago.[5] Hence it follows as a matter of course that, in the quotations referred to, the place value was being indicated with the help of the zero. But direct instances of the use of the zero from any early writing will be at once decisive and hence all the more welcome.

It must be remarked that almost all the literary instances should be interpreted as referring to the concept of the zero as a numeral, which must

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[1] *Epigraphia Indica*, ii, p. 29.

[2] Haraprasad Shastri, *Presidential Address to the Sanskrit and Prakrit Section, Proc. & Trans. of the Second Oriental Conference*, Calcutta (1922).

[3] Thibaut, *Astronomie, Astrologie, und Mathematik*, Strassburg, 1899.

[4] In fact the question has been put to the writer in a personal letter from Professor D. E. Smith.

[5] Fleet's interpretation of the word "ganitra" occurring in a writing of the first century A.D. as meaning "an instrument of reckoning" (Fleet, "The use of the abacus in India", *Journ. Roy. Asiat. Soc.*, 1911), cannot be taken as an instance of the use of the abacus as an instrument for arithmetical calculation. It has been stated that the man who was "expert in ganitra" was an astrologer and we know how even in modern times, the Indian astrologers are in the habit of 'casting the stars' in the form of a diagram, suggested probably by the zodiacal circle, which has no connection whatever with the mathematical calculations necessary for the purpose.

not be confounded with the symbol or the form of the symbol for the zero. We know for certain that the form of the symbol for zero has undergone changes in India from a mere dot to a small circle.[1] Who knows that there have not been other changes or that there were not other forms? Furthermore, any illustration of the form, to speak from our present knowledge, will have the difficulty that it might be explained away as meaning vacuity or disappearance. Even an instance like the one in Subandu's *Vāsabadattā*[2] (*c.* 620 A.D.) or the one in Śrīharsa's *Naiṣada-Carita*[3], where the zero has been expressly stated to be *śūnyabindy*[4] need not be considered too seriously as a decisive factor for the form of the zero symbol, being a mere point or dot. There are numerous instances in Indian literature[5] where *bindu* has been used in the sense of a small particle, as of water, oil, or gold, or of a comparatively larger spot, as in *bindu-mrga* (spotted deer) and *bindu-citra* (lit., spot-marked). Again there are positive palaeographical evidences that centuries before the time of Śrīharṣa (*c.* 12th century) who calls the zero a *śūnya-bindu*, the zero was represented in India by a small circle, as in the inscriptions of Bhojadeva[6] and of Mahīpāl.[7] All that can be safely inferred from the literary concept of the zero as a numeral is that there must have been a symbol and any further inference will not be always free from doubt.

Now the required instances are furnished by Varāhamihira's *Pañca-siddhāntikā* (505 A. D.). There he incidentally states two fundamental arithmetical operations by the zero (*kha*, *śūnya*, *ambara*), *viz.* addition and subtraction, in more than one place, *e.g.*, iii. 2, 17; iv. 7, 8, 11, 12; xviii. 35, 44, 45, 48, 51—the Roman figures indicating the chapters and the Hindu figures the verses. It is rightly stated that the value of a quantity does not change when zero is added to or subtracted from it. The true significance is not always evident in Thibaut and Dvivedi's translation, and in fairness to the Sanskrit commentator it must also be stated that he is more expressive. Hence we propose to give a fresh literal translation of the ślokas (omitting unnecessary sentences), their more complete translation being given in the footnotes.

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[1] Bühler, *On the origin of the Indian Alphabet*, p. 53, note; *cf.* the dot of the Bakshāli Mss and the small circle of the Gwalior inscription, dated Samvat 933 (876 A.D.).

[2] Hall's edition p. 181.

[3] Canto I, stanza 21.

[4] Varāhamihira has also one equivalent for zero as *bindu* not *śūnyabindu*; vide his *Pañca-siddhāntikā*, edited with English translation and Sanskrit commentary by Thibaut and Dvivedi, Benares, 1899, Chapter iv, śloka 7.

[5] For example see *Atharvaveda*, ix. i. 21; x. 10. 19; xii. 3. 29.

[6] The zero is found in two inscriptions of Bhojadeva: the first is dated 870 A.D. (Ojha, *Prāchīn Lipi Mālā*, p. 127); and the second is dated 876 A.D. (Hill, *The Early Use of Numerals in Europe*, 1915).

[7] *Vide* the Asni inscription of 917 A.D., *Indian Antiquary*, vol. 16, p. 174.

Translation

iii.2[a] Corresponding to the signs of the anomaly, the numbers of minutes to be deducted (from the sun's mean longitude) are eleven; eight sixes; seventy minus one; and (that) *plus zero* [*kha*]; nine sixes; and five squared.

iii.17 "The daily motion of the sun amounts to sixty (minutes) minus three, three, three, three, two, one; plus one, one, one, one; *minus naught* [*kha*], one, in turn." [Thibaut and Dvivedi].

iv.7[b] (To which *to be added* in succession) fifty plus one; five eights; five squared; four; thirty increased by four; fifty-six; five; *zero* [*śūnya*].

iv.8[c] The signs in Taurus are six; thirteen; nineteen; three eights; again thirty *added with zero* [*ambara*], five, nine, thirteen (in succession) minutes.

iv.11[d] The seconds are sixty diminished by eighteen, three, eighteen; zero [*śūnya*]; fifty less by three; four; fifty minus one; five.

vi. 12[e] In Aries the minutes are seven, in the last sign six; in Taurus six (repeated) thrice; five (repeated) thrice; four; four; in Gemini they are three, two, one, zero [*śūnya*] (each repeated) two times.

xviii.35 "Thirty-six *increased by* two; three, nine, twelve, nine, three, *zero* [*śūnya*] are the days. The motion in the eighth course is the same as in the seventh." [Thibaut and Dvivedi].

xviii.44[f] In Gemini (in) twice ten *plus* five squared, *zero* [*kha*], six, and three cubed (days, he passes through) half-hundred minus two; fourteeen; three cubed; and five eights (degrees).

xviii.45[g] In Cancer, (in days) four, one, three and four, multiplied by ten and *increased by* two, eight, *zero* [*śūnya*], six (respectively), (he passes through) five squared increased by one, halved, increased by one; and five squared degrees. Or symbolically:

days: $4\times10+2$; $1\times10+8$; $3\times10+0$; $4\times10+6$;
degrees: $5\times5+1$; $5\times5\times\frac{1}{2}$; $5\times5+1$; 5×5.

xviii.48[h] (In Libra, in days) twenty increased by one; *decreased by zero* [*kha*], ten, three, and then (the results) multiplied by two, (he passes through) the number of degrees less by three, eight; more by one, thirty (than the number of days, respectively). Or symbolically:

days: $(20+1)2$; $(20+0)2$; $(20+10)2$; $(20+3)2$;
degrees: $(20+1)2-3$; $(20-0)2-8$; $(20-10)2+1$; $(20-3)2+30$.

xviii.51[i] In Capricorn, (in days) twice ten *plus zero* [*kha*]; minus seven; increased by eighteen and twelve; (he passes through) the number of degrees less by one; more by one, one, twenty-six (than the number of days respectively). Or symbolically:

days: $2\times10+0$; $2\times10-7$; $2\times10+18$; $2\times10+12$;
degrees: $2\times10+0-1$; $2\times10-7+1$; $2\times10+18+1$; $2\times10+12+26$.

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[a] "Corresponding to the signs of the anomaly we have the following (aggregates of) minutes which have to be deducted or added to (from the sun's mean longitude); viz: 11, 48, 69, 69, 54, 25."

[b] "(To which to be added in succession) 51; 40; 25; 4; 34; 56; 5; 0."

[c] "The signs in Taurus are 6; 13; 19; 24; 30; 35; 39; 43 minutes."

[d] "The seconds are 42; 57; 42; 0; 47; 4; 49; 5."

[e] "In Aries the minutes are 7, in the last sign 6; in Taurus they are 6; 6; 6; 5; 5; 5; 4; 4; in Gemini they are 3; 3; 2; 2; 1; 1; 0; 0;"

[f] "In Gemini he passes in forty-five days through forty-eight degrees; in twenty days through fourteen degrees; in twenty-six days through twenty-seven degrees; in forty-seven days through forty degrees."

[g] "In Cancer he passes in forty-two days through twenty-six degrees; in eighteen days through twelve and a half degrees; in thirty days through twenty-six degrees; in forty-six days through twenty-five degrees."

[h] "(In Libra) he passes in forty-two days through thirty-seven [?-nine] degrees; in forty days through thirty-two degrees; in thirty-four days through sixty four degrees."

[i] "In Capricorn he passes in twenty days through nineteen degrees; in thirteen days through fourteen degrees, in thirty-eight days through thirty-nine degrees; in thirty-two days through fifty-eight degrees".

It is noteworthy that all the above verses are from those chapters of *Pañca-siddhāntikā* where are summarised the teachings of the *Pauliśa-siddhānta*. Thibaut remarks: "Varāhamihira has in no case obliterated the characteristic features of the Siddhantas he has to deal with, and whatever distinguishes those works from one another in the test of the *Pañca-siddhāntikā* really distinguished them in their original form."[1] Is it therefore safe to state that the method was due to the original *Pauliśa-siddhānta*? Unfortunately no means of verification of this conjecture has been left to us in the present age, the book not being now extant. It was known to Brahmagupta (*c.* 628 A.D.), to Bhaṭṭotpala (966 A.D.), and even as late as in the eleventh century to Alberuni. There are however numerous quotations in Bhaṭṭotpala's excellent commentary on Varāhamihira's *Bṛhat-samhitā*[2] from an "original Puliśa-siddhānta" and probably also from a different edition of the same work. There we find the use of the word numerals with the place value and the zero. In any case this conjecture will lead us to the conclusion that the zero was known in India before 400 A.D., for that is the higher limit for the time of composition of the original *Pauliśa-siddhānta* as set by Thibaut.[3] It may further be stated in passing, that Whitney believed that at the time of Āryabhata (476 A.D.), that Hindus had "invented their system of signs employed in decimal notation."[4]

An earlier instance of the use of the zero is found in *Chandaḥ-sūtra* of Piṅgala or Piṅgalanāga as he is otherwise called.[5] It is a manual of Vedic metres. One section of the book deals with the problem of determination of the number of possible variations for a metre with a given number of syllables. There have been formulated definite rules for the purpose of calculations and it will be interesting and rather surprising to know that the method yields a huge number of variations, such as $20{,}282{,}388 \times 10^{24}$ and even more.[6] It is in

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[1] *Pañca-siddhāntikā*, loc. cit.; Introduction, p. xvii; *cf.* p. xvi.

[2] Varāhamihira, *Bṛhat-saṃhitā*; edited by Kern, Calcutta (1895) and translated by him in *Journ. Roy. Asiat. Soc.*, (1870–75); Bhaṭṭotpala's Commentary edited by Dvivedi, Benares (1895).

[3] *Loc. cit.* p. lx. As much has been said in favor of as against the supposed identity of Puliśa, the author of the *Pauliśa-siddhānta* and an unknown Greek astronomer. But this much is absolutely certain: that this conjecture, even if it proves to be true, will not deprive the Hindus of their originality in the invention of the zero and the place-value for the decimal system of notation. For until now nothing has been discovered in the literature of the Greeks or of any other ancient nations which would justify their conception of the zero as a numeral. On the other hand Kern says that "*Pauliśa-siddhānta*, judging from the quotations, and rather numerous ones is so thoroughly Hinduised that few or no traces of its Greek origin are left." *Bṛhat-saṃhitā*, *loc. cit.*, Introduction, p. 49.

[4] *Journ. Amer. Orient. Soc.*, vol 6, p. 563 fn. The date 476 is that of Āryabhaṭa's birth.

[5] Piṅgala, *Chandaḥ-sūtra*: the text with the commentary of Halāyudha edited by Visvanath Sastri in *Bibliotheca Indica* (1871-74); also edited and commented by Weber, *Indische Studien*, viii (1863).

[6] Colebrooke, *Miscellaneous Essays*, ii (1873) p. 88.

this connection that Piṅgala has used the word *śūnya* in two successive sūtras.[1] These two sūtras, in fact the whole of the manual, are found restated in the Agni-Purāṇa.[2] There is no doubt that, by *śūnya*, Piṅgala was referring, not to the mere concept of nothingness, but to a definite symbol whose concept is akin to the concept of our zero numeral. Halāyudha, the earliest commentatator of *Chandaḥ-sūtra*, has explained the sūtras fully and has adduced illustrative examples as well. He is of opinion that, in the sūtras referred to, *śūnya* denotes the zero, and he has been supported by Weber who remarks that there cannot be any doubt about that.[3] Having no very weighty arguments or facts to suggest to the contrary, we cannot help but accept it. Now Piṅgala lived about the middle of the second century B.C.[4], and his book was popular amongst the Brāhmaṇas, the Vedic scholars of the age.[5] Hence it has to be admitted that the zero was known to the Brāhmaṇas of India in the second century B.C. The date of invention, it is highly probable, preceded this by a century or two at least.

It will be noticed that evidence from writings of authors posterior to Varāhamihira has not been collected here. Some of these writers are much more explicit and have spoken of even multiplication and division by zero. Among the earlier writings also there are two hymns of the Atharveda[6] where, most probably, there is reference to the zero as well as to positive and negative numbers. They will, however, form the subject of a future communication. For the present it may be stated that in the hymns referred to, the zero has been called *kṣudra* (trifling)[7] and positive and negative (numbers) have been denoted by *ṛca* and *anṛca* respectively.

My thanks are due to Professor David Eugene Smith of Columbia University for his interest in the preparation of this paper.

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[1] Ch. viii, sūtras 29, 30. Weber translates them as "Bei Eins eine Null" and "Zweimal bei einer Null." For his commentary of the sūtras see, *loc. cit.*, *pp.* 444-48.

[2] *Agni-Purāna*, Bangabāshī edition, Ch. 328–34.

[3] "Was denn auch in der That wohl keinem Zweifel unterliegt" (*Br.*p. (445). *Cf.* also p. 169.

[4] Winternitz, *Geschichte der Indischen Litteratur*, Bd. iii (1922) p. 28; compare also Max Müller, *Ancient Sanskrit Literature*, p. 75 et seq.

[5] This will be at once evident from its being raised to the level of a *Vedāṅga*, which mark it does not in reality deserve, from its being incorporated in the *Agni-Purāna* as already stated, and also from its being quoted in the *Bhāratīya-Nātya-Sūtra* (Ch. xv) and in the *Pariśiṣthas*. Vide Max Müller, *loc cit.*

[6] xix. 22, 23.

[7] *Cf.* Amarsiṅha's synonym for zero as *tucca*, meaning insignificant, negligible. Perhaps a better or more succinct definition is hardly possible even today. Amarsiṅha is the celebrated lexicographer of ndia of the fifth century of the Christian era.

NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

BY W. E. MILNE, University of Oregon.

The method of numerical integration here described has grown out of the practical experience of the writer in the course of a long series of integrations carried on for several years. In its fundamental principles this method does not differ essentially from other well-known methods,[1] but in practical operation it differs considerably. A careful trial of the method in comparison with others convinces me that it possesses distinct advantages in ease, speed, and simplicity. It is especially designed so that every step (except perhaps the substitution in the differential equation) may be readily performed on a calculating machine, thus preventing mental fatigue and insuring accuracy.

1. Let u_0, u_1, u_2, u_3, u_4, be five values of a function $u(x)$ at equally spaced values of x, which we may without loss of generality take to be $0, h, 2h, 3h, 4h$. A polynomial of fourth degree which takes the values u_0, u_1, u_2, u_3, u_4, is given by Newton's interpolation formula

$$P(x) = u_0 + \Delta u_0 \frac{x}{h} + \Delta^2 u_0 \frac{x(x-h)}{2h^2} + \Delta^3 u_0 \frac{x(x-h)(x-2h)}{6h^3} + \Delta^4 u_0 \frac{x(x-h)(x-2h)(x-3h)}{24h^4}. \tag{1}$$

From this we obtain by integration and reduction

$$\int_{x_2}^{x_4} P(x)dx = \frac{h}{3}\{u_4 + 4u_3 + u_2\} - 8h\Delta^4 u_0/720, \tag{2}$$

$$\int_{x_0}^{x_4} P(x)dx = \frac{4h}{3}\{2u_3 - u_2 + 2u_1\} + 224h\Delta^4 u_0/720. \tag{3}$$

Since

$$\Delta^4 u = h^4 \frac{d^4 u}{dx^4} + \textit{terms of higher order}$$

[1] The following references may be consulted in this connection:

(1) Von Sanden, *Practical Mathematical Analysis*, E. P. Dutton and Co., Chapters X and XI.

(2) Whittaker and Robinson, *The Calculus of Observations*, London, 1924, Chapter XIV.

(3) Runge und König, *Numerisches Rechnen*, Berlin, 1924, Kapitel X.

(4) *The Method of Numerical Integration in Exterior Ballistics*, Ordnance Textbook, Washington, D. C., 1919.

if $u(x)$ is analytic for the interval under consideration it will be possible to take h small enough that the last terms in (2) and (3) can be neglected. We shall then write in general

$$\int_{x_{n-2}}^{x_n} u(x)dx = \frac{h}{3}\left\{u_n + 4u_{n-1} + u_{n-2}\right\}, \tag{4}$$

$$\int_{x_{n-4}}^{x_n} u(x)dx = \frac{4h}{3}\left\{2u_{n-1} - u_{n-2} + 2u_{n-3}\right\}. \tag{5}$$

Formula (4) is Simpson's rule. Formula (5) is less accurate, as is seen from the remainder terms in (2) and (3) but for our purposes possesses the advantage that the integral can be calculated without the knowledge of u_n.

2. Suppose that we are to solve a differential equation

$$\frac{dy}{dx} = f(x, y) \tag{6}$$

with the initial value $y = y_0$ when $x = x_0$. Suppose also that a suitable interval h has been chosen, and that the values of y have been obtained for $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, and $x_3 = x_0 + 3h$. As the calculation of these initial values involves special methods we shall defer consideration of them to the next paragraph. The calculated values may be written as in the margin. To proceed with the computation we obtain a trial value of y_4 by use of the formula (5), which in this notation becomes

x	y	y'
x_0	y_0	y_0'
x_1	y_1	y_1'
x_2	y_2	y_2'
x_3	y_3	y_3'
x_4		

$$y_4 = y_0 + \frac{4h}{3}\left\{2y_3' - y_2' + 2y_1'\right\}. \tag{7}$$

We then obtain a trial value of y_4' by substitution in (6). Now we recalculate y_4 using formula (4), which becomes

$$y_4 = y_2 + \frac{h}{3}\left\{y_4' + 4y_3' + y_2'\right\}. \tag{8}$$

If the values given by (7) and (8) check to the desired number of decimal places we take this value as correct and proceed to the calculation of y_5 in a similar manner. If the values given by (7) and (8) do not check then either an error has been committed in calculation or the fourth differences of y' are so large that the remainder terms in (2) and (3) affect the results. Since the total discrepancy ϵ due to this cause is $\epsilon = 232h\Delta^4 y'/720$, the error ϵ_1 of formula (4) due to the fourth difference is

$$\epsilon_1 = \epsilon/29. \tag{9}$$

Consequently if the values given by (7) and (8) have a difference ϵ we first make sure that this is not due to an error of calculation, and if not, we calculate ϵ_1 by (9). If ϵ_1 is negligible to the number of decimal places desired we assume that the value of y_4 given by (8) is correct and proceed to the calculation of y_5 in a similar manner.

Of course it is necessary to obtain the corrected value of y_4' by direct calculation from the differential equation (6). The change in y_4' due to a change ϵ in y_4 is approximately $\epsilon\, \partial f/\partial y$, and the change ϵ_2 in y_4 due to this correction is therefore approximately $\epsilon_2 = (\epsilon h/3)(\partial f/\partial y.)$ If ϵ_2 is large enough to be significant the value of y_4 should be recomputed by (4) after y_4' has been corrected, but in many cases it is found that this step is unnecessary.

When the value of ϵ_1 is significant we must assume that $\Delta^4 y'$ is too large for the remainder terms in (2) and (3) to be ignored, and therefore we must take a smaller value for the interval h.

In actual practice these checks and corrections which it takes some time to describe can be performed very readily and (except when actual mistakes have been committed) take practically no additional time. It is desirable to record the values of the discrepancies ϵ in a column beside the values of y. Any suspicious fluctuation in the successive values of ϵ is an indication of a mistake in calculation, and then, and only then, we recheck our work far enough back to discover and remove the cause of the fluctuation.

As a very simple example let us obtain a few values of the integral of

$$\frac{dy}{dx} = 1.2y$$

with the condition that $y=1$ when $x=0$. The method of getting the first four values is explained later. All values after the first four are obtained by the process just explained.

x	y	ϵ	y'
−0 1	.88692		1.06430
.0	1.00000		1.20000
.1	1.12750		1.35300
.2	1.27125		1.52550
.3	1.43333	.00001	1.72000
.4	1.61607	1	1.93928
.5	1.82212	1	2.18654
.6	2.05443	1	2.46532
.7	2.31637	2	2.77964
.8	2.61169	1	3.13403
.9	2.94468	2	3.53362

The given differential equation has the solution

$$y, = e^{1.2x}$$

values of which may be obtained from existing tables. It is found that the values given by the method of numerical integration check in a satisfactory manner.

3. Various ways may be suggested for finding the first four values of the computation. One method which has proved very successful in practice is to calculate from the differential equation (6) the initial values y_0', y_0'', etc., from which we get the Taylor's series

$$y = y_0 + y_0'(x - x_0) + \frac{y_0''}{2!}(x - x_0)^2 + \cdots. \tag{10}$$

If as often happens, this series converges rapidly for the first two or three values of x, the corresponding values of y can be easily obtained. This method is available only when the solution is analytic at $x = x_0$.

A second method which often is very satisfactory is a method of successive approximations. For it, we require another integral formula, obtained from (1) by integration from x_2 to x_3. It is

$$\int_{x_2}^{x_3} P(x)dx = \frac{h}{24}\{-u_4 + 13u_3 + 13u_2 - u_1\} + \frac{11h\Delta^4 u_0}{720}. \tag{11}$$

We may assume that the remainder term is negligible, and with a slight change of subscripts write (11) in the form

$$\int_{x_0}^{x_1} u(x)dx = \frac{h}{24}\{-u_2 + 13u_1 + 13u_0 - u_{-1}\}. \tag{12}$$

To get started on the computation we assume trial values for y_{-1}, y_1, y_2 as follows:

$$y_{-1} = y_0 - hy_0', \quad y_1 = y_0 + hy_0', \quad y_2 = y_0 + 2hy_0',$$

and then calculate from (6) the corresponding trial values of y_{-1}', y_1', y_2'. From these we recompute y_2, y_1, y_{-1} by the formulas

$$\begin{aligned} y_2 &= y_0 + \frac{h}{3}\{y_2' + 4y_1' + y_0'\}, \\ y_1 &= y_0 + \frac{h}{24}\{-y_2' + 13y_1' + 13y_0' - y_{-1}'\}, \\ y_{-1} &= y_1 - \frac{h}{3}\{y_1' + 4y_0' + y_{-1}'\}. \end{aligned} \tag{13}$$

We again recompute y_{-1}', y_1', y_2' by (6) and y_{-1}, y_1, y_2 by (13) and continue the process until the values are unchanged by further recomputation.

For example in the problem of paragraph 2 we have

First approximation	$-.1$	.88000	1.05600
	0	1.00000	1.20000
	.1	1.12000	1.34400
	.2	1.24000	1.48800
Second approximation	$-.1$	.88720	1.06464
	0	1.00000	1.20000
	.1	1.12720	1.35264
	.2	1.26880	1.52256
Third approximation	$-.1$	.88691	1.06429
	0	1.00000	1.20000
	.1	1.12749	1.35299
	.2	1.27110	1.52532
Fourth approximation	$-.1$	.88692	1.06430
	0	1.00000	1.20000
	.1	1.12750	1.35300
	.2	1.27124	1.52549
Fifth approximation	$-.1$	.88692	1.06430
	0	1.00000	1.20000
	.1	1.12750	1.35300
	.2	1.27125	1.52550

The figures which are changed by recomputation are indicated by bold faced type.

4. This completes the discussion of the process. A few remarks may however be useful.

First, in actual practice we should make use of an additional check, the theory of which is as follows. Since Simpson's rule applies to an even number of intervals, we easily see that the values of y corresponding to even values of x are obtained by addition to preceding even values, while those corresponding to odd values are obtained by addition to preceding odd values. Hence it may come about that the accumulation of errors too small to be detected by the values of ϵ may produce an oscillation in the values of y. This may be checked by the use of an integral formula for an odd number of intervals. One of the simplest formulas is obtained from (1) by integration from x_1 to x_4 which gives

$$\int_{x_1}^{x_4} P(x)dx = \frac{h}{24}\{8u_4 + 31u_3 + 21u_2 + 13u_1 - u_0\} + \frac{3h}{720}\Delta^4 u_0. \quad (14)$$

If this formula is applied to every third line of the computation it is almost impossible for errors to escape detection.

Second, the division by 3 which must be performed in (7) and (8), except when h is divisible by 3, is done on the computing machine as follows. The dividend is on the dials of the machine as a result of the previous operation. The operator then divides by 3 mentally, recording the result on the keyboard, and then substracts three times, which affords a sure check. The quotient

is now on the keyboard ready for addition to (or subtraction from) the proper value of y.

Third, the proper value of h to secure a given degree of accuracy can be calculated theoretically by appropriate formulas of finite differences before the computation is begun, but in practice it is usually just as satisfactory to arrive at the best value by trial and error. Moreover it is often desirable to change the size of h in the course of a computation, because of a considerable change in the fourth differences.

Fourth, the method applies equally well to solutions of differential equations of the second order, and was in fact devised especially for a second order equation. The steps are as follows. We have given a differential equation

$$y'' = f(x, y, y') \tag{15}$$

and obtain the first four values by series or by successive approximations. Then we get a trial value of y' by the formula

$$y_4' = y_0' + \frac{4h}{3}\left\{2y_3'' - y_2'' + 2y_1''\right\}.$$

Then using this value of y_4' we compute a trial value of y_4 by Simpson's rule, then calculate y_4'' by (15), then recalculate y_4' by Simpson's rule, and continue until a satisfactory check is reached.

QUESTIONS AND DISCUSSIONS

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The Department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions

DISCUSSIONS

I. NEW PROPERTIES OF AN ORTHOCENTRIC SYSTEM OF TRIANGLES.

By A. A. BENNETT, Lehigh University.

Let T_1, T_2, T_3, be the vertices of an arbitrary triangle in a given plane. We shall introduce certain points in this plane associated with the triangle as follows, (where in each case $i = 1, 2, 3$),

M_i, the mid-point of the side $T_j T_k$, $i \neq j \neq k$,

T_0, the orthocenter of the triangle T_1, T_2, T_3,

M_i', the mid-point of $T_0 T_i$,

P_i, the foot of the perpendicular from T_i on $T_j T_k$,

N, the mid-point of $M_1 M_1'$, (let $N M_1 = n$).

T_0', T_1', T_2', T_3', P_1', P_2', P_3', the reflexions in N of the respective points in unprimed letters.

This system of points is called an *orthocentric system.*

The following relations are well-known and may be found for example in Altshiller-Court's *College Geometry.*

1. The lines M_1M_1', M_2M_2', M_3M_3', meet and bisect each other.

2. If one starts with the triangle T_1' T_2' T_3', instead of with the given triangle $T_1T_2T_3$, the same set of six points, M_1, M_1', M_2, M_2', M_3, M_3', would have been obtained but with the primed and unprimed letters exchanging rôles.

3. Each of the four points, T_0, T_1, T_2, T_3, is the orthocenter of the triangle of the other three. Similarly for T_0', T_1', T_2', T_3'.

4. T_h, $(h=0, 1, 2, 3)$, is the center of a circle of fixed common radius, $2n$, passing through T_i', T_j', T_k', h, i, j, k, all distinct. Similarly for the same n as originally defined, for the primed and unprimed letters interchanged.

5. N is the center of a circle, the *nine-point circle,* of radius n, through the twelve points, P_i, P_i', M_i, M_i'.

6. In each of the twelve points, P_i, P_i', M_i, M_i', meet two mutually perpendicular sides, one from the quadrilateral, $T_0\ T_1\ T_2\ T_3$ and one from the quadrilateral, $T_0' T_1' T_2' T_3'$.

It is easy to verify also the following,

7. The vector NT_0' is the sum of the vectors NT_1, NT_2, NT_3. Also in every case the vector from N to a vertex, is the sum of the vectors from N to the three medial points, M, or M', of the sides meeting in this vertex.

8. The point P_1 is the reflexion of M_1 in a line through N for which M_2 and M_3 are mutual reflexions. Similarly for the other points, P_2 and P_3.

Some new relations verified by computations with concomitants of the simultaneous cubic and quadric binary forms are the following:

There is a one-parameter linear family of cubic curves, K, each symmetric with respect to N, and each passing through the nine pointss, N, T_0, T_1, T_2, T_3, T_0', T_1', T_2', T_3'. One of these, K_M, meets the nine point circle, C, in the six medial points, M_i, M_i'. Another, K_p, meets C in the six pedal points, P_i, P_i'. A third, K_c, is a circular cubic. Furthermore the real asymptote, L, of K_c, is also the inflexional tangent at N to K_M, and is also the second polar with respect to the two circular points at infinity of the "medial line triple," NM_1, NM_2, NM_3.

There is a host of remarkable points and lines and also several classical cubic curves associated with a triangle. Most of these are so directly associated with the triangle, that they cannot be regarded as determined symmetrically for the entire orthocentric system. In contrast to these celebrated related figures it is of special interest to point out that the line L and the cubics, K_M,

K_p, and K_c, are common property of the related eight triangles, $T_0T_1T_2$, $T_1T_2T_3$, $T_2T_3T_0$, $T_3T_0T_1$, $T_0'T_1'T_2'$, $T_1'T_2'T_3'$, $T_2'T_3'T_0'$, $T_3'T_0'T_1'$, and are determined in the same position by any one of these eight triangles alone.

Since the reader may desire to check these results or continue the investigation along these same lines, we shall here list the more important relations in explicit form. To do this in an economical manner, we take the medial line triple and nine point circle as fundamental. It is of significance that here the computation deals essentially with binary forms, rather than with the ternary forms which are required for the study of the other special figures usually studied in connection with the triangle.

1. Nine point circle, $C=0$, $C\equiv x^2+y^2-1$,
2. Medial line triple, $F=0$, $F\equiv ax^3+3bx^2y+3cxy^2+dx^3$,
3. The line, $L=0$, $L\equiv(a+c)\,x+(b+d)y$,
4. The cubic, $K_M=0$, $K_M\equiv 4F-3CL$,
5. The pedal line triple, NP_1, NP_2, NP_3, written in product form $P=0$,

$$\begin{aligned}P\equiv\ &(8ad^2-12abd-36bcd-a^3+9ac^2-3a^2c+27c^3)x^3\\ &+(9a^2d-12bd^2+9c^2d-18abc-18acd-27bc^2-3a^2b+36b^2d)x^2y\\ &+(9ad^2-12a^2c+9ab^2-18bcd-18abd-27b^2c-3cd^2+36ac^2)xy^2\\ &+(8a^2d-12acd-36abc-d^3+9b^2d-3bd^2+27b^3)y^3,\end{aligned}$$

6. A second fundamental associated line, $L'=0$,

$$\begin{aligned}L'\equiv\ &(ad^2+ab^2-4bcd-a^2c+3c^3+2ac^2-2abd)x\\ &+(a^2d+c^2d-4abc-bd^2+3b^3+2b^2d-2acd)y,\end{aligned}$$

7. The cubic, K_p, $K_p=0$, where K_p can be written in either of the two forms, $P+9CL'$, and $-[(a-3c)^2+(d-3b)^2]F+9L'$.

II. On a Circle Attached to a Collapsible Four-Bar

By Helen Barton, Johns Hopkins University.

1. If in a quadrangle of four jointed rods, three of them are made to move in a plane about the fourth, which remains fixed, the resulting configuration is known as the general case of "Three Bar Motion".[1] If, however, we impose the condition that $p_1\pm p_2\pm p_3\pm p_4=0$, where p_1 is the length of the first rod, etc., we have the rational case. When there is to be any motion of the rods, two of the terms in the above relation will be positive and two negative. It is with this rational case that the problem here presented is connected.

[1] Morley, F. V. "An analytical Treatment of the Three-Bar Curve". *Proc. London Math. Soc.* Vol. 21. 1922. p. 140.

It is a known fact that if the four jointed rods are of such lengths that all are tangent to the same circle, then as they vary their relative positions, the rods will always be tangent to some circle. It is the purpose of this paper to determine the locus of the center of this circle and also to note certain characteristic properties of this collapsible quadrangle or four-bar.

Two cases naturally arise: (1) the circle is inscribed in the four-bar; (2) the circle is escribed to the four-bar.

2. In the first case, we know from the properties of the circle that the sum of the opposite sides of the circumscribed quadrangle must equal the sum of the other two sides. Thus, if we assign the numbers 1, 2, 3, 4 in order to the sides of the quadrangle, we have

$$p_1 + p_3 = p_2 + p_4. \tag{1}$$

These four jointed rods form a closed sequence or quadrangle, which may be regarded as a sequence of points and strokes. Thus if p_n = the length of any side, and θ_n = the angle between that directed side and the base line, then since the sequence of lines is closed, we have

$$\sum_{n=1}^{4} p_n e^{i\theta_n} = \sum_{n=1}^{4} p_n t_n = 0, \text{ where } t_n = e^{i\theta_n}.$$

If now we choose p_1 as the stationary rod and also as a segment of our base line, our equation becomes

$$p_1 + p_2 t_2 + p_3 t_3 + p_4 t_4 = 0\,. \tag{2}$$

This equation carries with it its conjugate equation, that is, the equation derived by replacing every term in (2) by its conjugate or image in the base line. Hence

$$p_1 + p_2 t_2^{-1} + p_3 t_3^{-1} + p_4 t_4^{-1} = 0. \tag{3}$$

To find the center of the inscribed circle, we must bisect the two interior angles at either end of p_1, and since this does not involve the side p_3, we shall eliminate t_3 and p_3 from equations (2) and (3). Eliminating t_3, we obtain

$$p_1^2 + p_2^2 + p_4^2 + p_1 p_2(t_2 + t_2^{-1}) + p_1 p_4(t_4 + t_4^{-1}) + p_2 p_4(t_2 t_4^{-1} + t_4 t_2^{-1}) = p_3^2\,. \tag{4}$$

If we represent the interior angle between p_1 and p_4 by 2α and the interior angle between p_2 and p_1 by 2β we have

$$t_2 = e^{\,i(\pi - 2\beta)} = \cos(\pi - 2\beta) + i \sin(\pi - 2\beta),$$

$$t_2^{-1} = e^{-i(\pi - 2\beta)} = \cos(\pi - 2\beta) - i \sin(\pi - 2\beta).$$

Hence $$(t_2 + t_2^{-1}) = 2 \cos(\pi - 2\beta) = -2 \cos 2\beta,$$

Similarly $$(t_4 + t_4^{-1}) = 2\cos(\pi + 2\alpha) = -2\cos 2\alpha.$$
$$(t_2 t_4^{-1} + t_4 t_2^{-1}) = 2\cos(2\alpha + 2\beta).$$

Making these substitutions in (4), we obtain

$$p_1^2 + p_2^2 + p_4^2 - 2p_1p_2\cos 2\beta - 2p_1p_4\cos 2\alpha + 2p_2p_4\cos(2\alpha + 2\beta) = p_3^2.$$

Squaring (1) we have

$$p_1^2 + p_2^2 + p_4^2 - 2p_1p_2 - 2p_1p_4 + 2p_2p_4 = p_3^2.$$

Hence $$p_1p_2(1 - \cos 2\beta) + p_1p_4(1 - \cos 2\alpha) - p_2p_4[1 - \cos(2\alpha + 2\beta)] = 0,$$

or, $$p_1p_2 \sin^2\beta + p_1p_4\sin^2\alpha - p_2p_4\sin^2(2\alpha + 2\beta) = 0.$$

It follows then from the law of sines that

$$p_1p_2r_1^2 + p_1p_4r_2^2 - p_1^2p_2p_4 = 0,$$

or

$$p_2r_1^2 + p_4r_2^2 - p_1p_2p_4 = 0, \tag{5}$$

when r_1, r_2 are the distances of the center of the inscribed circle from the two extremities of the rod p_1; r_1 being the one associated with the angle 2α and r_2 with the angle 2β.

If now, we choose the intersection of p_1 and p_4 as the origin and p_1 as coincident with the x axis, we have the following relations between r_1, r_2 and x and y, the rectangular coordinates of the center of the circle:

$$r_1^2 = x^2 + y^2, \quad r_2^2 = (p_1 - x)^2 + y^2.$$

Hence equation (5) becomes

$$\begin{aligned} &p_2(x^2 + y^2) + p_4[(p_1 - x)^2 + y^2)] = p_1p_2p_4, \\ &(x^2 + y^2)(p_2 + p_4) - 2p_1p_4x + p_1^2p_4 - p_1p_2p_4 = 0. \end{aligned} \tag{6}$$

Thus we see that the locus of the center of the inscribed circle is a circle, whose center is $(p_1p_4/(p_2+p_4),0)$ and radius equals $\sqrt{p_1p_2p_3p_4}/(p_2+p_4)$.

3. In the second case, where the circle is escribed to the four-bar, the procedure is very similar to the one just considered, Here we have the following relation existing between the sides

$$p_1 + p_4 = p_2 + p_3. \tag{7}$$

Now if we keep p_1 fixed as before, and vary the relative positions of the other three rods, it is evident that the rods may form a convex quadrangle or may become crossed, but the analytical treatment following holds for each type.

We shall use p_1 as the base line as before, then equations (2), (3), (4) hold as in the first case. Our angles, however, are slightly different, for to obtain the center (x, y) of the escribed circle, we must bisect the interior angle at one extremity of p_1 and the exterior angle at the other end.

By steps similar to those which led to equation (5), we obtain in this case

$$p_2r_1^2 - p_4r_2^2 + p_1p_2p_4 = 0. \tag{8}$$

Transforming to Cartesian coordinates, with the intersection of p_1 and p_4 as origin, and p_1 coinciding with the x axis, we obtain as the equation of the locus of the center of the escribed circle

$$(x^2 + y^2)(p_2 - p_4) - 2x(p_1p_4) + p_1p_2p_4 - p_1^2p_4 = 0. \tag{9}$$

Thus the locus in this case is also a circle, with center $(p_1p_4/(p_2-p_4),0)$ and radius $\sqrt{p_1p_2p_3p_4}/(p_2-p_4)$.

4. In each of the two cases considered, we note that the centers of the two loci fall on the fixed rod and that the centers divide the rod p_1 internally and externally in the ratio of p_4 to p_2.

If now we had kept p_2 fixed and allowed the other rods to vary their positions, the locus would have been a circle about a point on p_2, with radius $\sqrt{p_1p_2p_3p_4}/(p_1+p_3)=\sqrt{p_1p_2p_3p_4}/(p_2+p_4)$. That is, the radius of the circle, which is the locus, is a constant, no matter which rod is kept fixed. Hence, if we divide each of the four rods internally in the ratio of adjacent rods, the four points so obtained are at a constant distance from the center of the inscribed circle. Hence, as the rods vary their positions these four points always lie on a circle. The same holds true for the points obtained by dividing the rods externally in the ratio of adjacent rods.

5. It has been shown that:

given four jointed bars of such lengths that they are all tangent to the same circle,

(1) as three of these bars move about the fourth, the locus of the center of the circle, whether inscribed or escribed, is a complete circle with center on the fixed bar;

(2) in the inscribed case, the center of the locus divides the fixed bar internally in the ratio of adjacent bars; in the escribed case, the center of the locus divides the fixed bar externally in the ratio of adjacent bars;

(3) the four points of tangency of the maximum inscribed circle (or escribed) always lie on a circle with constant radius

$$\frac{\sqrt{p_1p_2p_3p_4}}{p_2+p_4}\left(\text{or } \frac{\sqrt{p_1p_2p_3p_4}}{p_2-p_4}\right),$$

no matter what the relative positions of the rods may be.

III. A Problem of Regions

By H. A. Robinson, Johns Hopkins University

1. Questions as to the number of ovals of plane curves called attention to the following analysis. The results obtained will enable one to find the number of regions formed by a number of planes in space, or more generally to the number of regions formed by a number of m-dimensional flat space cuts in an $m+1$ dimensional space.[1]

If a line has no *cuts* on it, it may be said to consist of one 1-dimensional region. One point or 0-dimensional cut [0-cut] on a line divides the line into two regions, two cuts into three regions, and obviously n cuts (no two cuts coinciding) into $n+1$ regions.

A plane free of cuts may be said to form one 2-dimensional region. One line or 1-dimensional cut [1-cut] divides the plane into two regions; two cuts, four regions[2]; three cuts, seven regions—one of which is closed[3] and the other six open; four cuts, eleven regions—three of which are closed and eight open; etc. Now it may be readily observed that the added nth line or 1-cut, intersects the existing 1-cuts in n−1 points or 0-cuts. The n 1-dimensional regions formed by the $n-1$ 0-cuts on the nth 1-cut pass through n existing 2-dimensional regions, and hence add n new regions to the plane. Therefore the number of regions which n 1-cuts form in a plane satisfies the equation

$$I^1_n = I^1_{n-1} + I^0_{n-1}$$

where I represents the number of regions; the superscript, the dimension of the cutting space; and the subscript, the number of cuts. Since $I^0_{n-1} = n$, we have

$$I^1_n - I^1_{n-1} = n.$$

Solving this partial difference equation and adjusting the constant, we find

$$I^1_n = 1 + \tfrac{1}{2}\, n(n+1).$$

A three way space free of plane or 2-cuts may be said to form one 3-dimensional region. One 2-cut divides the space into two regions; two cuts, four regions; three cuts, eight regions; four cuts, fifteen regions—one of which is closed and fourteen open; five cuts, twenty-six regions—four of which are

[1] *Cf.* Steiner, "Über die Teilung der Ebene und des Raumes," *Crelle*, vol. 1, p. 349. A. J. Kempner.

[2] Through the whole article, it will be assumed that every cutting space will intersect every other cutting space the maximum number of times. Thus every line in the plane will intersect every other line in the plane. Also no three lines shall meet in a point, for if they did we would not have the maximum number of regions.

[3] By closed regions we mean regions which are entirely bounded by cutting spaces.

closed; etc. Now it may be noted that the added nth 2-cut, intersects the existing $n-1$ 2-cuts in $n-1$ 1-cuts. These $n-1$ 1-cuts form on the nth 2-cut I^1_{n-1} plane regions and hence add I^1_{n-1} new regions to the three way space. Therefore the number of regions formed by n 2-cuts in the space in question may be represented by the following equation

$$I^2_n = I^2_{n-1} + I^1_{n-1},$$

the solution of which is (constant adjusted),

$$I^2_n = \tfrac{1}{6}\, n(n+1)(n+2) - \tfrac{1}{2}\,(n-2)(n+1)\,.$$

There is no logical reason why the investigation should stop here. An $m+1$ way space free of m-cuts may be thought of as forming one $m+1$ dimensional region. One m-cut divides the space into two regions; two cuts, four regions; etc. The first $m+1$ cuts form no closed regions; the next cut would form one closed region; the succeeding, $m+2$ closed regions; etc. The added nth m-cut intersects the existing $n-1$ m-cuts in $n-1$ $(m-1)$-cuts. These $n-1$ $(m-1)$-cuts form on the nth m-cut I^{m-1}_{n-1} regions and hence add I^{m-1}_{n-1} new regions to the $m+1$ way space. Therefore the number of regions formed by n m-cuts in an $m+1$ way space may be represented by the following equation

$$I^m_n = I^m_{n-1} + I^{m-1}_{n-1}.$$

Let us tabulate the results found. Table A gives the number of regions produced by a given number of cuts; Table B, the number of open regions

Table A

The Total Number of Regions Formed by I cuts

n	I^0_n	I^1_n	I^2_n	I^3_n	I^4_n	......
0	1	1	1	1	1	
1	2	2	2	2	2	
2	3	4	4	4	4	
3	4	7	8	8	8	
4	5	11	15	16	16	
5	6	16	26	31	32	
6	7	22	42	57	63	
...	...	....	....	....	....	......

Table B

The Number of Open Regions Formed by I cuts

n	oI^0_n	oI^1_n	oI^2_n	oI^3_n	oI^4_n	......
0	1	1	1	1	1	
1	2	2	2	2	2	
2	2	4	4	4	4	
3	2	6	8	8	8	
4	2	8	14	16	16	
5	2	10	22	30	32	
6	2	12	32	52	62	
...	...	....	....	....	....	......

Table C

The Number of Closed Regions Formed by I cuts

n	cI^0_n	cI^1_n	cI^2_n	cI^3_n	cI^4_n	......
0	0	0	0	0	0	
1	0	0	0	0	0	
2	1	0	0	0	0	
3	2	1	0	0	0	
4	3	3	1	0	0	
5	4	6	4	1	0	
6	5	10	10	5	1	
...	...	....	....	....	....	......

formed; and Table C, the number of closed regions. Upon examination of Table C we note that its matrix contains the numbers of Pascal's triangle, *i.e.*, the binomial coefficients (with the exception of a border of ones on the left hand side). Each element of Table B is a linear combination of those found in C. Now the general law, as may be read from the tables, for the total number of regions is

$$I_n^m = \sum 2^k C_{n-k-1}^{m-k+1},$$

where C_s^r means the coefficient found in the sth row and the rth column of *Pascal's matrix* (*i.e.*, Pascal's triangle made into rectangle form by the addition of zeros), and k runs from 0 to the particular value that will make first the superscript or subscript zero. In order to illustrate the notation, let us consider the following example. Given six planes in a three way space, of which every plane intersects every other plane.[1] The number of regions formed is

$$\begin{aligned} I_6^2 &= 2^0C_5^3 + 2^1C_4^2 + 2^2C_3^1 + 2^3C_2^0 \\ &= 1\cdot 10 + 2\cdot 6 \ \ + 4\cdot 3 \ \ + 8\cdot 1 \ \ = 42. \end{aligned}$$

2. The above has dealt with a system of cutting spaces that intersect each other in *one* point, *one* line, etc. We have denoted the number of regions by I. Let us now consider circles (and their logical extension to higher spaces) as our cutting spaces in a plane and let us denote the number of regions formed in the plane by II_n^1. Since two circles intersect in at most *two* real points, we will here denote the number of regions by II.

A circle free of *cuts* may be said to consist of one arc or region. One *two point* or (0)-cut on a circle divides the circle into two regions; two cuts, four regions; three cuts, six regions; etc. Obviously n cuts form $2n$ regions.

A plane free of circle-cuts or (1)-cuts may be said to consist of one two-dimensional region. One (1)-cut divides the plane into two regions; two cuts, four regions; three cuts, eight regions; four cuts, fourteen regions; etc. We note now that the existing $n-1$ (1)-cuts intersect the nth (1)-cut in $n-1$ (0)-cuts. These $n-1$ (0)-cuts on the circle form II_{n-1}^0 arcs and hence add II_{n-1}^0 new regions to the plane. Therefore

$$II_n^1 = II_{n-1}^1 + II_{n-1}^0,$$

the solution of which is

$$II_n^1 = n(n-1) + 2.$$

[1] This problem is given in many puzzle books: How many pieces of cheese will six straight cuts of a knife divide a cake into?

When circles are *opened out* into lines the number of regions lost is equal to the number of closed regions found in the case of lines in a plane. Hence we have a way to pass from line to circle regions.

In a three way space, spheres [(2)-cuts] would be used as the cutting spaces.

The extension may be made to an $m+1$ way space. The number of regions formed by n (m)-cuts would be given by

$$II_n^m = II_{n-1}^m + II_{n-1}^{m-1}.$$

When the (m)-cuts are *opened out* into m-cuts the number of regions lost is equal to the number of closed regions found in case of m-cuts. Hence

$$II_n^m = I_n^m + \text{Closed } I_n^m = [\sum 2^k C_{n-k-1}^{m-k+1}] + C_{n-1}^{m+1}.$$

Table D gives the number of regions produced by a given number of (i)-cuts in an $i+1$ way space.

TABLE D

The Total Number of Regions Formed by II cuts

n	II_n^0	II_n^1	II_n^2	II_n^3	II_n^4	......
0	1	1	1	1	1	
1	2	2	2	2	2	
2	4	4	4	4	4	
3	6	8	8	8	8	
4	8	14	16	16	16	
5	10	22	30	32	32	
6	12	32	52	62	64	
...	...	....	....	....	....	......

It would be interesting to have an extension of this notion to A_n^m, where $A>II$. [A is the number of times an initial cutting space would intersect another initial cutting space.]

IV. A Modified Method for Cube Roots and Fifth Roots

By L. S. Dederick, St. Stephen's College

In a recent number of this Monthly (*1925*, 377) Mr. D. H. Lehmer gave a short method of extracting cube roots and fifth roots with the help of the calculating machine, the essential feature being to express the number as a product of factors of the form $1+p.10^{-n}$ where p is an integer less than 10. The purpose of this note is to present a method which is similar but even shorter. The difference is that the factors are the *reciprocals*[1] of a similar set

[1] It seems probable that this is the method intended in the article on logarithms in the *Encyc. Brit.* to which Mr. Lehmer gives a reference. There is in that article an obvious error, which Mr. Lehmer apparently aims to correct by a change of sign. It seems more probable, however, that the correction should be a transfer of the factor to the denominator.

of simple numbers, given in the accompanying table, and hence may be taken out by *multiplication* instead of division. A rule for the 'factoring' may be formulated thus:

Multiply or divide the number by such a power of 10 *that the result shall lie between 1 and 10. Multiply the result by the smallest number n in the table that will leave it greater than 1. Repeat this process until the result is sensibly unity.*

The factors then are a power of 10 and the reciprocals of the numbers by which we have multiplied. The cube roots and fifth roots of these reciprocals are accordingly the numbers tabulated, and the factors of the required root are such of these as correspond to the multipliers used.

The shortening of the work is greater than merely the substitution of multiplication for division. The first multiplication will in any case produce a result less than 1.2. The multiplications after that will be by numbers greater than or equal to 0.9. Each of these multiplications may be performed by *subtracting* a multiple of the number in the machine by a number of one digit. Thus to multiply by .94 we subtract .06 of the number. The 6 will appear in red on the counting dials and should be left there, since the next multiplier is regularly in the next decimal place. This means that the counting dials need be cleared (if at all) only after the first multiplication. After that a succession of red digits on these dials such as 13609 means $[.9\times.97\times.994\times.99991]^{-1}$. For convenience of reference, the latter part of the table is made to agree with this, the entry in the first column being $1-n$, the same digit which appears on the counting dials. Of course after each multiplication the new product is transferred to the keys.

As in the other method, after half of the factors are determined, the remaining half can be found by inspection. It makes the work shorter, however, as well as more accurate, to observe that at this stage we may substitute $1+\frac{1}{3}x$ for $\sqrt[3]{1+x}$ and $1+\frac{1}{5}x$ for $\sqrt[5]{1+x}$. This enables us to replace this second group of factors by a single one of which the root may be written without even consulting the table.

These simplifications have made it appear worth while to compute the accompanying table to 16 significant figures. This provides for work involving two complete periods on an eight-column machine or a period and a half on one with ten columns. The latter case is mentioned because, if the number of figures does not exceed the number of columns of keys by more than one half, the work of multiplying is much shorter than in the contrary case.

The following examples are on an eight-column machine.

Example 1. To find the cube root of 165.97234. Multiplying this by .01 and 0.7 gives 1.1618064. Multiplying by 0.9, 0.96, etc. causes the red digits 14378 to appear on the counting dials, the next ones being 875. From the table we get the corresponding factors of the cube root as $4.6415888\times1.1262479\times1.0357442\times1.0137003\times1.0010020\times1.0002334$; and a final factor 1.0000296 is obtained by dividing .00008875 by 3. The product of these seven factors is 5.4955593, the required cube root.

Example 2. To find the fifth root of 4.8278165. The first multiplier is .23 and reduces the number to 1.1103978. For this 0.9 is too small and .91 (appearing as .09) gives 1.0104620. This illustrates the very unusual case where the next multiplier .99 appears in the same decimal place, namely as .01 on the dials. The mechanical effect is merely to convert the red 9 into a *black* 9. The rest of the factors present nothing unusual. In taking the factors from the table we merely need to remember that a black 9 signifies two factors belonging to the same decimal place, one with the digit 9, the other with the digit 1. The factors here then are 1.3416970×1.0190411×1.0020121×1.0000715 giving the root 1.3700934. If this is tested by raising to the fifth power, a correction is found necessary, the corrected value being 1.3700933.

RECIPROCALS OF CUBE AND FIFTH ROOTS

n	$1\div\sqrt[3]{n}$	$1\div\sqrt[5]{n}$
.0001		6.3095734 44801932
.001		3.9810717 05534973
.01	4.6415888 33612779	2.5118864 31509580
.1	2.1544346 90031884	1.5848931 92461113
.12	2.0274006 65191133	1.5281421 35815799
.14	1.9258567 85554179	1.4817481 47204298
.16	1.8420157 49320193	1.4426999 05907214
.18	1.7710976 15304352	1.4091119 53304333
.2	1.7099759 46676697	1.3797296 61461215
.23	1.6321399 23033617	1.3416969 50852690
.26	1.5667831 20190969	1.3091979 91621230
.3	1.4938015 82185722	1.2722596 36539392
.35	1.4189834 11970384	1.2336341 72516721
.4	1.3572088 08297453	1.2011244 33981431
.45	1.3049558 80389621	1.1731606 76311841
.5	1.2599210 49894873	1.1486983 54997035
.6	1.1856311 01496688	1.1075663 43248290
.7	1.1262478 80443606	1.0739409 23785779
.8	1.0772173 45015942	1.0456395 52591273
.9	1.0357441 68651286	1.0212956 87600135
$1-n$		
.09	1.0319362 51301859	1.0190411 49735922
.08	1.0281837 22701926	1.0168161 47821955
.07	1.0244851 88140280	1.0146199 80122523
.06	1.0208393 02540953	1.0124519 68893663
.05	1.0172447 68191101	1.0103114 59317936
.04	1.0137003 32595567	1.0081978 18497167
.03	1.0102047 86449813	1.0061104 34499387
.02	1.0067569 61723556	1.0040487 15456574
.01	1.0033557 29847986	1.0020120 88709965
$.0^{2}9$	1.0030181 26952430	1.0018097 84617414
$.0^{2}8$	1.0026809 77976795	1.0016077 25346310
$.0^{2}7$	1.0023442 81854107	1.0014059 10354030
$.0^{2}6$	1.0020080 37520976	1.0012043 39099701
$.0^{2}5$	1.0016722 43917573	1.0010030 11044186
$.0^{2}4$	1.0013368 99987618	1.0008019 25650083
$.0^{2}3$	1.0010020 04678364	1.0006010 82381717
$.0^{2}2$	1.0006675 56940580	1.0004004 80705128
$.0^{2}1$	1.0003335 55728539	1.0002001 20088070

$1-n$	$1\div\sqrt[3]{n}$	$1\div\sqrt[5]{n}$
$.0^{3}9$	$1.0^{3}3001$ 80126095	$1.0^{3}1800$ 97264198
$.0^{3}8$	$1.0^{3}2668$ 08977442	$1.0^{3}1600$ 76845085
$.0^{3}7$	$1.0^{3}2334$ 42281541	$1.0^{3}1400$ 58830201
$.0^{3}6$	$1.0^{3}2000$ 80037352	$1.0^{3}1200$ 43219017
$.0^{3}5$	$1.0^{3}1667$ 22243836	$1.0^{3}1000$ 30011004
$.0^{3}4$	$1.0^{3}1333$ 68899954	$1.0^{3}0800$ 19205634
$.0^{3}3$	$1.0^{3}1000$ 20004668	$1.0^{3}0600$ 10802377
$.0^{3}2$	$1.0^{3}0666$ 75556939	$1.0^{3}0400$ 04800704
$.0^{3}1$	$1.0^{3}0333$ 35555728	$1.0^{3}0200$ 01200088
$.0^{4}9$	$1.0^{4}300$ 01800126	$1.0^{4}180$ 00972064
$.0^{4}8$	$1.0^{4}266$ 68088977	$1.0^{4}160$ 00768045
$.0^{4}7$	$1.0^{4}233$ 34422282	$1.0^{4}140$ 00588030
$.0^{4}6$	$1.0^{4}200$ 00800037	$1.0^{4}120$ 00432019
$.0^{4}5$	$1.0^{4}166$ 67222244	$1.0^{4}100$ 00300011
$.0^{4}4$	$1.0^{4}133$ 33688900	$1.0^{4}080$ 00192006
$.0^{4}3$	$1.0^{4}100$ 00200005	$1.0^{4}060$ 00108002
$.0^{4}2$	$1.0^{4}066$ 66755557	$1.0^{4}040$ 00048001
$.0^{4}1$	$1.0^{4}033$ 33355556	$1.0^{4}020$ 00012000
$.0^{5}9$	$1.0^{5}30$ 00018000	$1.0^{5}18$ 00009720
$.0^{5}8$	$1.0^{5}26$ 66680889	$1.0^{5}16$ 00007680
$.0^{5}7$	$1.0^{5}23$ 33344222	$1.0^{5}14$ 00005880
$.0^{5}6$	$1.0^{5}20$ 00008000	$1.0^{5}12$ 00004320
$.0^{5}5$	$1.0^{5}16$ 66672222	$1.0^{5}10$ 00003000
$.0^{5}4$	$1.0^{5}13$ 33336889	$1.0^{5}08$ 00001920
$.0^{5}3$	$1.0^{5}10$ 00002000	$1.0^{5}06$ 00001080
$.0^{5}2$	$1.0^{5}06$ 66667556	$1.0^{5}04$ 00000480
$.0^{5}1$	$1.0^{5}03$ 33333556	$1.0^{5}02$ 00000120
$.0^{6}9$	$1.0^{6}3$ 00000180	$1.0^{6}1$ 80000097
$.0^{6}8$	$1.0^{6}2$ 66666809	$1.0^{6}1$ 60000077
$.0^{6}7$	$1.0^{6}2$ 33333442	$1.0^{6}1$ 40000059
$.0^{6}6$	$1.0^{6}2$ 00000080	$1.0^{6}1$ 20000043
$.0^{6}5$	$1.0^{6}1$ 66666722	$1.0^{6}1$ 00000030
$.0^{6}4$	$1.0^{6}1$ 33333369	$1.0^{6}0$ 80000019
$.0^{6}3$	$1.0^{6}1$ 00000020	$1.0^{6}0$ 60000011
$.0^{6}2$	$1.0^{6}0$ 66666676	$1.0^{6}0$ 40000005
$.0^{6}1$	$1.0^{6}0$ 33333336	$1.0^{6}0$ 20000001
$.0^{7}9$	1.0^{7} 30000002	1.0^{7} 18000001

Example 3. To find the cube root of 6.7304592 1819061. This when multiplied by .16 gives 1.0768734-74910498. The next multiplier is obviously 1-.07. Before applying it to the first eight digits we observe that the first ten will be affected. Hence we make a mental note of the ninth and tenth, namely 74. After subtracting .010768734 seven times we shift the carriage, put on the keys 749105 and subtract this seven times from the last six digits. At each stage, if we carry in mind a group of not more than four digits from one part of the subtraction to the other, we can effect all the factoring multiplications without any writing or shifting of columns. After the preliminary .16 we get the factors indicated by the digits 07149063 4841112. Only five of these are taken from the table. The last is 1.0000000 11613704, found by dividing the figures 34841112 by 3. The required cube root is 1.8880561 2016121. It is convenient here to use 16-figure factors while taking the original number and the root to 15 figures.

V. A New Calculation of π

By C. C. Camp, University of Illinois

In the May number of the Monthly (*1925*, 253) Professor Bennett gave two hitherto unpublished equations for $\pi/4$. That article is partly responsible for the calculation which follows although my formula was arrived at in January, 1924. Several other formulas were then derived, including his form (8), p. 254, which did not seem at that time to be a sufficient improvement over those used by Rutherford and Machin. In any such calculation one must consider not only the total number of terms necessary for a given number of decimals but also the ease in computing.

Machin's formula requires about .93 n terms where n is the number of decimals employed. The formula[1] used in the present calculation, namely

$$(\pi/4)=8\arctan(1/10)-\arctan(1/239)-4\arctan(1/515)$$

requires about .898 n terms while the forms given by Professor Bennett require about 1.125 n and 1.118 n terms, respectively.

As for the numerical calculation with Gregory's series, none could be easier than that for arctan (1/10) and the others are readily computed with a Monroe calculator.

The labor may be lessened by using the coefficients as numerators in the several terms to avoid later multiplication. One might combine terms before doing the divisions by 3,5,7, etc. In any case for the last two arctangents it is best to divide by 239^2 or 515^2 successively on the machine, and one may well write alternate terms on different sheets to separate the positive and negative terms. The maximum error was estimated as 12 in the 56th decimal place but the actual error in $\pi/4$ was zero there. The series used were $(8/10-8/3\cdot 10^3+8/5\cdot 10^5-\cdots)-(1/239-1/3\cdot 239^3+1/5\cdot 239^5-\cdots)-(4/515-4/3\cdot 515^3+4/5\cdot 515^5-\cdots)$ and the several terms are exhibited below:

[1] A sketch of the history of this formula is given by H. J. Heyman in *Archivio di Storia della Scienza*, vol. 6 (1925), p. 113. The earliest known reference to it is in a manuscript by S. Klingenstierna (1698–1765) which has recently been found at Upsala. The manuscript is dated "Londini d. 7 Aprilis 1730." Editor.

$8/10$	.8				
$8/5 \cdot 10^5$	.000016				
$8/9 \cdot 10^9$	.00000000088888888888	8888888888	8888888888	8888888888	888889−
$8/13 \cdot 10^{13}$	.00000000000006153846	1538461538	4615384615	3846153846	153846
$8/17 \cdot 10^{17}$	.00 ... 0470	5882352941	1764705882	3529411764	705882
$8/21 \cdot 10^{21}$	.00 ...	0380952380	9523809523	8095238095	238095
$8/25 \cdot 10^{25}$	.00 ...	032000			
$8/29 \cdot 10^{29}$	.00 ...	02	7586206896	5517241379	310345
$8/33 \cdot 10^{33}$	.00 ...		02424242	4242424242	424242
$8/37 \cdot 10^{37}$	.00 ...		0216	2162162162	162162
$8/41 \cdot 10^{41}$	.00 ...			0195121951	219512
$8/45 \cdot 10^{45}$	.00 ...			017777	777778−
$8/49 \cdot 10^{49}$	.00 ...			01	632653
$8/53 \cdot 10^{53}$	.00 ...				0151−
$1/3 \cdot 239^3$	.00000002441659178708	3803627411	8923012459	5182064392	070692
$1/7 \cdot 239^7$	.00 ... 0320	7130657784	6947170443	5835310750	411777
$1/11 \cdot 239^{11}$	.00 ...	0625	5044509921	5653492927	733316
$1/15 \cdot 239^{15}$	.00 ...		01405	8540188257	236882−
$1/19 \cdot 239^{19}$	.00 ...			03401	623797−
$1/23 \cdot 239^{23}$	.00 ...				01−
$4/3 \cdot 515^3$	.00000000976151103310	0368771583	9973448689	9899669969	412915
$4/7 \cdot 515^7$	.00 ... 05	9471836258	2567308933	7280932316	570314
$4/11 \cdot 515^{11}$	.00 ...		5380068270	8155093792	305323
$4/15 \cdot 515^{15}$	.00 ...			0560867527	794550
$4/19 \cdot 515^{19}$	.00 ...				006295−
Total	.80001603506705325550	7465581417	1216940390	7584283444	679417
	Negative Terms				
$8/3 \cdot 10^3$	.00266666666666666666	6666666666	6666666666	6666666666	666667−
$8/7 \cdot 10^7$	.00000011428571428571	4285714285	7142857142	8571428571	428571
$8/11 \cdot 10^{11}$	.00 ...0727272727	2727272727	2727272727	2727272727	272727
$8/15 \cdot 10^{15}$	.00 ... 053333	3333333333	3333333333	3333333333	333333
$8/19 \cdot 10^{19}$	.00 ... 04	2105263157	8947368421	0526315789	473684
$8/23 \cdot 10^{23}$	.00 ...	03478260	8695652173	9130434782	608696−
$8/27 \cdot 10^{27}$	.00 ...	0296	2962962962	9629629629	629630−
$8/31 \cdot 10^{31}$	.00 ...		0258064516	1290322580	645161
$8/35 \cdot 10^{35}$	.00 ...		022857	1428571428	571429−
$8/39 \cdot 10^{39}$	.00 ...		02	0512820512	820513−
$8/43 \cdot 10^{43}$	.00 ...			01860465	116279
$8/47 \cdot 10^{47}$	.00 ...			0170	212766−
$8/51 \cdot 10^{51}$	.00 ...				015686
$8/55 \cdot 10^{55}$	.00 ...				01
$1/239$	.00418410041841004184	1004184100	4184100418	4100418410	041841
$1/5 \cdot 239^5$	.00 ...0025647231	4424647365	7052071108	8299523979	598313
$1/9 \cdot 239^9$	.00 ...	0043669315	2440391897	4984975066	914976−
$1/13 \cdot 239^{13}$	.00 ...		0092658216	2415509570	671442
$1/17 \cdot 239^{17}$	.00 ...			0217163464	991096−
$1/21 \cdot 239^{21}$	.00 ...				053880−
$4/515$	.00776699029126213592	2330097087	3786407766	9902912621	359223
$4/5 \cdot 515^5$	.00 ...0002208278	4880234598	0316727246	9383127306	256882
$4/9 \cdot 515^9$	.00 ...	0174402	3853155275	3584633996	931836
$4/13 \cdot 515^{13}$	.00 ...		017164	1642376126	520071−
$4/17 \cdot 515^{17}$	.00 ...			01865	899190
Total	−.01461787166960494589	1804735597	2459729897	8349299067	033893
Positive	.80001603506705325550	7465581417	1216940390	7584283444	679417
Sum, $\pi/4$	.78539816339744830961	5660845819	8757210492	9234984377	645524
$\pi =$	3.14159265358979323846	2643383279	5028841971	6939937510	5820975−

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

Einführung in die Determinantentheorie einschliesslich der Fredholmschen Determinanten. By GERHARD KOWALEWSKI. Second, abridged edition. Berlin, Walter de Gruyter & Co., 1925. 298 pages.

The reviewer's pleasure at hearing that a new edition of Kowalewski's admirable book on determinants is available was considerably tempered when he discovered the nature of the revision. The original work stood in a class by itself not because of a particularly noteworthy exposition of the elements of determinants, though that feature was worthy of praise, but because it brought together in one volume a treatment of analogous concepts from three different fields. The ordinary algebraic determinant found a place in mathematics as a device for solving systems of linear algebraic equations. The "infinite" determinant found a place as a device for solving systems of infinitely many linear equations in infinitely many unknowns. The Fredholm determinant found a place as a device for solving linear integral equations. It was highly fitting that these three concepts should be treated in neighboring pages. And considering the comparative newness of the latter two notions at the time of the first edition of Kowalewski's work (1909), the author deserved great praise for his presentation. In view of this the reviewer had hopes of finding in a second edition, after an interval of sixteen years, some advancement along two different lines. First it was to be expected that the subjects treated, which during the sixteen years have been very much alive, should be recast and brought up to date. Second it was at least within reason to hope that some attempt might be made to *unify* the three concepts mentioned by means of a generalization, of which the three cases treated would be special instances. The latter would have been a particularly acceptable and noteworthy contribution.

It is then a disappointment to note that the edition under review is a mere *abridgement* of the first edition. For this, not the author but the publishers are to blame, according to a statement in the brief preface. Under pressure, the author has reduced the number of pages from 540 to 298. Entire chapters which have been eliminated are those which treated of elementary divisors, determinants of infinitely high order, and Hilbert's characteristic functions of a real symmetric kernel.

Of the chapters which remain, the first ten dealing with the pure theory of the algebraic determinants with applications to systems of linear equations appear to be essentially the same as in the original edition.

Chapter XI on resultants and discriminants, reduced from 32 pages to 9, contains a brief but elegant treatment of Sylvester's dialytic method, and an indication of Bezout's form of the resultant for two forms of the same degree n, details being given for $n=2$ and 3.

Chapter XII on linear and quadratic forms contains most of the material of the corresponding chapter in the first edition, and is particularly pleasing.

Chapter XIII on functional determinants contains a definition of these determinants and a proof of their law of multiplication together with certain applications of this law in about 6 pages. Another 6 pages are devoted to a detailed proof that under suitable restrictions the Jacobian can be obtained as a *limit* of a quotient of two determinants, the elements of the denominator

$$\frac{d(u_1, \cdots u_n)}{d(x_1, \cdots x_n)}$$

being suitably restricted increments of the independent variables x_i, and the elements of the numerator being corresponding increments of the functions u_i. The remaining 12 pages of the chapter have to do with systems of functions from the function-theoretic point of view. The proofs depend largely upon the theory of point sets, and bear little relation to the remainder of the book. In the interest of the much desired condensation, this material might have been replaced by a bare statement of facts without serious loss. All the proofs are easily available elsewhere in at least as satisfactory form.

Chapter XIV on Wronskian and Gramian determinants contains much of the material of the first edition on linear dependence of functions. The notion of a normalized orthogonal system is presented in an elegant fashion. It is to be regretted that space did not permit the inclusion here of the elements of the geometry of function space which the author has developed in memoirs which unfortunately are inaccessible to many students.

Chapter XV on some geometric applications of determinants contains the material of the first edition with the exception of the geometric interpretations of the Jacobian.

Chapter XVI on linear integral equations is apparently identical with the corresponding chapter in the original.

The second edition like the first is remarkably free from errors. A few discrepancies have arisen from the abridgement. For example the chapter under the heading *Resultants and Discriminants* contains no reference to the latter. Again the reference in the final chapter to von Koch's theory of infinite determinants is of little value since the chapter on infinite determinants has been eliminated.

Notwithstanding our regret that the revision is an abridgement, it must be confessed that the book fits its title quite as well as it did before, and it will be a very valuable aid to students.

Of special interest to Americans is the fact that the dedication is to Professor Alexander Ziwet.

L. L. DINES

Lebesguesche Integrale und Fouriersche Reihen. By L. SCHLESINGER AND A. PLESSNER. Berlin, Walter de Gruyter & Co., 1926. viii+229 pages.

The present book is the outcome of a course of lectures given by Professor Schlesinger at Giessen in 1921–2. It is intended to be an introductory treatment of the theory of Lebesgue integration, and is consequently more restricted and more elementary than the treatises of Hausdorff, Carathéodory, and Hahn. In point of difficulty, it stands between these and de la Vallée-Poussin's monograph in the Borel series. The book is divided into six chapters, whose titles are self-explanatory: The fundamental concepts of the theory of sets, the measure of sets of points, functions of real variables, the Lebesgue integral, functions of one and of two variables, Fourier series. The last of these is inserted as an illustration of the importance of the Lebesgue theory in the investigation of a classic problem of analysis. The entire book is thorough, accurate, readable, and well documented.

Since the presentation of the subject-matter does not depart materially from the standard treatments, it is unnecessary to make much detailed comment. It seems to be in place to note certain omissions which are due to the tastes and aims of the authors. Apparently the authors are not disturbed by the battles being waged over the Zermelo principle: they allow it to do its work unseen. They pass over Baire's classification of functions very lightly, no doubt because they propose to focus attention on the theory of integration. Nothing is said concerning the existence or the construction of non-measurable sets. It seems to the reviewer that a few pages devoted to these subjects would provide illumination and stimulation for the student without leading him too far afield.

Typographically the book has been well prepared. Such minor errata as the reviewer noted can be corrected easily as one reads.

M. H. STONE

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) title of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Acta Mathematica, volume 47, no. 4: "An extension of Poincaré's last geometric theorem" by George D. Birkhoff, 297-311; "Functions of a complex variable with assigned derivatives at an infinite number of points, and an analogue of Mittag-Leffler's theorem" by Philip Franklin, 371-385.

Bulletin of the American Mathematical Society, volume 32, no. 1, January-February, 1926: "The Heaviside operational calculus" by J. R. Carson, 43-68; "Note on a fundamental lemma concerning the limit of a sum" by H. J. Ettlinger, 69-70; "Note on rational plane cubics" by C. A. Nelson, 71-76; "A trivial Tauberian theorem" by W. A. Hurwitz, 77-82.

Journal of Mathematics and Physics, Massachusetts Institute of Technology, volume 5, no. 3, March, 1926: "The harmonic analysis of irregular motion (Second paper)" by Norbert Wiener, 158-189.

Mathematische Annalen, volume 95, no.4, February, 1926; "The operational calculus" by Norbert Wiener, 557-584.

Rendiconti del Circolo Mathematico di Palermo, volume 49, no. 2, May-August, 1925: "Trigonometric realms of rationality" by H. Hancock, 263-276; "Asymptotic satellites near the straight line equilibrium points (elliptical case)" by D. Buchanan, 299-304.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE MATHEMATICS CLUB OF COOPER UNION, New York City

The officers for the year 1925-1926 were: Fred Miller, '26, president; Thaddeus Slonczewski, '26, vice president; Walter Judson, '26, secretary and treasurer. Meetings were held at intervals of three or four weeks throughout the year, with the following program:

Nov. 11, 1925. "The use of mathematics in the inspection of gauges and standards" by Mr. Charles H. Lehmann, telephone engineer.

Dec. 9. "The theory and use of the planimeter, with an exhibition of instruments" by Bert Speier '26.

Jan. 6, 1926. "Methods of constructing magic squares" by Fred Mertens '29.

Jan. 27. "Constructions with straight-edge only; constructions with compasses only" by Samuel Lubkin '27.

Feb. 17. "Diophantine equations" by Thaddeus Slonczewski '26.

Mar. 17. "New uses of the slide-rule" by Professor W. E. Breckenridge.

Apr. 7. "The object of a mathematics club" by Professor H. W. Reddick. Discussion of plans for next year, led by Professor W. J. Pickett. Election of officers.

(Report by Mr. Judson)

THE MATOON MATHEMATICS CLUB of Park College, Parkville, Missouri.

The officers of the Club for the year 1925–1926 were: Ora Gates, president; Edna Buckley, vice president; Gladys McClave, secretary and treasurer.

The Club was organized primarily for those who have chosen mathematics as either a major or minor. The attendance at all meetings was very good and a greater interest in mathematics has resulted.

The program for the first semester of the year 1925–1926 was the following:

October 8. "The fourth dimension" by Margaret Creegler.

October 22. "De Moivre's theorem" by Professor R. A. Wells.

November 12. "History of geometry" by Marian Brown.

November 19. "Proofs of Pythagorean theorem" by John Waterman.

December 10. "Development of formula for finding length of sides of a right triangle" by Herma Hudson.

January 14. "Modern views of space" by Professor R. A. Wells.

(Report by Miss Gladys McClave, secretary)

THE MATHEMATICAL CLUB of Rutgers University, New Brunswick, New Jersey

The following papers were read before the Club during the academic year 1925–1926:

"The anharmonic ratio" by Professor Richard Morris.

"Parallel lines" by Raymond J. Seeger.

"The Brachistochrone" by Professor S. E. Brasefield.

"Sturm's functions" by Joseph P. Bogdan.

"The cubic equation" by Professor W. V. N. Garretson.

"Cross ratio" by Frank W. Malsbury.

"Trilinear coordinates" by Professor Richard Morris.

"Paradoxes of probability" by R. J. Seeger.

"Nomography" by Herman B. Dresser.

"Relativity" by Howard B. Waxwood, Jr.

"The rectification of the ellipse" by Professor W. E. Breazeale.

"Archimedes' spiral" by R. J. Seeger.

(Report by Professor Morris)

PI MU EPSILON, Bucknell University, Lewisburg, Pennsylvania

Officers for the year 1925–1926: Director, Prof. H. S. Everett; Vice-Director, W. I. Miller, '26; Secretary, M. Pauline Lindley, '26; Treasurer, P. C. Wallace, '26; Librarian, Katheryne E. Miller, '26.

Program for the year 1925–1926:

Oct. 26. Business meeting, election to membership.

Nov. 11. Annual initiation and banquet.

Dec. 17. "Groups, an exposition of the field" by Prof. H. S. Everett.

Jan. 21. "Our pioneer mathematics professors, who they were and what they did" by Prof. W. C. Bartol, head of the department for forty-seven years.

Feb. 24. "The origin of certain mathematical symbols" by Miss Katherine Gaventa, '27.

Mar. 25. "Atomic structure" by Professor V. B. Hall.

May 26. Annual picnic and election of officers. At this meeting a prize (slide rule) was voted that member of the integral calculus classes who attained the highest standing in regular freshman and sophomore mathematics. Awarded to V. W. McHail, '28.

The following officers were elected for 1926–1927: Director, Prof. H. S. Everett; Vice-Director, Christopher Mathewson, Jr., '27; Secretary, Agnes Dunbar, '27; Treasurer, K. W. Horsman, '27; Librarian, Evelyn Deen, '27.

THE MATHEMATICS CLUB of the New Jersey College for Women, New Brunswick, New Jersey

During the year 1925–1926, the following papers were read:

"Continued fractions" by Lois Schenck.

"The relation of mathematics to English" by Henrian Emerson.

"Benjamin Peirce" by Dorothy Brown.

"The Pythagorean theorem" by Vera Joslin.

"The cycloid" by Elizabeth Baier.
"Pythagoras" by Ruth Palmer.
"A problem in analytic geometry" by Professor E. P. Starke.
"Bernoullian numbers" by Ruth Thompson.
"Remarkable calculators" by Emily Mott.
"Calculus of variations" by R. J. Seeger.
"The origin of mathematics" by Muriel Blackford.
"Anharmonic ratios" by Professor Richard Morris.
"Indeterminate equations" by Ruth Ruegg.
"Triangles and circles" by Professor Richard Morris.
"Work on the United States Coast and Geodetic Survey" by Professor A. A. Titsworth.
"Egyptian mathematics" by Anna Pokorny.

(Report by Professor Morris)

Pi Mu Epsilon, Hunter College, New York City

The Hunter College Chapter of Pi Mu Epsilon has had a successful year. The first semester wa. devoted to the study of relativity. Each member of the chapter was required to do certain reading on the subject. Reports and talks were made at the meeting by individuals. Meetings were held monthlys More frequent meetings were not held for the reason that we did not wish to interfere with Mathematics Club, a larger organization open to all and to which all the members of Pi Mu Epsilon belong. For this same reason, Pi Mu Epsilon at Hunter College has made of itself primarily an organization for study rather than for the delivery of papers.

At the midwinter dinner of the American Mathematical Society, certain members of the chapter presented a playlet entitled "A Drama in Relativity." For this entertainment they received the officia thanks of the Society.

The highwater mark of the year's accomplishment was reached when in February, Professor Edward Kasner of Columbia University was initiated as an active member of the Hunter College Chapter. At this time, the following undergraduates were also initiated: Misses Esther Gurwitz, Esther Lacher, Lillian Parker and Sylvia Sider.

The second semester was devoted to the study of statistics. The meetings were held as in the fall.

The chapter is again offering a ten dollar gold piece as a prize to that student in the graduating class at Hunter College High School, who presents the best record in mathematics.

The officials for the year 1925–1926 were: Director, Professor Tomlinson Fort; Vice-Director Rosalind Honig; Secretary, Agnes Corcoran; Treasurer, Mary Draper.

(Report by Professor Tomlinson Fort)

The Mathematics Club of Wellesley College, Wellesley, Massachusetts

The officers for the year 1925–1926 were: President, Elizabeth Maxon, '26; vice president, Nina Hammond, '26; faculty executive member, Miss Lennie Copeland; senior executive member, Charlotte Banta, '26; junior executive member, Dorothy Graef, '27; secretary and treasurer, Blanche Weatherhead, '27.

The following meetings were held.

October 16. Business meeting. Game-problem contest.

November 10. "Use of alignment charts" by Ruth Mason, '26 and Margaret Lane, '26. "Graphs used in statistics" by Helen Sawin, '27 and Frances Baume, '27. "Semi logarithmic and logarithmic graphs" by Miriam Dice, '27 and Grace Loveland, '27.

December 11. Display of mathematical instruments and models.

February 12. Lecture by Mr. A. Harry Wheeler of Worcester, Massachusetts, "An esthetic side of geometry."

March 19. "The mathematics of life insurance" by Margaret Fairbanks, '27. "The mathematics of navigation" by Isabel McKerracher, '27. "The mathematics of commerce" by Emily Frame, '26. "The mathematics of religion and philosophy" by Eleanor Loomis, '26.

April 16. "The mathematics of the physiologist and physician" by Ruth Foljambe, '27. "The mathematics of biology" by Dorothy Beaton, '27. "The mathematics of art and architecture" by Clara Meade, '26. "The mathematics of domestic arts" by Amy Kenny, '27.

May 14. Election of officers. Supper.

(Report by Miss Blanche Weatherhead, secretary)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3217. Proposed by R. H. Sciobereti, Berkeley, California.

Find a curve such that the radius of curvature at any point is proportional to the reciprocal of the normal.

3218. Proposed by F. M. Garnett, Savannah, Georgia.

A rectangular lighter $a=40$ ft. by $b=20$ ft. moves upstream due west, at the uniform speed $v_1=8$ miles per hour. What time will be required for a swimmer who begins at the southeast corner of the lighter to swim around it while it is in motion, if his rate upstream $v_2=10$ miles per hour, and downstream $v_3=16$ miles per hour?

3219. Proposed by Philip Fitch, Denver, Colorado.

Construct a polygon similar to a given polygon and having the reciprocal of its area equal to the sum of the reciprocals of the areas of a certain number of given polygons.

3220. Proposed by W. L. Ayres, University of Pennsylvania.

Find the value of

$$2^n-(n-1)2^{n-2}+\frac{(n-2)(n-3)}{2!}2^{n-4}-\frac{(n-3)(n-4)(n-5)}{3!}2^{n-6}+\cdots$$
$$+(-1)^{r+1}\frac{(n-r+1)(n-r)\cdots(n-2r+3)}{(r-1)!}2^{n-2r+2}+\cdots,$$

where the number of terms is the greatest integer in $(n+2)/2$ and n is a non-negative integer.

3221. Proposed by H. E. Trefethen, Colby College.

A variable rectangle has a diagonal of constant length and two sides lying upon two fixed perpendicular straight lines. Determine geometrically the locus of the foot of the perpendicular from the vertex opposite the fixed vertex upon the diagonal which does not pass through that vertex.

3222. Proposed by Norman Anning, University of Michigan.

Under what conditions is it possible to choose five points in space such that the straight line joining any two shall be perpendicular to the plane which contains the remaining three?

3223. Proposed by Paul Capron, U. S. Naval Academy.

A circle of radius b and a straight line at a distance a from the center of the circle, lie in the same plane; the circle is revolved about the line generating a torus. A plane Π is passed through the axis l, intersecting the torus in two circles S_1 and S_2; a plane Σ is passed perpendicular to Π and containing a common interior tangent of S_1 and S_2. Show that Σ intersects the torus in two circles.

3224. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Find the locus of the center of gravity of the variable triangle determined by three skew lines in a plane turning about a fixed axis.

3225. Proposed by C. N. Schmall, New York, N.Y.

If $ABCD$ is a cyclic quadrilateral, and P any point in its plane prove that

$$PA^2 \cdot \Delta BCD + PC^2 \cdot \Delta ABD = PB^2 \cdot \Delta ACD + PD^2 \cdot \Delta ABC.$$

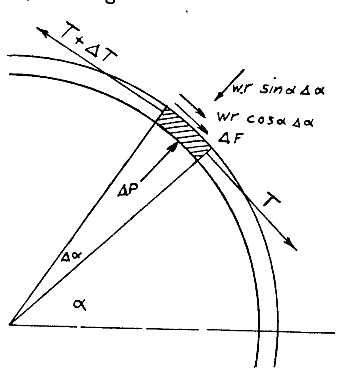
SOLUTIONS

3139 [1925, 261]. Proposed by C. N. Mills, Illinois State Normal University.

A rope weighs w pounds per foot. How many coils of the rope must be taken around a rough cylinder of radius r in order that a weight may support another n times as great, k being the coefficient of friction between the rope and the cylinder.

SOLUTION BY MICHAEL GOLDBERG, Washington, D. C.

From the figure it is seen that



$$\Delta T = \Delta F + wr \cos\alpha\, \Delta\alpha,$$
$$= k\Delta P + wr \cos\alpha\, \Delta\alpha. \qquad (1)$$

$$(T + \Delta T + T) \sin(\Delta\alpha/2) + wr \sin\alpha\, \Delta\alpha = \Delta P. \qquad (2)$$

If we eliminate ΔP from (1) and (2) and pass to the limit, there results the differential equation

$$dT/d\alpha - kT = wrk \sin\alpha + wr \cos\alpha \qquad (3)$$

whose complete integral is

$$T = \frac{wr}{(1+k^2)}\left[(1-k^2)\sin\alpha - 2k\cos\alpha\right] + Ce^{k\alpha}. \qquad (4)$$

When $\alpha = \alpha_1 = 0$, $T = T_1 = C - 2wrk/(1+k^2)$.
When $\alpha = \alpha_2$, $T = T_2 = nT_1$.
Equation (4) can now be written

$$\frac{e^{k\alpha_2}}{1+k^2} = \frac{nT_1 - wr\sin(\alpha_2 - \beta)}{T_1(1+k^2) + 2wrk}, \qquad (5)$$

where $\sin\beta = 2k/(1+k^2)$, $\cos\beta = (1-k^2)/(1+k^2)$; α_2 must then be found by trial from (5).

3145 [1925, 385]. Proposed by W. J. Sidis, New York City.

Prove that if a number of n digits, expressed in the scale of r, is divisible by any factor of $r^n - 1$, that divisibility is not altered by a cyclical permutation of the digits of the original number.

SOLUTION BY ORMOND STONE, Clifton Station, Va.

Let $\overline{\alpha_1 \cdots \alpha_n}$ be the number referred to in which $\alpha_1, \cdots, \alpha_n$ are the n digits. Then

$$\overline{\alpha_2 \cdots \alpha_n \alpha_1} = \overline{\alpha_2 \cdots \alpha_n} \cdot r + \alpha_1$$
$$= \overline{\alpha_1 \cdots \alpha_n} \cdot r - \alpha_1(r^n - 1),$$

from which the proposition follows immediately.

Also solved by HARRY LANGMAN, L. C. MATHEWSON, and A. PELLETIER.

3146 [1925, 385]. Proposed by L. H. Burns, Student, Yale University.

Show, by elementary geometrical methods (preferably suitable for use in teaching a class in elementary solid geometry), that the volume of a regular icosahedron of edge a is

$$5a^3(3+\sqrt{5})/12,$$

and obtain a similar formula for the volume of a regular dodecahedron of edge a.

SOLUTION BY N. PETROFF, Student, Cooper Union.

The following formulae for regular polygons of side a will be used:

Area of equilateral triangle, $a^2\sqrt{3}/4$ (1)

Radius of circumscribed circle, $a/\sqrt{3}$ (2)

Area of regular pentagon, $a^2\sqrt{25+10\sqrt{5}}/4$ (3)

Radius of circumscribed circle, $a\sqrt{5+\sqrt{5}}/\sqrt{10}$ (4)

Length of diagonal, $a(1+\sqrt{5})/2$ (5)

A. Regular Icosahedron. Consider five faces having V for a common vertex. The five bases of these faces form a regular pentagon of side a, and the perpendicular from V to this pentagon passes through its center C and the opposite vertex V' of the icosahedron. Let A be a vertex of the pentagon, R_i and r_i the radii of the circumscribed and inscribed spheres, then VAV' is a right angle. Hence $VA^2=VV'.VC$, or $a^2=2R_i\sqrt{a^2-\overline{CA}^2}$. Inserting the value of CA given by (4) we have after reduction

$$R_i = \frac{a}{4}\sqrt{10+2\sqrt{5}}. \tag{6}$$

Now consider one of the pyramids whose base is a face and whose lateral edges are radii R_i. The length of the altitude is r_i and its foot is the center of the equilateral triangle which forms the base. Hence r_i, the length given by (2), and R_i form a right triangle. Using (6), we obtain from this right triangle

$$r_i = \frac{a}{2}\sqrt{\frac{7+3\sqrt{5}}{6}}. \tag{7}$$

The volume of the solid is now obtained from (7) and (1)

$$V_i = \frac{5a^3}{12}(3+\sqrt{5}). \tag{8}$$

B. Regular Dodecahedron. We may use the same analysis as before. Here there are three faces having a common vertex at V which is the vertex of a pyramid whose base is an equilateral triangle of side given by (5). Here CA is given by (2) and (5). After a reduction, we find

$$R_d = a(\sqrt{3}+\sqrt{15})/4,\quad r_d = a\sqrt{\frac{25+11\sqrt{5}}{40}},\quad V_d = \frac{a^3}{4}(15+7\sqrt{5}).$$

NOTE BY OTTO DUNKEL. If the middle points of the faces of a regular polyhedron P be joined by segments of straight lines, we obtain a second regular polyhedron. Let P' be a polyhedron similar to this second one. Let R and r denote the radii of the circumscribed and inscribed spheres of P, and let a, ρ, b denote the length of a side of a face, the radius of the circumscribed circle and the apothem. Also let the same letters accented denote the corresponding parts of P'. Then from similar triangles of a figure, it will be easily seen that

$$\frac{r}{\rho} = \frac{r'}{\rho'} = \frac{1}{\sqrt{\left(\frac{aa'}{4bb'}\right)^2 - 1}}. \tag{9}$$

If P is an icosahedron, then P' is a dodecahedron and the ratio in (9) has the value $(3+\sqrt{5})/4$. This result may be used to compute the volumes of the two solids.

Also solved by HARRY LANGMAN, A. PELLETIER, S. D. TURNER, and the PROPOSER.

3149 [3145; 1925, 433; 1926, 389]. Proposed by W. D. Cairns, Oberlin, Ohio.

The center of gravity of any zone of a certain surface of revolution lies midway between the bases of the zone. What is the surface?

A solution of this problem bearing the incorrect number **3145** appeared in the August-September number. Also solved by MICHAEL GOLDBERG, A. PELLETIER, J. B. REYNOLDS, and the PROPOSER.

3150 [3146; 1925, 433]. Proposed by W. C. Eells, Whitman College.

For what integral values of s and n is it true that the nth power of any integer leaves the same remainder when divided by s as does the integer itself when divided by s? (Generalized from problem 873, *School Science and Mathematics*, March 1925).

SOLUTION BY H. L. OLSON, Michigan State College.

It is evidently assumed that n is a positive integer. First, let s be a prime number; we must have $x^n - x$ divisible by s, for all integers x. Hence either x or $x^{n-1}-1$ must be divisible by s. According to Fermat's theorem, if x is not divisible by s, then $x^{s-1}-1$ is divisible by s. Since there are numbers x (primitive roots) such that no lower power of x than x^{s-1} will leave a remainder 1 when divided by s, it follows that $n-1$ must be a multiple of $s-1$, i.e., $n=k(s-1)+1$, where k is a positive integer or 0.

Next let $s=s_1s_2 \cdots s_t$, a product of distinct prime numbers; then x^n-x must be divisible separately by $s_1, s_2 \cdots s_t$, for all integers x. Hence $n-1$ must be divisible by $s_1-1, s_2-1, \cdots, s_t-1$, and therefore by their least common multiple, m; *i.e.* $n=km+1$, where k is a positive integer or 0.

Finally, let $s=s_1^{\epsilon_1}s_2^{\epsilon_2} \cdots s_t^{\epsilon_t}$, where at least one of the exponents, say $\epsilon_r, >1$. If x is divisible by s_r, $x^{n-1}-1$ is prime to s_r, and hence, unless x is divisible also by $s_r^{\epsilon_r}$, x^n-x is not divisible by $s_r^{\epsilon_r}$. Hence no number s of this form satisfies the conditions of the problem.

Also solved by W. E. ROTH.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

DR. TOBIAS DANTZIG has been appointed assistant professor of mathematics and mechanics at the University of Maryland.

Professor J. E. DAVIS, of Pennsylvania State College, has been appointed associate professor of mathematics at Drexel Institute.

Professor W. C. EELLS, of Whitman College, has been granted leave of absence for the academic year 1926–27 to serve as research assistant in educational measurements at Stanford University.

Mr. IRA GWINN has been appointed to a professorship at Morningside College.

Dr. R. W. Hartley has been appointed professor of mathematics at Southwestern College, Memphis.

Dr. G. W. Hess has been appointed associate professor of mathematics at Howard College, Birmingham.

Dr. K. W. Lamson, of Columbia University, has been appointed assistant professor of mathematics at Lehigh University.

Dr. B. Z. Linfield has been appointed associate professor of mathematics at the University of Virginia.

Dr. W. L. Machmer, of the department of mathematics at the Massachusetts Agricultural College, has been made dean of that college.

Dr. B. C. Patterson has been appointed assistant professor of mathematics at Washington and Jefferson College.

Professor Georgia E. Robinson, of Shorter College, has been appointed professor of mathematics at Jefferson City Junior College.

Mr. H. A. Robinson has been appointed head of the department of mathematics at Agnes Scott College.

Miss Jessie M. Short has been promoted to an assistant professorship of mathematics at Reed College.

Assistant Professor J. K. Whittemore has been promoted to an associate professorship of mathematics at Yale University.

At the University of Michigan, W. M. Coates, instructor in mathematics, has returned after two years' study of aeronautics in Germany and Austria. The following new instructors have been appointed: L. W. Cohen, R. H. Marquis, M. J. Thompson, J. H. Bushey, W. R. Jones.

A limited number of fellowships for advanced study in various subjects, including mathematics, at French Universities will be awarded for the 1927–28 by the Institute of International Education. Each will carry a stipend of $1200 and will be tenable for one year, with possible renewal if circumstances are favorable. Applications must be received not later than January 1, 1927. Full information and application blanks may be obtained from the executive secretary, Stephen P. Duggan, 522 Fifth Avenue, New York City.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7–8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Annapolis. Md., December 4.

MICHIGAN, Ann Arbor. Mich.. April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

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THE AMERICAN MATHEMATICAL MONTHLY

By H. S. EVERETT, Bucknell University

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PERSONAL MENTION—NECROLOGY

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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TENTH SUMMER MEETING OF THE ASSOCIATION

The tenth summer meeting of the Mathematical Association of America was held at Ohio State University, Columbus, O., on September 7–8, 1926, preceding and in conjunction with the summer meeting of the American Mathematical Society. The total attendance was 151, including the following members of the Association.

O. P. Akers, Allegheny College.
W. E. Anderson, Miami University.
N. H. Anning, University of Michigan.
C. L. Arnold, Ohio State University.

Gladys L. Banes, Butler University.
Grace M. Bareis, Ohio State University.
I. A. Barnett, University of Cincinnati.
H. M. Beatty, Ohio State University.
Ethelwynn R. Beckwith, Milwaukee-Downer College.
E. T. Bell, California Institute of Technology.
Susan R. Benedict, Smith College.
A. A. Bennett, Lehigh University.
W. W. Bigelow, U. S. Coast and Geodetic Survey.
Henry Blumberg, Ohio State University.
R. L. Borger, Ohio University.
C. F. Bowles, South Dakota State School of Mines.
C. T. Bumer, Ohio State University.
W. H. Bussey, University of Minnesota.

W. D. Cairns, Oberlin College.
C. C. Camp, University of Illinois.
J. W. Campbell, University of Alberta.
A. L. Candy, University of Nebraska.
V. B. Caris, Ohio State University.
R. D. Carmichael, University of Illinois.
E. W. Chittenden, University of Iowa.
E. H. Clarke, Hiram College.
A. Cohen, Johns Hopkins University.
Lennie P. Copeland, Wellesley College.
C. C. Craig, University of Michigan.
W. H. Cramblet, Bethany College.
Rufus Crane, Ohio Wesleyan University.
F. Cumming, Columbia University.

J. M. Davis, University of Kentucky.
S. C. Davisson, Indiana University.
Arnold Dresden, University of Wisconsin.

Arnold Emch, University of Illinois.
G. C. Evans, Rice Institute.
H. S. Everett, Bucknell University.

Fay Farnum, Iowa State College.
Peter Field, University of Michigan.
B. F. Finkel, Drury College.
W. B. Fite, Columbia University.
F. A. Foraker, University of Pittsburgh.
W. B. Ford, University of Michigan.
T. C. Fry, Bell Tel. Laboratories, New York.

C. A. Garabedian, University of Cincinnati.
W. V. N. Garretson, Rutgers University.
D. C. Gillespie, Cornell University.
O. E. Glenn, University of Pennsylvania.
B. C. Glover, Otterbein College.
J. W. Glover, University of Michigan.
C. F. Gummer, Queen's University.

W. M. Hamilton, U. S. Naval Observatory, Washington.
C. T. Hazard, Purdue University.
E. R. Hedrick, University of California, Southern Branch.
T. R. Hollcroft, Wells College.

M. H. Ingraham, Brown University.

Dunham Jackson, University of Minnesota.

A. J. Kempner, University of Colorado.
E. C. Kiefer, James Millikin University.
H. W. Kuhn, Ohio State University.

Anna D. Lewis, Lake Erie College.
F. A. Lewis, University of Alabama.
Flora E. Le Stourgeon, University of Kentucky.
C. A. Lindemann, Bucknell University.

C. C. MacDuffee, Ohio State University.
T. E. Mason, Purdue University.
A. S. Merrill, University of Montana.
A. D. Michal, Princeton University.
J. N. Michie, Texas Technological College.
I. L. Miller, South Dakota State College of A. and M. Arts.
W. I. Miller, University of Pittsburgh.
C. N. Mills, Illinois State Normal University.

W. L. Miser, Vanderbilt University.
C. L. E. Moore, Mass. Inst. of Technology.
C. C. Morris, Ohio State University.
M. E. Mullings, University of Cincinnati.

R. L. Newlin, Ohio Wesleyan University.

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H. L. Rietz, University of Iowa.
E. D. Roe, Jr., Syracuse University.
S. A. Rowland, Ohio Wesleyan University.
E. C. Rupp, Denison University.

W. A. Shewhart, Bell Tel. Laboratories, New York.
J. A. Shohat, University of Michigan.
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C. H. Sisam, Colorado College.
H. E. Slaught, University of Chicago.
L. L. Smail, Lehigh University.
Clara E. Smith, Wellesley College.
G. W. Spenceley, Miami University.
Elizabeth T. Stafford, Texas Tech. College.
K. D. Swartzel, University of Pittsburgh.

J. C. Tinner, Wilberforce University.
M. O. Tripp, Wittenberg College.
Bird M. Turner, West Virginia University.

H. S. Vandiver, University of Texas.

T. O. Walton, Kalamazoo College.
J. H. Weaver, Ohio State University.
R. A. Wells, Park College.
W. D. A. Westfall, University of Missouri.
R. L. Wilder, University of Michigan.
F. B. Wiley, Denison University.
C. O. Williamson, College of Wooster.
H. E. Wolfe, Indiana University.
Frances M. Wright, University of Oklahoma.

B. F. Yanney, College of Wooster.
C. H. Yeaton, Oberlin College.
J. W. Young, Dartmouth College.
E. I. Yowell, University of Cincinnati.

W. A. Zehring, Purdue University.

The arrangements for this meeting were as fine as could be desired. The campus and buildings of Ohio State University lend themselves most admirably to the purposes of such a gathering. In particular, the new administration building with its auditorium, faculty club rooms, and dining hall provided every needed facility, while the dormitories placed at our disposal made a home-like gathering place for evening and morning intercourse. The joint dinner on Wednesday evening was attended by 130 members and guests. Professor Jackson, president of the Association, presided and Professors H. W. Kuhn, E. T. Bell, M. H. Ingraham, Ethelwynn R. Beckwith, W. B. Fite and H. E. Slaught were the speakers representing the various interests of the two organizations. On Tuesday evening a reception was held in the faculty club rooms where the welcome of the University was extended by Mr. G. W. Eckelberry, as Assistant to the President, and replies were made by Professors G. C. Evans, vice president of the Society, and Dunham Jackson, president of the Association. At the final sessions of the Association a resolution of appreciation was voted to the University, the members of the department of mathematics, especially Professors Rasor and Kuhn, and to the ladies of the department who assisted so greatly in the entertainment. This sentiment was also voiced by the speakers at the dinner.

At the joint sessions on Wednesday morning, Professor E. T. BELL represented the Association and contributed an interesting and instructive paper on "Successive generalizations in the theory of numbers." This paper will be published in full in the MONTHLY. The other paper was given by Professor E. W. CHITTENDEN, representing the Society, on "The metrization problem and related problems in the theory of abstract sets." Vice-president Evans of the Society presided to introduce Professor Chittenden and President Jackson of the Association to introduce Professor Bell. At this session the following joint resolution was presented and carried by unanimous rising vote:

Resolved, That the American Mathematical Society and the Mathematical Association of America, in joint session at Columbus on September 8, 1926, express their deep sense of loss in the death of Professor Bohannan, on June 20, 1926, and their profound regret that the meetings on the campus so long animated by his personality could not be graced by his presence; and direct that copies of this resolution be transmitted to Mrs. Bohannan and to the President of the Ohio State University.

At the two sessions of the Association on Tuesday the following papers were presented; abstracts of the papers are given, numbered in accordance with the numbers of the papers:

(1) "An outline of the history and applications of the calculus of residues," by Professor W. B. FORD, University of Michigan.

(2) "Waring's problem: A chapter in the additive theory of numbers," by Professor A. J. KEMPNER, University of Colorado.

(3) "Sets of linear equations in positive unknowns," by Professor C. F. GUMMER, Queen's University.

(4) "A general operational analysis," by Mr. W. O. PENNELL, Chief Engineer, Southwestern Bell Telephone Company, St. Louis, Mo.

(5) "On the value of mathematical figures and models," by Professor ARNOLD EMCH, University of Illinois.

(6) "Mathematics in modern engineering practice," by Dr. TOBIAS DANTZIG, Baltimore, Md. (by invitation).

1. The calculus of residues as developed by Cauchy and extensively applied by him to the evaluation of definite integrals and the study of infinite series was outlined first in Professor Ford's paper. Its further applications to the following fields were then briefly discussed, chiefly by means of examples: (*a*) The determination of the remainder in the Maclaurin sum formula, (*b*) the study of the convergence of Fourier series and other allied developments of mathematical physics, (*c*) the problem of the analytic continuation of a function of a complex variable, (*d*) the determination of asymptotic developments, (*e*) applications to the theory of numbers.

2. Professor Kempner's paper will appear in an early issue of the MONTHLY.

3. A system of linear homogeneous equations in which the number of unknowns exceeds the rank of the matrix has a solution involving one or more undetermined quantities. This indeterminate solution may or may not include solutions in which all the unknowns have positive values. A system may be called *simple* if the number of unknowns is one more than the rank. The solution is then determinate except for a common factor; so that the existence of positive solutions is proved or disproved by the signs of the determinants appearing in the ordinary solution formula.

If a system in general has positive solutions it is shown that a *simple* system with positive solutions may be obtained by *abbreviating* the given system, that is, by leaving out the terms in one or more unknowns. In case such a simple abbreviated system has a matrix of rank less than that of the given system, other simple systems may be obtained involving fresh unknowns so that the matrix of coefficients has a greater rank.

It is then found that *a necessary and sufficient condition for the existence of positive solutions is that there exist simple systems with positive solutions whose combined matrix has the given rank.*

The result is extended to finite systems of equations with a denumerably infinite set of unknowns, and also to the continuously infinite case where the sums of terms are replaced by definite integrals. A geometrical interpretation is possible throughout. A test similar to the foregoing applies when only some of the unknowns are required to be positive.

4. Mr. Pennell's paper formed an extension of his paper which appeared in the MONTHLY (*1926*, 293–307).

5. In his paper on the value of mathematical models and figures, which was illustrated by sixty screen projections, Professor Emch pointed out that the reason for choosing the title for his paper was not to put up a defense for the use of figures and models in mathematical instruction, but to show the possibilities of visualization in certain lines of more advanced mathematical teaching and research.

It happens not unfrequently that after the effective construction of a model or of a figure, a close examination of the finished product reveals or suggests the existence of new properties of the forms investigated, not anticipated before the construction. Another important factor of such work lies in the strengthening of the geometrical imagination and mathematical intuition in general. The use of physical images of certain mathematical forms and relations is an incomparable aid in fortifying mathematical memory. The acquisition of a concrete picture of a certain theory, or of certain forms, is a powerful means of mathematical orientation. As a matter of fact, mathematical symbols and formulas must be classed in this category of concrete pictures of mathematical

relations. Geometrical figures and models serve, in a general way, the same purpose. They are merely heuristic auxiliaries and not means in themselves.

According to Professor Emch the large number of projections which might be increased indefinitely should prove convincing enough to show that mathematical visualization even in more advanced fields is worthy of serious consideration.

6. Dr. Dantzig discussed in general the possible field of application of mathematics to engineering and the practical difficulties which are hampering a wider application of mathematical methods to problems of industry. A few incidents and observations based on the experience of the author as a research and consulting engineer were related to substantiate his conclusions and recommendations, which have for their object the furthering of mathematical methods in the field of engineering.

The fundamental problem of strength in the design of machines and structures was taken as an illustration and the parallel development of the theory of elasticity and that of strength of materials was traced since the time of Galileo. Such illustrious mathematicians as Euler, Cauchy, Poisson, and Clebsch developed powerful methods and solved completely a number of important cases, after reducing the general problem to the solution of a system of partial differential equations under given boundary conditions.

The author then discussed the reason why the engineer has failed to take full advantage of the remarkable results achieved by the theory of elasticity and came to the conclusion that some of the difficulties are insurmountable and are inherent in the nature of mathematical methods. Mathematics is probably destined to keep centuries ahead of its practical applications. On the other hand educational institutions and professional organizations can do a great deal in clearing up the minor obstacles. The author made the following recommendations based on his observations and contacts with the practical engineer.

Regarding the mathematics curriculum in colleges and technical schools the author was opposed to the presenting of the subject as a mere weapon in attacking technical problems without regard to the cultural value of mathematics. However, a special course in "Mathematical topics in engineering" would be of tremendous value to the prospective engineer.

The establishment of correspondence and extension courses in mathematics was another step recommended by the author to assist the technical man to a deeper understanding of the mathematical principles underlying his work and in reënforcing and reconstructing his mathematical education.

Finally one of the departments of such an organ as the MONTHLY could be made to serve as a clearing house for technical problems susceptible of mathematical treatment thus broadening the influence of the Association and placing the benefits of mathematical methods at the disposal of wider circles.

The American Mathematical Society held sessions on Wednesday and Thursday for the reading of papers. Seventy-one papers, besides that on the joint program, were read at these sessions, or presented by title.

On Wednesday afternoon Mr. C. H. Singer of The Banta Publishing Co. and Professor E. R. Hedrick spoke of the possibilities of the monotype machine for the setting of mathematical articles. The various items of information as to what sorts of notation are readily available and what sorts are particularly expensive merit a full presentation in our mathematical journals, so that they may be available to all who may prepare papers for printing.

The members of Pi Mu Epsilon met at a luncheon Wednesday noon in the private dining room at the Faculty Club.

Business Meeting of the Association

At the business meeting on Tuesday afternoon, called through the Monthly, the amendment proposed by the Trustees and printed in the April issue of the Monthly was carried by a vote of 40 to 18, after considerable discussion of both sides of the question. By this action Article III, Section 3, of the By-Laws now reads: "The President shall be elected by the Association's members biennially for a term of two years and shall be ineligible for re-election. The Vice-Presidents shall be elected by the Association's members annually, etc."

Business Meeting of the Trustees

Eight members of the Board were present at this session.

The following sixty-four persons, on application duly certified, were elected to individual membership:

H. C. Barber, A.M. (Amherst). Head of Dept., Charlestown High School, Boston, Mass.

Theodore Bennett, Ph.D. (Illinois). Instr., Univ. of Ill., Urbana, Ill.

W. J. Berry, M.S. (Colorado). Jr. Physicist, U. S. Bureau of Standards, Washington, D. C.

Jeanette Bickford, A.M. (Radcliffe). Head of Dept., High School, Framingham, Mass.

J. H. Binney, A.M. (Texas). Asst. Prof, A. and M. Coll. of Texas, College Station, Texas.

L. W. Blau, A.B. (W.T.S.T.C.). Tutor, Physics, Univ. of Texas, Austin, Texas.

W. M. Borgman, Jr., B.S. (Michigan). Instr., Coll. of the City of Detroit, Detroit, Mich.

M. Lucile Bristol, Geom. and Alg., Derham Hall High School, St. Paul, Minn.

J. G. Chaney, A.B. (Southwestern). Instr., High School, Lubbock, Texas.

A. L. Christian, A.B. (Baylor). Penelope, Texas.

P. T. Copp, A.M. (Ohio State). Prof., Univ. of Detroit, Detroit, Mich.

J. W. Davis, A.M. (Yale). Jr. Master, Dorchester High School for Boys, Dorchester, Mass.

V. C. d'Unger, B.S. (Little Rock). Instr., Little Rock Coll., Little Rock, Ark.

L. A. Dye, A.M. (Rochester). Instr., Univ. of Rochester, Rochester, N. Y.

D. C. Eipper, A.B. (Harvard). Head of Dept., Math. and Physics, Berkshire School, Sheffield, Mass.

C. H. Fischer, B.S. (Washington Univ.). Instr., Beloit Coll., Beloit, Wis.

George Frecheville, M.A. (Cantab.). Research, Agric. Economics Research Inst., Univ. of Oxford, Oxford, Eng.

Raymond Garver, A. M. (Montana). Instr., Univ. of Rochester, Rochester, N. Y.

L. M. Graves, Ph.D. (Chicago). Asst. Prof., Univ. of Chicago, Chicago, Ill.

L. D. Haertter, University High School, Univ. of Minn., Minneapolis, Minn.

R. G. Hein, A.B. (Wisconsin). Teacher, High School, Waupun, Wis.
A. F. Hendricks, B.S. (Valparaiso). Pres. Emeritus, Will Mayfield Coll., Marble Hill, Mo.
Etta M. Henry, B.S. (Michigan). Teacher, Ceredo-Kenova High School, Kenova, W. Va.
W. C. Holbrook, N. E. Sales Mgr., Henry Holt & Co., Boston, Mass.
C. F. Holmes, M.S. (Howard). Instr., State Normal School, Elizabeth City, N. C.
J. C. Hoshauer, B.S. (Bucknell). Head of Dept., High School, Williamsport, Pa.

E. C. Kennedy, E.M. (Texas). Austin, Texas.
E. C. Klipple, A.B. (Texas). Instr., Pure Math., Univ. of Texas, Austin, Texas.

C. C. Lao, A.B. (Yale-in-China). Instr., Yale-in-China, Changsha, China.
C. L. Leiper, Grad. (U.S.N.A.). Prof., U. S. Naval Acad., Annapolis, Md.
F. A. Lewis, A.M. (Alabama). Asso. Prof., Univ. of Ala., University, Ala.
J. V. Longenecker, M.S. (Iowa). Actuary, Farmers and Bankers Life Ins. Co., Wichita, Kans.
Henry Lucey, A.B. (Amherst). Head of Dept., Jr. and Sr. High Schools, Pittsfield, Mass.
C. T. Male, M.C.E. (Union). Instr., Union Coll., Schenectady, N. Y.
J. A. Marsh, A.M. (Amherst). Head of Dept., High School of Commerce, Boston, Mass.
W. I. Miller, A.M. (Bucknell). Fellow, Univ. of Pittsburgh, Pittsburgh, Pa.
H. F. Mitchell, B.S. (Chicago). Tulsa, Okla.
W. D. Morgan, St. Paul, Minn.
A. F. Moursund, A.B. (Texas). Instr., High School, Austin, Texas.
C. A. Murray, A.M. (Texas). Instr., Stephen F. Austin State Teachers Coll., Nacogdoches, Texas.

G. F. Nelson, A.M. (Texas). Teacher, Math. and Physics, High School, Sour Lake, Texas.
C. V. Newsom, A.B. (College of Emporia). Head of Dept., High School, Salina, Kans.
F. E. Newton, Ph.B. (Yale). Head of Dept., Phillips Acad., Andover, Mass.
Edith M. Nutt, A.B. (Wellesley). Teacher, Math. and German, High School, Natick, Mass.

L. J. Paradiso, A.M. (Ohio State). Instr., Lehigh Univ., Bethlehem, Pa.
Jonathan Pickett, Ranger, Texas.
R. B. Plymale, A.M. (Mercer). Instr., Freshman Math., Mercer Univ., Macon, Ga.
J. E. Powell, M.S. (Ohio State). Instr., Mich. Agric. Coll., East Lansing, Mich.

Walter Roberts, B.S. (Pennsylvania). Head of Dept., West Phila. High School, Philadelphia, Pa.
Inez Rundstrom, Prof., Gustavus Adolphus Coll., St. Peter, Minn.
E. C. Rupp, M.S. (Denison). Asst. Prof., Denison Univ., Granville, Ohio.

Pauline Shirley, A.B. (Baylor). Teacher, High School, Wichita Falls, Texas.
A. K. Solum, A.B. (St. Olaf). Instr., St. Olaf Coll., Northfield, Minn.
Ruth L. Spaulding, A.B. (Smith). Teacher, High School, Hartford, Conn.
J. H. Sturdivant, A.B. (Texas). Instr., Univ. of Texas, Austin, Texas.

J. D. Tamarkin, Ph.D. (Petrograd). Asst. Prof., Dartmouth Coll., Hanover, N. H.
Adela M. Thom, A.B. (Kansas). Instr., High School, Okmulgee, Okla.

W. P. Udinski, Ph.D. (Illinois). University Junior Coll., San Antonio, Texas.

A. H. Wait, A.M. (Wisconsin). Adj. Prof., Texas Tech. Coll., Lubbock, Texas.
A. D. Washburn, A.B. (Harvard). Head of Upper School and Head of Dept. of Math. and Sci., The Rivers School, Brookline, Mass.
E. D. Wells, A.M. (Minnesota). Instr., Coll. of Eng., Univ. of Minn., Minneapolis, Minn.
Anna M. Whitney, A.B. (Minnesota). Head of Dept., High School, Yakima, Wash.
Ellen E. Wiley, A.B. (St. Lawrence). Asst. Prof., Middlebury Coll., Middlebury, Vt.
J. E. Willis, Jr. Astronomer, U. S. Naval Observatory, Washington, D. C.

The resignation of Professor W. B. Ford as editor-in-chief of the Monthly, after four years' service, was accepted with regret and with an expression of deep appreciation of the high quality of scholarship and editorial ability shown in his administration. The Trustees appointed Professor W. H. Bussey of the

University of Minnesota as editor-in-chief, beginning with the volume for 1927 and feel very fortunate in securing Professor Bussey for this side of the Association's work.

The Trustees accepted invitations to hold the summer meetings of 1928 and 1929 at Amherst College and the University of Colorado, respectively. The same action was taken by the Council of the Society.

The resignation of Professor R. D. CARMICHAEL as chairman of the Committee on Standard Departments of Mathematics in Colleges was accepted and Professor TOMLINSON FORT was elected chairman, the president to fill the committee vacancy.

In recognition of the effort of the officers of the Louisiana-Mississippi Section to arouse the interest of teachers of mathematics, the president and secretary were authorized to send a letter to Professor S. T. Sanders, the chairman of the section, urging the cooperation of the high schools, normal schools and colleges of the two states, suggesting that they invite all secondary teachers of mathematics to attend the section meetings, that they devote some part of each program to questions of interest and concern to the secondary teachers, that they urge all these to subscribe to the *Mathematics Teacher*, that they urge all the collegiate teachers to join the Mathematical Association and invite any secondary teachers who wish to keep in touch with the collegiate field also to join the Association, the initiation fees of such teachers being waived provided they are already subscribers to the *Mathematics Teacher*, that they urge each school or institution to contribute toward the expenses of the teachers of mathematics in attending the meetings of the section, or at least to pay the expenses of one official delegate, the Trustees confidently believing that such expenditure of funds is sure to increase the worth and effectiveness of the teachers and thus to become a paying investment.

Later announcements will be made on business in progress concerning the Chace publication of the Ahmes papyrus, a proposed re-printing of the Report of the National Committee on Mathematical Requirements, and a proposed bureau of information as to vacancies in colleges.

W. D. CAIRNS, *Secretary-Treasurer*

THIRD ANNUAL MEETING OF NEBRASKA SECTION

The third annual meeting of the Nebraska Section of the Mathematical Association of America was held at Cotner University, Bethany, Nebraska, on April 30, 1926. in joint session with the Mathematics Section of the Nebraska Academy of Science.

The following papers were presented:

(1) "Definitions," by Professor R. M. McDILL, Hastings College.

(2) "Some summable trigonometric series," by Mr. G. D. Nichols, University of Nebraska (by invitation).

(3) "Note on fundamental units in a pure cubic field," by Professor T. A. Pierce, University of Nebraska.

(4) "Definitions of parallel curves," by Professor M. G. Gaba, University of Nebraska.

1. In his paper on definitions, Mr. McDill made the plea that in mathematics definitions be short enough and be specific enough to be held in the mind by the pupil, that they be intelligible, even if it is impossible to make them both intelligible and free from all other criticism, that they stay in their place which is to furnish words to describe realities, though of course these realities may be thought realities which are idealized from physical realities, and that they do not insinuate principles. If we must assume certain principles as so self-evident as to seem fundamental, let us be honest and bold in acknowledging them as assumptions and not try to cover them under the cloak of definitions.

2. Using the Cesàro definition for the "sum," $\bar{\sigma}$, of a non-convergent series, the following theorem was established:

$$\sum_{r=1}^{r=\infty} \frac{\phi(r)}{\psi(r)} \sin\,[c + (r-1)d]x \quad \text{and} \quad \sum_{r=1}^{r=\infty} \frac{\phi(r)}{\psi(r)} \cos\,[c + (r-1)d]x$$

are k-ply indeterminate, where c and d are real positive, rational constants, and where

$$\phi(r) = a_s r^{\alpha_s} + a_{s-1} r^{\alpha_{s-1}} + \cdots + a_0,$$
$$\psi(r) = b_t r^{\alpha_t} + b_{t-1} r^{\alpha_{t-1}} + \cdots + b_0,$$

a_i, b_i, α_i, being any real, rational constants, and the α's being such that if they are arranged in descending order of magnitude $k-1 \leqq (\alpha_s - \alpha_t) < k$; $\psi(r) = 0$ having no positive integral roots; and the angle being restricted so that $dx \neq 2n\pi$.

A special case of the cosine series was treated, namely that in which the coefficient is a polynomial in r of the second degree. Thus, for $P(r) = r^2 + 3r$,

$$2^3 \sin^3 \frac{x}{2} \cdot \sigma = \lim_{n=\infty} \Bigg\{ [\Delta^2 P(0) - 2\Delta P(0)] \sin \frac{x}{2} + \sum_{r=0}^{r=n-3} \frac{(n-r)(n-r+1)(n-r+2)}{(n+1)(n+2)(n+3)}$$
$$\Delta^3 P(r) \sin \frac{2r+3}{2} x + \frac{60[3P(n) - 3P(n-1) + P(n-2)]}{(n+1)(n+2)(n+3)} \sin \frac{2n-1}{2} x$$
$$- \frac{24[3P(n) - P(n-1)]}{(n+1)(n+2)(n+3)} \sin \frac{2n+1}{2} x + \frac{P(n)}{(n+1)(n+2)(n+3)} \sin \frac{2n+3}{2} x \Bigg\}$$
$$= [\Delta^2 P(0) - 2\Delta P(0)] \sin \frac{x}{2} - \sigma \sin \frac{x}{2} \sigma = -\frac{3}{4} \csc^2 \frac{x}{2} \cdot$$

Similar results would hold for polynomials of higher degree, but the actual process of obtaining the "sum" becomes more difficult the higher the degree of the polynomial.

3. In his note Mr. Pierce gave the results of some expansions by Jacobi's continued fraction algorithm. The numbers constituting the minimal basis of a pure cubic field were expanded, and fundamental units obtained in a large number of special fields.

4. Mr. Gaba based the definition of parallel curves on their relation to a common evolute and thus avoided a chance for ambiguity and misunderstanding.

At the business meeting the report of the nominating committee was adopted to the effect that the officers for the coming year are to be:

W. C. Brenke of the University of Nebraska, Chairman; Ellen H. Frankish of Omaha North High School, Omaha, Secretary-Treasurer; J. M. Howie of Nebraska Wesleyan University, Member of the Executive Committee.

Emma E. Hanthorn, *Secretary-Treasurer.*

ON THE MECHANICAL HANDLING OF STATISTICS[1]

By VICTOR JOHNS, 50 Broad Street, New York, N.Y.

The art of mechanical accounting—an accounting procedure that comprises the use of electric machines and perforated cards,—the application of which is now world-wide, germinated from the requirements of the United States Census Bureau. At the close of the compilation of the tenth census in 1880 the attention of Dr. Herman Hollerith, an engineer, who, as special agent of the bureau had won early recognition as an able and accomplished statistician, was drawn to the need of mechanical aid for census tabulation. For this purpose Dr. Hollerith developed a system of recording the descriptive data of each individual, or unit of inquiry, by punching holes in strips of paper, and later in cards. These perforations were adapted to control the

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[1] The present article, which was prepared at the invitation of the editor, may be regarded as a sequel to the paper by Professor L. L. Locke, entitled, "The History of Modern Calculating Machines, an American Contribution," (1924, 422–429). The machines here described are essentially concerned with the handling of mathematical statistics, and are now extensively used throughout the world. Inasmuch as American colleges and universities, generally speaking, are developing courses in the scientific handling of statistics, and in many cases are installing the mechanical equipment necessary for such work, it is hoped that the present article may prove of interest and value to readers of the Monthly. Mr. Johns is an official of the International Business Machines Corp., of New York.—Editor.

tabulating machine—an electrically actuated mechanism in the form of a counting or adding device. Single classifications of data, or classifications in desired groups, were obtainable. The next use made of this tabulator, and of the supplementary machines devised for its application, was in the tabulation of mortality statistics in the city of Baltimore; it was also employed by the Bureau of Vital Statistics of New Jersey and by the Board of Health of New York City.

At the beginning of the organization of the eleventh United States census in 1890 a commission of three experienced statisticians was appointed by the Superintendent of Census to make a practical test of all tabulating systems available for use in the United States Census Bureau. The report of this commission showed that three methods were investigated and that the punched card tabulating machine method not only was found to be far more rapid than any other, but that in accuracy it was far ahead of the others. As a result, Dr. Hollerith's system of tabulation was selected for compiling the returns of the eleventh U. S. Census. The success attained attracted widespread attention, not only in the United States but also in foreign countries. About this time the same method was used in the compilation of the Austria-Hungarian census.

The cards used at the time were perforated by a machine known as the keyboard punch, a device that operated somewhat like a pantograph. The machine which counted or tallied the items punched in the cards was known as the tabulator. It resembled a small upright piano and consisted of two parts: (1) the press, or circuit-closing device, which was located in a position corresponding to the right-hand end of the keyboard; (2) the dials, or counters, which were located in a position corresponding to the upper face where the

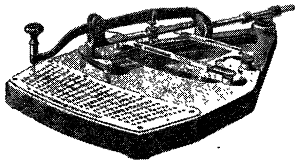

FIG. 1
FIRST HOLLERITH
KEYBOARD PUNCH

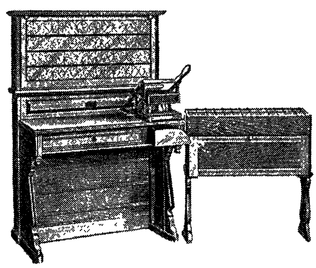

FIG. 2
ORIGINAL MODEL OF THE POPULATION
TYPE HOLLERITH ELECTRIC
TABULATOR

music rack is set. A sorting box stood as a separate unit to the right of the machine. The bed plate of the press was formed by a series of mercury-filled holes, or cups, agreeing in number and arrangement with the holes that might be made on the cards. Above this was a reciprocating box provided with a number of contact points which corresponded in position with the center of the mercury cups, so that when a card was placed in the press, and the handle brought down, these points formed circuits in positions conforming to the holes in the punched records; the results were registered on the dials.

If, while certain facts were being tabulated, it was desired to sort or arrange the cards according to any data—as, for example, nationality—a sorting box was employed. This device was suitably divided into twenty-four compartments, each of which was closed by a lid held against the tension of a spring catch which formed the armature of a suitable magnet. These magnets were connected with the binding posts of the press, according to the data by which the cards were to be sorted. When a card was put in the press the armature corresponding with the given record was attracted, thus releasing the corresponding lid which remained open until the card was deposited in that division and the lid again closed by hand.

The use of electric tabulating machines for compiling population statistics presented the need of other mechanical devices for compiling agricultural, manufacturing and similar classes of data that dealt with quantities and amounts. Accordingly, a so-called integrating machine was developed. This device added digits from one to nine in each column, the accumulations being determined by the location of the holes in the card. This device was the predecessor of the modern electric tabulating machine now used extensively for railroad accounting, sales analysis, shop cost accounting, and similar purposes. Since tabulating machines can be used wherever figure-facts are called upon to assist in directing the operations of a business, the principle of punched hole accounting is represented in practically every field of human activity.

During the process of tabulating the eleventh census (1890) the attention of Dr. Hollerith was called to the need of some machine or device to aid in auditing railroad freight accounts and in calculating commodities statistics. Experiments were made with Hollerith tabulators at the office of the auditor of freight accounts of the New York Central Railroad. The test proved that the work could be done to advantage by this method. In the application of this system to railroad and commercial accounting, it was found desirable also to make radical changes in the apparatus for punching the cards, as well as in the apparatus by means of which the amounts were recorded.

One of the first units of the Hollerith system to be changed was the punching machine. In the pantograph type of punch, the fields, or groups of characters, on the cards were arranged in a sequence proceeding from the upper left-hand corner of the cards across the top and then from right to left across the bottom of the cards. In the work of punching cards for agricultural and railroad statistics, etc., it was found to be more convenient to have the card arranged so that the fields or classes of information would be in vertical columns—to facilitate the punching and reading of the cards as well as the tabulating. For this purpose the key punch was developed. While various improvements have been made from time to time on this device (illustrated by Figure 3) it is substantially the same in principle today as when devised. In 1923 an electrically actuated key punch was placed on the market. This machine requires a very light touch on the part of the operator and in many instances it has increased the production of card punching from ten to twenty percent.

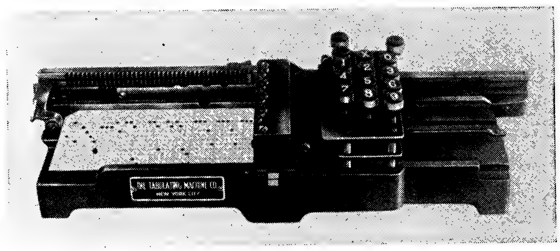

Fig. 3
Key Punch

Another recent development is the electric duplicating key punch, which can be used in the same manner and for the same purposes as those mentioned above, or it can be used to duplicate a portion, or all, of the perforations in a master card. When any information common to a number of cards is to be duplicated, a master card is first punched and inserted in the rear carrier of the machine. The cards on which the data are to be duplicated are

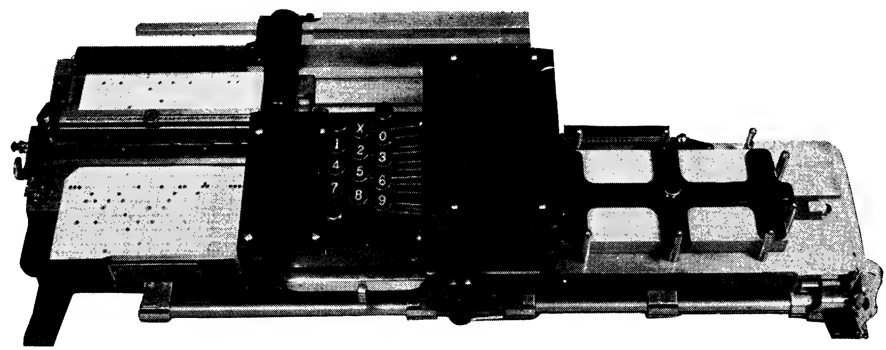

Fig. 4
Electric Key Punch with Duplicating Attachment

then automatically fed from the card magazine to the punching device; the information on the master card is automatically reproduced on each of the desired number of cards. When the carriage reaches a point at which the individual data are to be manually punched, the machine stops and these data are recorded in the regular manner by the operator.

Another perforating device is the gang punch, which is used for recording those facts that are common to a number of cards, such as the month, day, year, etc. It is equipped with a number of movable punches which can be changed easily and set for any desired combination. The object of this machine is to provide a short cut in transcribing data. A number of cards—from ten to twelve—can be punched at one operation when the recorded data are similar for all the cards.

In order to check the accuracy of the punching, a so-called verification key punch was devised. As soon as the cards have been perforated the work is verified by merely inserting the punched cards, one at a time, in this device in the same manner as placing a fresh card in the key punch. The comparison is made by the operator pressing the keys in the same sequence as that followed when the data were transcribed from the original records. If the punching is correct the carriage and card will move forward to the next position. If the punching is incorrect, or if the wrong key of the verification key punch is depressed, the carriage will remain stationary.

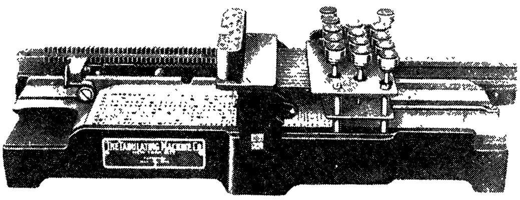

FIG. 5
VERIFICATION KEY PUNCH

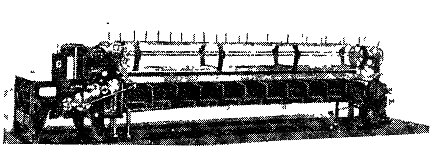

FIG. 6
ORIGINAL MODEL OF THE HOLLERITH ELECTRIC SORTING MACHINE

The unit which has made the operation of electric tabulating machines both practical and successful is the commercial tabulating card. While the number of columns on the cards has been changed to increase the recording capacity, the size of the cards, however, remains the same; which indicates on what a solid foundation of reason Dr. Hollerith's early work was based.

The development of the art of punched hole accounting continued; in 1902 the population and vital statistics of the United States were sorted and tabulated on machines having an automatic feed.

The sorting machine was redesigned, as shown by Figure 6, becoming fully automatic. Its next development was to place it in a vertical instead of horizontal position (Figure 7) as a means of conserving floor space. A stack of 400 or 500 cards is placed in the hopper at the top and automatically sent one by one into the twelve pockets below; the pocket into which a certain card falls depends upon the position of the hole in the column being sorted. Recently there was put on the market a new model (Figure 8), which operates almost silently and sorts cards at the rate of 400 per minute. Where conservation of floor space is not a matter of importance this machine is preferable from the operator's standpoint inasmuch as the pockets are all equidistant from the floor, which position brings all the sorted cards within easy reach.

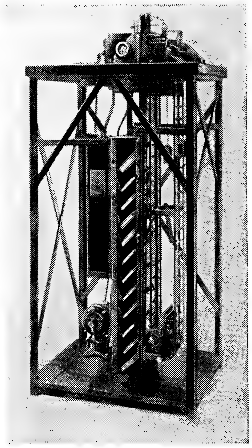

Fig. 7
Electrical Vertical Sorting Machine

At the time the sorter was redesigned a change was also made in the electric tabulating machine. The new principle involved—and still embodied in these machines—was this: As the cards drop from the hopper down through the

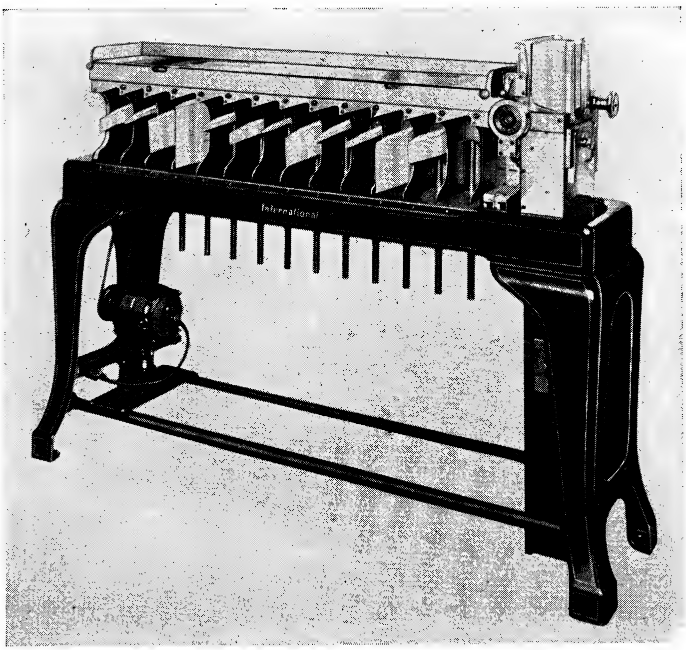

Fig. 8
Recent Form of Sorting Machine

feed mechanism they pass between a row of small wire brushes and a row of fixed metal contacts corresponding in position to the columns of the cards, so that the punched holes, as the cards pass between the brushes and the metal contacts, cause electrical connections to be made, thus energizing magnets which in turn move the adding wheels. In this manner the amounts and quantities punched in tabulating cards are automatically totaled. Since as many as five different classifications of data can be added simultaneously on an electric tabulating machine at the rate of 150 amounts per minute for each classification—in other words, since an electric tabulating machine is capable of yielding *750 additions per minute*,—it is obvious that the power of this machine to handle great masses of numerical data is phenomenal.

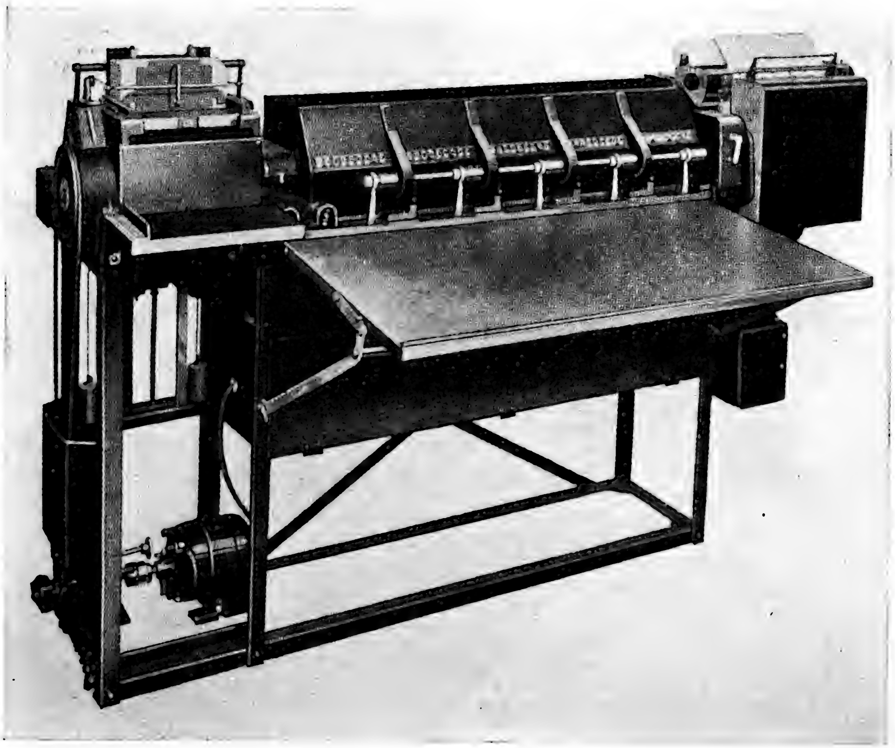

FIG. 9
ELECTRIC TOTAL PRINTING AND LISTING TABULATING MACHINE

There has been a constant improvement in this unit, the automatic control and the total-printing and listing features referred to below probably being the most important.

The automatically controlled tabulator was devised to eliminate the use of stop cards. Prior to 1922 it was necessary, when feeding various groups of

cards into the tabulating machine, to insert at the end of each group a stop card which, after the last card of a group had been registered, would stop the machine and the total then be recorded. The automatic control now in use does away with this extra card by automatically stopping the tabulating machine at the end of each group.

The introduction of the listing and total-printing feature was of major importance. With this device it is unnecessary for the operator to copy the figures shown on the counters, as the machine automatically prints on a sheet of paper the figures thus registered. It can be made to list simultaneously, in itemized detail and at the rate of 75 cards per minute, as many as seven different groups of data while accumulating the totals of five; when an itemized list is not desired, it accumulates and prints the totals of from one to five different groups at the rate of 150 cards per minute.

There is also the digit or card-counting tabulating machine which performs the functions of both the sorter and tabulator. This device is principally used in the compilation of employees and vital statistics. It is equipped with fourteen counters, twelve supplying individual totals, and two giving sub- or accumulated totals. It handles cards at the rate of 250 per minute.

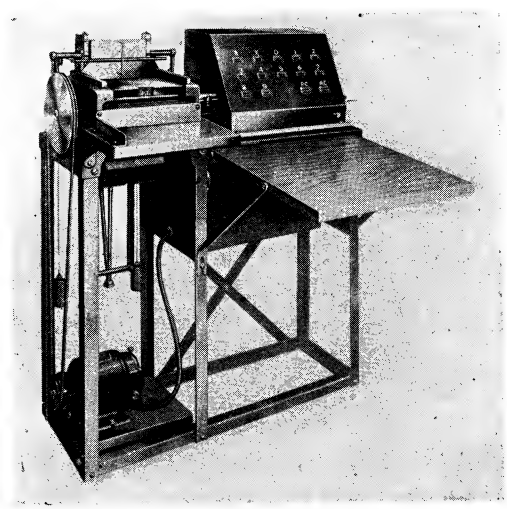

Fig. 10
Electric Card Counting Tabulating Machine

The uses to which these mechanical accounting machines can be applied are unlimited: industry employs them for the maintenance of its records, for making production, cost, sales and many other analyses; insurance companies use them in the various phases of their statistical work; the United States and foreign governments use them for compiling agriculture and population censuses and for preparing multitudinous reports.

This equipment is also to be found in many of the great American colleges where it serves both practical and educational purposes: at the executive end it is used to prepare payrolls, to analyze direct labor charges and accounts payable, and to compile school and special study statements; at the educational end it is used for demonstration purposes in accounting courses. The University of Michigan, the LaSalle Extension, Johns Hopkins, Cornell and Columbia Universities are numbered among the users.

Electric tabulating machines, in fact, are used to record, aggregate, tabulate and print in report form all data capable of denotation by figures. They have brought to every angle of accounting work not only the facilities for saving time, labor, and money, but an absolutely new range of possibilities as well.

THE GREAT TREASURE HOUSE OF CHINESE AND EUROPEAN MATHEMATICS

By Père LOUIS VANHÉE, S. J., Brussels

The recent appearance in this MONTHLY of several interesting and valuable articles upon early Chinese mathematics suggests the desirability of calling attention to a notable source book that appeared in China in 1889 but which seems to be little recognized in the west. It is known as *The Great Treasure House of Chinese and European Mathematics* and was edited by Ch'en Wei-ki. It consists of a selection from some of the best mathematical treatises published in China, both native and European, and is made up of a hundred parts, some of these being classical treatises and others being mere compilations. The following list with comments, affords an idea of the scope of this monumental compendium:

1. (Part 1). The *lo-shu* and *ho-t'u*; the *Chou-peï Suan-King*;[1] Measures. It is a general custom in China to have a mathematical treatise begin with the *lo-shu* and *ho-t'u*, figures of fabled origin,[2] the former being the world's first known magic square and the latter being used for divination. The *Chou-peï Suan-king* is well known to the western world through Biot's translation[3] and has only a sentimental value. It pretends to be a dialogue between Chóu-Kung and Shang Kao, shortly before 1100 B. C., but this is probably a literary fiction devised for the purpose of giving the impression of an advanced state of mathematical knowledge in that period. The style is obscure and the work is merely a set of vague statements about numbers derived from the circle and the square.[4] The Pythagorean theorem is, with their usual obstinacy in such matters, claimed by the Chinese, but the impartial historian must exert a great deal of charity to sanction this appeal to national pride.

2. (Parts 2–17). Euclid's *Elements*, the first six books being the translation made by the celebrated mandarins Sü Kuang-ch'i[5] and Li Chi-ts'ao (died in 1631), under the direction of the far more celebrated Matteo Ricci

[1] Smith, *History*, I, 28, 29; This MONTHLY (1925, 499–504).

[2] Smith, I, 29.

[3] *Journal Asiatique* (1841), p. 595.

[4] See the extracts in Smith, I, 31.

[5] Known after his adoption of Christianity as Paul Sü (Hsü, Siu) (1562–1634). Smith, I, 304.

(1552–1610). This remarkable Jesuit scholar, intimately associated with the savants of the Empire, sought to give to the Chinese a sound mathematical basis upon which to build their own science and philosophy. Although he died before the influence of his work was fully appreciated the world has not failed to pay to his memory the tribute that is so justly due. Himself a pupil of the great Clavius (1537–1612), whose edition of Euclid appeared (1574) three years before Ricci left for China, it was this text that he followed in making the translation. He first explained to Sü Kuang-ch'i the meaning of each proposition; he then dictated the translation; the minister then took this to his home and rewrote it in the polished style of the Chinese scholar; and finally Ricci revised the text for publication. This intimate association between two great scholars, one from the east and the other from the west, gave to the Chinese civilization this great classic, known in their tongue as the *Ki-ho-yüan-pen*.[1] The style is remarkable, reading as pure Chinese and not as a translation from the Greek or Latin, and thus the work is in marked contrast to most oriental translations from the European languages, hastily made as they are by immature writers and immediately thrown on the market merely with the idea of ready sale. The remaining books of the *Elements* were translated from the English by Wylie (1815–1881) with the assistance of Li Shan-lan (1814–1884).

3. (Parts 18–22). "Calculation with the pen." This relates to ordinary written computation as distinct from that on the abacus. It contains but little theory, being a collection of all sorts of possible as well as imaginary applications.

4. (Parts 23–25). Proportion. A collection of problems from various sources.

5. (Part 26). The *Ts'iê-shuai* method, an extension of the method of proportion. The expression means literally "borrowing of terms" and is used to indicate problems in which the data are not sufficient to enable the computer to write the proportions at once, quite as in the case of an indeterminate equation.

6. (Part 27). Excess and deficiency. This term, familiar in medieval Europe as a synonym of *Regula falsi positionis* or "Rule of False Position," is an old one in China. It is usually employed there with respect to problems like the following:

> "A certain sum of money is divided among a certain number of men. If each takes 7 taels, 4 taels remain; but if each should demand 9 taels, there would be 12 taels too few. Required the number of men and the amount of money."

[1] The *Ki-ho* means "how much," "quantity," "magnitude," and is the most elegant way of translating our word "geometry." In spite of the influx of new terms, the native form is still used, the syllable *hsiao* (science) being added, giving the familiar form *Ki-ho-hsiao*. The syllables *yüan-pen* signify "original volume," or "first text," *yüan* meaning "origin" and "*pen*" meaning "volume."

"A crowd of people wish to be taken across a stream. If each boat carries 13, there will be 12 people left; but if each boat carries 18, there will be one boat too many. Required the number of boats and the number of people."

7. (Part 28). Equations, *Fang-ch'eng*. The term *Fang* means "to compare," and *ch'eng* means "system" or "form." The title refers to the solution of numerical linear equations with two or more unknowns. No theory is given, showing that the Chinese had merely borrowed a mechanical method from foreign sources.[1]

8. (Parts 28–30). *Kou-ku* (Right triangle). This is the title of the last section of the celebrated *K'iu-ch'ang Suan-shu* (Arithmetic in Nine Sections), one of the oldest of the Chinese classics.[2] It is a subject frequently included in the mathematical works of China, where problems involving the Pythagorean theorem are a favorite pastime. It involves nothing of any difficulty or novelty.

9. (Part 31). Plane trigonometry, evidently merely a translation from European sources.

10. (Part 32). Measure of heights and distances. A very simple application of proportion, with the first principles of trigonometry.

11. (Part 33, 34). The circle. The quadrature problem seems to have interested Chinese scholars from very early times. Many names of such writers are found in the Chinese encyclopedias. It is the writer's belief that all the circle theory contained in these works was derived from western sources. Indeed, Greek and other foreign writers are frequently mentioned. It is unfortunate that the Chinese reader is misled by such titles as "Method of Jartoux," "Method of Yu," and "Method of Li." In the case of Jartoux, (1668–1720), the well-known French Jesuit missionary, the error is made the worse by the use of his Chinese name, Tu, and by the impression which is given that Li (who was merely an assistant of Wylie's) and Yu are to be ranked in the same class with a man whose nine series for quadratures are so highly appreciated in the east.

12. (Parts 35, 36). Surfaces. A series of applied problems requiring only a small amount of elementary mathematics.

13. (Parts 37, 38). Solids. Similar in treatment to **12.**

14. (Part 39). Spherical trigonometry. China received her knowledge of this subject from two sources; (1) the Arabs in the 13th century, and (2) the Jesuits in the 17th century. The first of these sources did not appreciably contribute to her mathematics; the second gave her a set of practical formulas

[1] This, however, may not be a correct inference, since it was not an uncommon thing even for European scholars in the early Renaissance period to conceal the method of solution of certain types of problems.

[2] Smith, I, 31.

the basis of which she only vaguely comprehended and the potency of which she failed to recognize by making any advance in the theory.

15. (Parts 40–50). "The algebra of the celestial element," a name derived from the fact that in China the unknown was represented by the character *t'ien* (heaven). This algebra apparently of Hindu or Arabic origin, was adapted (with numerous linguistic curiosities) to the Chinese language and appears in the works of such prominent writers as Li Yeh (1178–1265) and Yang Hui (*c.* 1260). The compiler embodied in his work everything that the Chinese had attempted in this line, including 70 of the 170 problems in Yeh's *Ts'ê yuen hai king* (*Sea Mirror of the Circle Measurement*) (1249);[1] the treatment of progressions by Chu Shih-Kié (Chóu Che-Kié, *c.* 1300) and Li Shan-lan (1815–1884); square root according to Li Juan and Chiu Pai-Ki (1819–1869); and finally a conjectured explanation by Hao of the ancient method of finding roots.

16. (Parts 51, 52). Quadrilateral algebra of Chu Shih-Kié (Chóu Che-Kié) (*c.* 1300) so called because it employs for the unknowns four Chinese characters, *t'ien*, *ti*, *jen*, and *wu* (*heaven*, *earth*, *man*, and *thing*). By a special disposition of these characters it is able to indicate powers without resorting to exponents. Such a use of four letters may possibly be due to the Hindu or Arab influence.[2]

17. (Part 53). The *Chui* method. The term *Chui* is obscure and its meaning has been the subject of much dispute. It seems to have been first used in a work by Tsu Ch'ung-chih (430–501)[3] the *Chui-Shu* (*Book of Chui*) and probably referred to the treatment of the calendar. In later times, however, it has been employed by both Chinese and Japanese writers to refer to the analytic treatment of the quadrature of the circle, and is so used in this collection. It is owing to this change in meaning that certain writers have attributed to Tsu-Ch'ung-chih the analytic method of finding the value of π.

18. (Parts 54, 55). Elementary algebra. An adaptation of the *Ts'ie-ken-fang* of the Jesuit scholars of the 17th century.

19. (Parts 56, 57). Logarithms. This feature was introduced into China by the Polish Jesuit Smogolenski[4] (1611–1656) who, knowing that the Chinese of that period were not prepared for the theory, gave only the tables[5] and the mechanical rules for using them. In the 19th century the influence of the west led to a demand for the underlying principles, and these were adapted from English works, but with an absurd claim for originality for three Chinese writers, Ku, Li, and Tsou.

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[1] Smith, I, 270.

[2] Compare the Hindu use of various colors for the unknown.

[3] Smith, I, 143.

[4] In Chinese, Mu Ni-ko. This is derived from the original Polish form—Nicolas Smogulecki, *Ni-ko* being from *Nico*las, and *Mu* being from the syllable *Smo*.

[5] Described in his *T'ien-pu Chen-yüan*.

20. (Parts 58–79). *T'ai-Shu*, a name given to advanced algebra. This includes the more elementary parts of modern algebra and contains numerous applications to geometry, conics, and other branches.

21-23. (Parts 80–96). Differentiation, integration, and mechanics. Translations from English texts, including that of the American scholar, Elias Loomis on conics, and the calculus.

24. (Part 97). The slide rule.

25. (Part 98). Logarithmic tables. The Jesuits had, as stated under **19**, introduced logarithms in the 17th century. So remarkable did this invention seem that the emperor, K'ang-Hsi, had a small table prepared for his own use. The compilation known as the *Shu-li Tsing-yün*, a part of the *Lü-li Yüan-yüan* printed at Peking in 1713, reproduced the Vlacq tables of 1628.[1]

26, 27. (Parts 99, 100). Trigonometric functions, natural and logarithmic. Reproductions from European sources.

The compilation, prepared for the purpose of stimulating mathematical work at a time when European influence was at its maximum and indicative of the radical changes that have since taken place, has great historical interest. In its treatment of the subject there is always apparent a lack of the judicial mind and there are constant indications of a characteristic spirit. Yé, for example, recognizes the value of the work of Diophantus, but he asserts that the ancient Chinese algebra was the source rather than the outcome of the works of the Hindus, Greeks, and Arabs, and that foreign writers improved upon the earlier works of the far east. As a kind of source book for the history of Chinese mathematics, however, the work has a value which justifies making it known to western scholars.[2]

THE KINETICS OF LEARNING

By H. J. ETTLINGER, University of Texas

The laws of certain specified types of the learning process, which may be roughly represented by learning to operate a typewriter, may be embodied in a first order linear differential equation (or difference equation) with constant coefficients. From this relation follows a simple set of equations which express chiefly by means of exponential functions (or binomial expressions) the quanti-

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[1] Smith, I, 436; II, 524.

[2] Professor D. E. Smith, in translating this article for the MONTHLY from Père Vanhée's manuscript, materially revised and shortened it. The original article has since appeared in *Archivio di Storia della Scienza* for May, 1926.

tative aspects of the learning process and exhibit it as a form of "growth." It is not unlikely that equations of this kind may prove useful in analyzing personnel in factory and offices, as well as in predicting future performances involving the above type of learning process from a limited amount of data obtained by observation.

We shall make the following assumptions:

1. The *attainment*, Y, may be measured in terms of the number of successful acts per unit of time (practice period).

2. There exists a *limit of attainment*, L, measured in terms of the number of successful acts per unit period of time.

3. The difference $y = L - Y$, limit of attainment minus attainment is the *margin of attainment.*

4. The amount of practice acts, X is a continuous variable.

5. When $X=0$, the attainment, Y, is zero.

The *Law of Learning* may be stated as follows:

The relative rate of decrease of the margin of attainment with respect to the amount of practice is constant.

If y' is the derivative of y with respect to X, we may write the Law of Learning as a linear differential equation of the first order,

$$(y'/y) = -k, \tag{1}$$

where k is a constant. Integrating (1) we obtain

$$\log y = -kX + C.$$

When $X=0$, $y=L$, hence

$$y = Le^{-kX}.$$

Since $y = L - Y$, we have for the equation connecting Y and X,

$$L - Y = Le^{-kX} \tag{2}$$

or

$$Y = L(1 - e^{-kX}) \; ; \; X \geqq 0.$$

The graph of the equation (2) is shown in figure 1. It is the typical *growth* curve or compound interest law. We note that when $X=0$, $Y=0$, and as X increases without limit, Y approaches L as a limit.

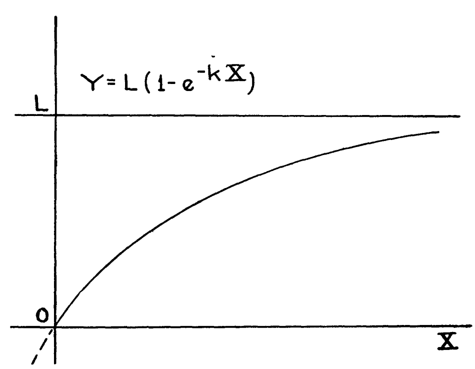

FIG. 1

Now from (2) we obtain

$$\frac{L}{Y} = \frac{1}{1 - e^{-kX}} = 1 + \frac{e^{-kX}}{1 - e^{-kX}} = 1 + \frac{1}{kX} - \frac{e^{kX_1}}{2 + kXe^{kX_1}}$$

by applying the Law of the Mean[1] to the term $e^{-kX}/(1-e^{-kX})$, where X_1 is an intermediate value of X, $0<X_1<X$. The equation

$$\frac{L}{Y} = 1 + \frac{1}{kX} \text{ or } Y + kXY = kLX,\ X > 0, \tag{3}$$

may be regarded as a first approximation to (2). The graph of (3) shown in figure 2 is a *hyperbola*, with a vertical asymptote at $X=-1/k$ and a horizontal asymptote at $Y=L$. Furthermore it passes through the origin since $Y=0$ when $X=0$.

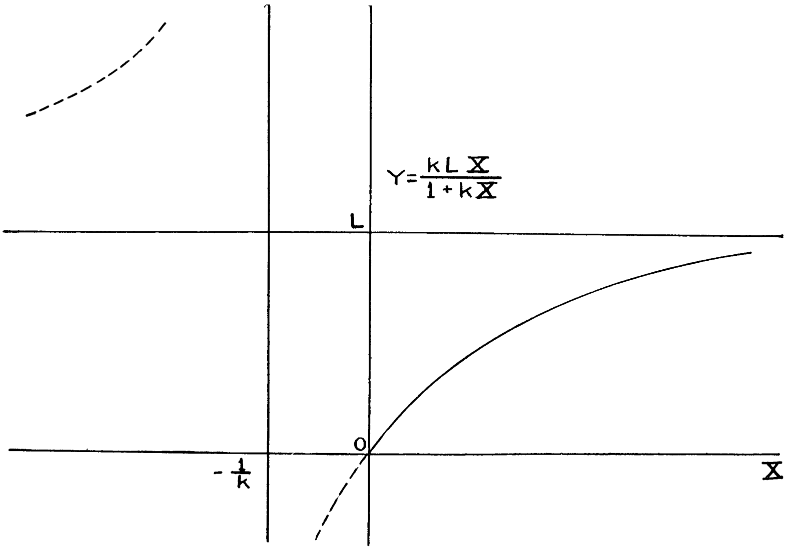

FIG. 2

This hyperbolic curve was obtained empirically by the psychologist Thurstone[2] for a learning curve from an extended series of typewriter learning data. As a practical method for determining (2) empirically, one may find the value of k and L in equation (3) by the use of the equation preceding (3). Plotting $1/Y$ against $1/X$, we obtain the vertical intercept as $1/L$ and the slope of the straight line as $1/kL$. From these, k and L may be computed and substituted into (2).

If we set $t=1/Y$, $t_L=1/L$, where t is the attainment measured in number of units of time required to perform unit successful act, we have

$$t = t_L(1 - e^{-kX})^{-1}. \tag{4}$$

[1] This result may also be obtained by the well known method of integration by parts as applied to approximations. See, for example, Porter and Ettlinger, *Elementary Calculus*, The Century Company, 1925, pp. 209–210.

[2] Thurstone, The Learning Curve Equation, *Psychological Monographs*, Studies from the Psychological Laboratories of the University of Chicago, vol. xxvi, no. 3, whole no. 114 (1919) pp. 1–51.

If we interpret t as the time rate of change of time of practice T, with respect to the amount of practice, X, we have a second differential equation

$$\frac{dT}{dX} = \frac{t_L}{1 - e^{-kX}}$$

which may be integrated to yield

$$T = \frac{1}{kL} \log \frac{e^{kX} - 1}{e^{kP} - 1}, \tag{5}$$

where P is the value of X corresponding to $T = 0$.

Equation (5) may be called with Thurstone[1] the *time-amount* curve. We may solve this equation for e^{kX} and substitute in (4) and obtain

$$t = t_L + (t_P - t_L)e^{-kLT} \tag{6}$$

where t_P is the time corresponding to $X = P$. Equation (6) is Thurstone's *time-time curve*,[2] that is, the attainment measured in time units, in terms of time of practice.

The reciprocal of (6) gives the *speed-time* curve,

$$Y = \frac{LY_0}{Y_0 + (L - Y_0)e^{-kLT}} \tag{7}$$

represented in figure 3.

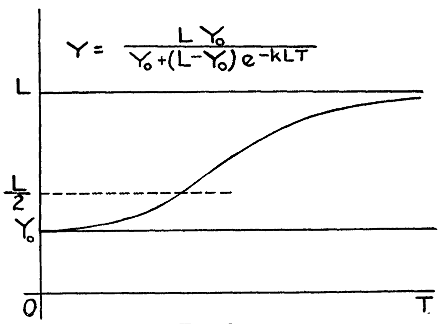

FIG. 3

By the methods of elementary calculus it may be shown that curve (7) has a point of inflection at $Y = L/2$. This result is more easily obtained from the inverse equation of (7),

$$T = \frac{1}{kL} \log \frac{Y(L - Y_0)}{Y_0(L - Y)} \cdot$$

The slope of curve (7) is the rapidity with which the speed of attainment increases with time of practice. The point of inflection yields the value for which this rate is a maximum.

As pointed out above, a limited number of observations will permit the determination of L, the limit of attainment. The determination of this quan-

[1] *Loc cit.*, p. 21.

[2] *Loc cit.*, p. 21. Thurstone is unable to obtain an equation of this type in explicit form from his hyperbolic approximation. For a complete comparison between the present equations and those obtained by Thurstone, see my paper entitled "Representation of the learning process by means of a growth curve," read before the Psychology Seminar of the University of Texas, cf. *Journal of Experimental Psychology*, October, 1926.

tity in advance would give important information as to what may be expected of the future performance of the individual. This may prove useful to efficiency experts. It may also be of interest in another field, *viz.*, athletic coaching. The instructor is not interested in the present attainment of an individual, but in what may be expected at the height of the season after a large amount of practice.

The assumption has been made above that when $X=0$, $Y=0$. In many learning processes this condition is not fulfilled. We may have a certain initial attainment without any practice. In this case, when $X=0$, $Y=Y_1$. The latter case, however, reduces to the former by setting $\overline{Y}=Y-Y_1$.

Equation (1) may be turned into a Law of Forgetting by changing the sign of k and properly interpreting the quantities involved.

It may be pointed out that equation (1) should in reality be a difference equation with intervals of unity for the independent variable. Such an equation may be written $\Delta y_i = -ky_i$, or $y_{i+1} = y_i(1-k)$, $y_n = y_0(1-k)^n$. (8)
But $y_0=L$, and $y_n=L-Y_n$, hence

$$Y_n = L[1-(1-k)^n], \tag{9}$$

where n is the number of practice periods.

Equation (9) is defined only for integral values of n, but in the main the properties of the curve, if we join these points by straight lines, will be very much like those of (2). For $n=0$, $Y_0=0$. If k is positive and less than unity, as n increases without limit, Y_n will approach L as its limit. For large values of n we have from the equation $(L/Y)=1/[1-(1-k)^n]$, an approximation similar to (3) of the type,

$$Y = kLn/(1+kn).$$

Similarly we may obtain other relations analogous to those obtained for the differential equation.

QUESTIONS AND DISCUSSIONS

EDITED BY TOMLINSON FORT, Hunter College, Park Ave. and 68th St., New York, N. Y. AND BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. A VECTOR PROOF OF THE THEOREM OF CORIOLIS

By E. L. REES, University of Kentucky

The following is a very neat vector proof of the theorem of Coriolis. We are to show that the absolute acceleration of a point in motion relative to a

moving body is made up of the following components: its relative acceleration; the acceleration of the point of the body with which it instantaneously coincides; and a component depending on the relative velocity of the point and the angular velocity of the body.

Let $\boldsymbol{p}$ be the position vector of any point P fixed in the body and $\boldsymbol{q}$ the vector from P to the moving point Q. The position vector of Q is then given by $\boldsymbol{r}=\boldsymbol{p}+\boldsymbol{q}$.

$\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ are of course functions of the time. Differentiating twice and indicating time derivatives by dots we have $\ddot{\boldsymbol{r}}=\ddot{\boldsymbol{p}}+\ddot{\boldsymbol{q}}$.

To introduce the relative motion of Q we refer $\boldsymbol{q}$ to the mutually perpendicular unit vectors $\boldsymbol{i}'$, $\boldsymbol{j}'$, $\boldsymbol{k}'$ fixed in the body. A vector expressed in terms of $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$ we denote by a primed letter. Thus we may write

$$= \Sigma \boldsymbol{q} \cdot \boldsymbol{i}'\boldsymbol{i}',$$

in which $\boldsymbol{i}'$, $\boldsymbol{j}'$, $\boldsymbol{k}'$ of the coefficients are to be regarded as functions of the time. Differentiating this equation, we have

$$\dot{\boldsymbol{q}}' = \Sigma \dot{\boldsymbol{q}} \cdot \boldsymbol{i}'\boldsymbol{i}' + \Sigma \boldsymbol{q} \cdot \dot{\boldsymbol{i}}'\boldsymbol{i}' = \dot{\boldsymbol{q}} + \Sigma \boldsymbol{q} \cdot \boldsymbol{w} \times \boldsymbol{i}'\boldsymbol{i}'$$

$$= \dot{\boldsymbol{q}} + \Sigma \boldsymbol{q} \times \boldsymbol{w} \cdot \boldsymbol{i}'\boldsymbol{i}' = \dot{\boldsymbol{q}} + \boldsymbol{q} \times \boldsymbol{w}.$$

By a second application of this formula, we have for the second derivative

$$\ddot{\boldsymbol{q}}' = \ddot{\boldsymbol{q}} + 2\dot{\boldsymbol{q}} \times \boldsymbol{w} + \boldsymbol{q} \times \dot{\boldsymbol{w}} + (\boldsymbol{q} \times \boldsymbol{w}) \times \boldsymbol{w}.$$

Now let Q coincide with P so that $\boldsymbol{q}=0$ and $\dot{\boldsymbol{p}}'=\dot{\boldsymbol{q}}$. The last equation then becomes

$$\ddot{\boldsymbol{q}}' = \ddot{\boldsymbol{q}} + 2\dot{\boldsymbol{q}}' \times \boldsymbol{w}.$$

Solving for $\ddot{\boldsymbol{q}}$ and substituting in the equation above, we get

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{p}} + \ddot{\boldsymbol{q}}' + 2\boldsymbol{w} \times \dot{\boldsymbol{q}}',$$

which proves the theorem. The component represented by the last term is known as the acceleration of Coriolis.

II. Notes on Some Uses of the Line at Infinity and of Imaginaries

By A. D. Campbell. University of Arkansas

In these notes we shall show some interesting uses of the line at infinity (l_∞) and of imaginaries, first in the general discussion of simultaneous real quadratic equations in two unknowns and secondly in the study of real plane

cubic curves in Euclidean geometry. We worked out complete results in these two problems with a graduate student, Mr. Sam Byrd. We shall give two illustrations of our method.

We note that the discussion in algebra of two simultaneous real quadratic equations in two unknowns and of their common solutions corresponds to the discussion in geometry of two conics in the real affine plane projective geometry and of the points of intersection of these two conics. Moreover the two conics may be considered as fundamental conics of a pencil of conics. Therefore we may replace the algebraic problem of finding typical forms to which all such simultaneous quadratic equations can be reduced by the geometric problem of finding typical forms to which all pencils of conics in the real affine geometry can be reduced. We define a class of pencils of conics as the set of all pencils projectively equivalent to a given typical pencil. We note that the classes of pencils of conics in the real affine geometry can be derived as subclasses of the classes in the real general projective geometry.[1] We shall now use this last mentioned fact. The first illustration of our method will be the finding of all the classes of pencils of conics in the real affine geometry (together with their typical pencils) that are subclasses of the class in the real general projective geometry that has for a typical pencil:[2]

$$\lambda(x^2 + y^2) + \mu(x^2 + z^2) = 0, \tag{1}$$

x, y, z being homogeneous coordinates referred to a triangle of reference. (To show more clearly how our geometrical discussion solves also the algebraic problem, we shall put our results in the form of simultaneous quadratic equations rather than of pencils of conics.) Each pencil of this class consists of all the conics that intersect in two pairs of conjugate imaginary points, P_1, P_2 and P_3, P_4 that form the vertices of a complete quadrangle. The degenerate conics of the pencil are two conjugate imaginary line pairs P_1P_3, P_2P_4 and P_1P_4, P_2P_3 and one real line pair P_1P_2, P_3P_4 that form the pairs of opposite sides of this complete quadrangle.

We have the following cases arising from the different possible positions of l_∞ relative to the degenerate conics of a pencil in the above class: (*a*) l_∞ does not coincide with a side of the complete quadrangle P_1, P_2, P_3, P_4, nor pass through a vertex or a diagonal point, (*b*) l_∞ passes through the diagonal point $D(P_1P_4, P_2P_3)$, (*c*) l_∞ passes through D and the diagonal point $D'(P_1P_3, P_2P_4)$, (*d*) l_∞ is the side P_1P_2, (*e*) l_∞ passes through the diagonal point $D''(P_1P_2, P_3P_4)$, (*f*) l_∞ is the line DD''.

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[1] See Veblen and Young *Projective Geometry* vol. II, pp. 70, 71.

[2] See L. E. Dickson "On Families of Quadratic Forms in a General Field" in the *Quarterly Journal of Mathematics*, vol. 39, pp. 316–333.

Case (a) gives (if we choose the x- and y-axes suitably and take two of the degenerate conics of the pencil for fundamental conics and their equations as the typical simultaneous quadratic equations we are seeking)

$$xy = 0, \left(\frac{x}{a+i} + \frac{y}{b+i} - 1\right)\left(\frac{x}{a-i} + \frac{y}{b-i} - 1\right) = 0, \tag{2}$$

where $i=\sqrt{-1}$, with seven other equivalent cases in which the denominators are as follows:[1]

$(-a-i, -b-i, -a+i, -b+i), (a+i, -b+i, a-i, -b-i),$
$(-a+i, b+i, -a-i, b-i), (b+i, a+i, b-i, a-i),$
$(-b+i, -a+i, -b-i, -a-i), (-b+i, a+i, -b-i,$
$a-i), (b+i, -a+i, b-i, -a-i).$

To get all the equivalent cases we first note that the pencil of conics corresponding in geometry to (2)[2] is equivalent to any pencil (2′) with the same denominators as (2) but referred to a different set of axes from those to which (2) is referred. Therefore to find all the pencils of type (2′) that are equivalent to (2), besides the cases where $a'=a$ and $b'=b$, we have merely to find the pencils (2′) that are equivalent to (2) and are also referred to the same axes as (2). To find the transformation that will send (2) into (2′) we note which degenerate conic of (2) goes into the degenerate conic $xy=0$ of (2′) and which goes into

$$\left(\frac{x}{a'+1} + \frac{y}{b'+1} - 1\right)\left(\frac{x}{a'-i} + \frac{y}{b'-i} - 1\right) = 0.$$

Cases (b) through (f) give us respectively the following cases:

$$xy = 0, \left(\frac{x}{a+i} + \frac{y}{-a+i} - 1\right)\left(\frac{x}{a-i} + \frac{y}{-a-i} - 1\right) = 0, \tag{3}$$

with one equivalent case in which the denominators are:

$$(-a+i, a+i, -a-i, a-i).$$

$$xy = 0, (ix + iy - 1)(ix + y + 1) = 0. \tag{4}$$

$$y = 0, (y + ix - 1)(y - ix - 1) = 0. \tag{5}$$

$$x(x-1) = 0, [(1+a)x + iy - 1][(1+a)x - iy - 1] = 0, \tag{6}$$

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[1] The transformation $x=-x'$, $y=-y'$ gives the first equivalent case. Similar transformations give the other cases.

[2] We shall call this pencil the pencil (2).

with equivalent cases with coefficients: $(1-a, i, 1-a, -i)$,

$$\left(\frac{1}{1+a}, i, \frac{1}{1+a}, -i\right), \left(\frac{a-1}{a}, i, \frac{a-1}{a}, -i\right), \left(\frac{a+1}{a}, i, \frac{a+1}{a}, -i\right),$$

$$x(x-1) = 0, \quad (y-i)(y+i) = 0. \tag{7}$$

As an illustration of how we applied the above method to the study of real plane cubic curves in the Euclidean geometry, we shall consider the class of cubics in the real ordinary projective geometry with a typical cubic:

$$y^2 z = x^3, \tag{8}$$

referred to a triangle of reference. Each cubic of this class has a cusp P_1 with its tangent l_1, a point of inflection P_2 with its tangent l_2 and its harmonic polar[1] l_1. In the cubic (8) P_1 is (0, 0, 1), P_2 is (0, 1, 0), l_1 is $y=0$, l_2 is $z=0$. We have the following cases according to the different positions of l_∞ relative to the cubic: (*a*) l_∞ is l_1, (*b*) l_∞ is l_2, (*c*) l_∞ is the line $l_3\,(P_1P_2,)$ (*d*) l_∞ passes through P_1 but is not l_1 or l_3, (*e*) l_∞ passes through P_2 but is not l_2 or l_3, (*f*) l_∞ passes through the point $P_3(l_1l_2)$ but is not l_1 or l_2, (*g*) l_∞ cuts the cubic in three real and distinct points but does not pass through P_1 or P_2 or P_3, (*h*) l_∞ touches the cubic but does not pass through P_1 or P_2 or P_3, (*i*) l_∞ cuts the cubic in one real and two conjugate imaginary points but does not pass through P_1 or P_2 or P_3.

The above cases (*a*) through (*f*) give us respectively the following typical cubics in the real affine geometry:[2]

$$y = x^3, y^2 = x^3, y^2 x = 1, y^2 x = (y+1)^3, y^2 x = (x+1)^3, y^2(y+1) = x^3.$$

Cases (*g*), (*h*), (*i*) give us:[3]

$$y^2(x+y-1) = \alpha x^3, \alpha \neq 0. \tag{9}$$

According as α makes the cubic $y^3+y^2x-\alpha x^3=0$ (which gives the intersections of the curve with l_∞) have three real and distinct roots in y/x, or two equal roots, or one real and two conjugate imaginary roots, do we have cases (*g*), (*h*), (*i*) respectively. Just as in the study of simultaneous quadratic equations, so here we can readily prove that (9) is not equivalent to any (9′) with $\alpha' \neq \alpha$.

We now wish to arrive at the cubic curves in the Euclidean geometry. We take as x'-axis in a set of rectangular axes for (9) the original x-axis of (9) and as y'-axis a line making an angle $\pi/2-\omega$ with the old y-axis. Then we perform the two transformations:

$$x = x' - y' \cot\omega, \quad y = y' \csc\omega. \tag{10}$$

$$x' = ax'', y' = by'', ab \neq 0, \tag{11}$$

[1] See Hilton *Plane Algebraic Curves*, page 95.

[2] Compare Salmon *Higher Plane Curves*, Third Edition, Art. 209.

[3] The line l_2 will have an equation of the form $ax+by-1=0$, and the resulting cubic $y^2(ax+by-1)=x^3$ is reducible to (9).

with no case of $a = \pm 1$, $b = \pm 1$. Using (10) and (11) on (9) we get:

$$b \csc \omega y'' = (ax'' - b \cot \omega y'')^3, \tag{12}$$

where (12) is not equivalent to a similar cubic (12′) unless $a' = \pm a$, $b' = \pm b$, $\omega' = n\pi \pm \omega$.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Espaces courbes. Critique de la relativité. By C. BURALI-FORTI AND T. BOGGIO. Torino, Sten Editrice, 1924. xxiv+255 pages.

This book by two known Italian mathematicians makes one feel sad. It is an example of how intolerance can mislead even powerful minds in a field where we would least expect it.

To explain the situation it will be necessary to make first a few general remarks. The two recognized methods in geometry, the synthetic method (of which Euclid's treatment of the so-called elementary geometry is an outstanding example) and the analytic, have each its advantages and its defects. The advantage of the synthetic method is that it deals directly with the objects of the study while the corresponding defect of the analytic method is that it introduces extraneous things like the axes of coordinates. On the other hand, the analytic method has the advantage that it uses the powerful instrument of formulas—it makes the formulas work for us. It is natural that as early as Leibniz attempts were made to devise a new method—a direct geometrical analysis—which would embody the advantages of the other two without having their defects. We cannot go into the most interesting history of this subject; it will suffice to remark that at present there is no lack of systems of direct geometrical analysis; the trouble is rather that there are too many of them. The number of different geometrical languages is immense and their use is often accompanied by manifestations of partisanship, or a sort of nationalism, lack of tolerance toward other systems, and pugnacity. The most militant of these systems originated in connection with an attempt at unification of notations; C. Burali-Forti, one of the authors of the book under review, is the moving spirit of this school.

It must be said, however, that the situation is by no means hopeless; no great importance can be attached to the use of one or another system of notations; this is largely a matter of taste and, even more, of habit; it may be mentioned parenthetically that the reviewer has been using in his work various of these systems and he finds that it is easy (and useful) to learn how to pass from one system of notations to another.

In the treatment of curved spaces which the advent of the so-called general relativity theory brought into such prominence, the importance of direct methods is even more pronounced than in ordinary geometry. The use of extraneous elements leads in this field to the introduction of distinctions which have nothing to do with the geometric objects and are due only to the system of representation used; for instance, we begin to see double; instead of one vector we see two, a covariant vector and a contravariant vector. The authors of the book under review make it their purpose to get rid of all extraneous features in the theory of curved spaces and this purpose is very commendable but it must be said at once that in spite of some good ideas (they recognize, for instance, the importance for the theory of what they call homographies, i.e. linear and multilinear vector functions) their attempt results in a *failure*.

The situation may be best characterized by stating that the authors have not succeeded in introducing the most fundamental concept in the theory of curved space—the curvature, or the Riemann, tensor—in an *intrinsic* or absolute way, i.e. without the use of extraneous or arbitrary things. In their attempt to eliminate extraneous things they stopped half way: they got rid of coordinates but instead of studying curved space directly they use a representation of it on a Euclidean space, a representation which, as the authors themselves recognize, involves a certain degree of arbitrariness. But the really strange thing is that because in *their* treatment this tensor is introduced with the aid of notions which have no intrinsic significance the authors conclude that the tensor itself is of no or little importance. On this point (which is the central point in their criticism of the application of geometry of curved space to physics) Burali-Forti and Boggio are behind those geometers who while using coordinates succeed in discriminating as to which expressions have a meaning independent of them. And it must be remarked that, of course, it *is* possible to introduce the Riemann tensor intrinsically and that, in fact, the authors themselves were not so very far from it when they introduced the Riemann curvature. It would be sufficient to notice that the Riemann curvatures at a given point (which they introduce intrinsically) determine completely the Riemann tensor at that point.

Outside of this main line of attack on the relativity theory the authors bring forth against this theory all possible arguments without finding anything to say in its favor. Most of these arguments cannot be taken seriously; to discuss them here would mean to enter into polemics which is not the purpose of this review. It must be said, however, that against some points in the theory of relativity the authors raise objections which are not entirely without justification as, for example, in the case of the postulate according to which all true laws of nature ought to be expressible as covariant relations, especially in the application of this postulate to the energy tensor.

The book cannot be recommended as interesting or instructive reading; in addition to the drawbacks already mentioned it is not self-sufficient as it presupposes the knowledge of the contents of a textbook on vector analysis by Burali-Forti and Marcolongo. Those who specialize in differential geometry might find some worthwhile ideas; besides, the effort of recognizing familiar things under a new form might prove useful as a sort of mental gymnastic. But those who will use it must be warned that it is dangerous to take anything at its face value. To give but one example: the authors state the theorem, not in general true, that if two spaces (of dimension higher than two) have the same metrics one can be obtained from the other by a translation or a symmetry in the containing space — and "prove" it (p. 168).

On the whole, it is to be regretted that such a book has been published. Not so much because it is apt to increase the existing confusion as to the so-called relativity theory (I do not think that it will appreciably increase this confusion because plenty of other books and pamphlets display much the same type of arguments against the relativity theory without displaying so many formulas and are therefore more accessible to the public). It is more to be regretted that some sound ideas and a little sound criticism have been buried among things of doubtful value. But most of all it is to be regretted that it actually may hamper the spreading of the method of direct geometrical analysis in general, and vector analysis in particular.

G. Y. RAINICH.

The Geometry of René Descartes. Translated from the French and Latin by DAVID EUGENE SMITH AND MARCIA L. LATHAM. With a facsimile of the first edition, 1637. Chicago, The Open Court Publishing Co., 1925. xiii+246 pages. Price $4.00.

American mathematicians are fortunate in having within their reach through this publication a facsimile of Descartes' original *La Géométrie*, accompanied by a very good translation into English. It is with regret that the announcement was received recently of the death of one of the translators, Marcia L. Latham of Hunter College. Descartes' *Geometry* came to be highly esteemed soon after its appearance in 1637, and to our day it has maintained its place as a classic. It enjoys the distinction of being the book from which Isaac Newton acquired his first knowledge of geometry. And yet one experiences some difficulty in stating, in a few words, exactly wherein the great novelty of the work lies. One cannot say truthfully that Descartes was "the" inventor of analytical geometry, for Fermat about the same time accomplished substantially the same; to be sure, Fermat's treatise did not appear in print until 42 years after that of Descartes. One cannot say that Descartes first employed coordinates, because their use goes back to Greek time. It is not

true that Descartes displayed great inventive power in introducing into algebra our exponential notation for positive integral powers; before him a term like $5a^4$ was written $5a4$ by Hérigone and $5a^{\text{IV}}$ by Hume. It is rather a happy combination of various innovations in one book of compact form, exhibiting the novel geometric use of an equation with more than one unknown, in the solution of difficult applied problems, which gives it celebrity.

FLORIAN CAJORI.

Principii di Filosofia Naturale, Teoria della Gravitazione. By SIR ISAAC NEWTON; translated and edited by FEDERIGO ENRIQUES AND U. FORTI. Rome, A. Stock, 1925. 218 pages. Price, 16 lire.

This is the third volume of the series "Per la Storia e la Filosofia della Matematiche" edited by Professor Enriques. It consists of a translation of Newton's *Philosophiae naturalis principia mathematica*, which appeared in 1687, together with a brief biography of Newton and an appendix of about fifty pages of notes. In this appendix there is given a historical survey of the following topics: (1) The concept of mass or quantity of matter; (2) Force and the laws of motion; (3) Motion, space, and time, absolute and relative; (4) Newton's mechanics as a cosmic science. In the third of these "notes" the historical development of the Einstein theory is sketched.

DAVID EUGENE SMITH.

Il "Metodo" di Archimede e le origine dell' Analisi Infinitesimale nell' Antichità. By ENRICO RUFINI. Rome, A. Stock, 1926. viii+293 pages. Price, 22.50 lire.

This is the fourth volume of the series "Per la Storia e la Filosofia delle Matematiche" edited by Professor Federigo Enriques. It constitutes a monument unconsciously erected to his own memory by the author, who died on November 3, 1924, at the early age of thirty-four, just after completing his manuscript,—a man of whom Professor Enriques speaks very feelingly with respect to his versatility of genius and the breadth of his scholarly interests: "Le Matematiche, le Lettere e la Filosofia lo interessavano quasi in egual grado, e questa varia cultura doveve poi avviarlo assai naturalmente alla storia della scienza." A list of his works was published in December, 1924, in the *Archivio della Storia della Scienza.*

This volume makes a departure from the line followed in the first three numbers of the series, all of which were translations of certain classics, with comments by Professor Enriques or the immediate editor of the volume in question. Dr. Rufini has not been content with translating the Archimedean "Method"; he has introduced it by a "Parte prima" of eighty-eight pages on the origin and development of the infinitesimal analysis in the period closing

with Archimedes. Furthermore, in Part III the author discusses the integrations of Archimedes, devoting nearly a hundred pages to the subject and referring briefly to the influence of the ancients upon the early writers on the calculus in the sixteenth and seventeenth centuries.

The test of the "Method" is based chiefly upon the Kliem translation of Heath's *Archimedes*. There is a brief bibliography of the history of Greek mathematics and a list of the translations of the "Method."

The first part treats of the following topics: (1) The Pythagorean geometry; (2) The critical attitude of Parmenides of Elea; (3) The polemics of Zeno of Elea; (4) The infinitesimal as suggested by Democritus of Abdera; (5) The quadrature of the circle of Antiphon and Bryson (with recognition of but not complete acceptance of the opinion that undue credit has been given to the latter); (6) Eudoxus of Cnidus and the critical (by a curious blunder in the contents, *mitica* (mythical) is used for *critica*) systematizing of the study of the infinitesimal; (7) Eudoxus and the proof by exhaustion; (8) Observations on the method of exhaustion; (9) The infinite and the infinitesimal according to Aristotle; (10) Book XII of Euclid's *Elements*; (11) The work of Archimedes; (12) The mechanical method of Archimedes.

The second part contains the translation of the work in question, and the third relates, as already stated, to Archimedes's integrations, all of which are familiar to non-Italian readers in the translations of Heath, Kliem, and Ver Eecke.

David Eugene Smith.

ARTICLES IN CURRENT PERIODICALS

American Journal of Mathematics, volume 48, no. 2, April, 1926: "The theory of the binary octavic" by A. M. Whelan, 73–100; "The Borel summability of Fourier series" by M. H. Stone, 101–112; "Self dual space curves" by T. R. Hollcroft, 113–124; "Expansions in terms of certain polynomials connected with the gamma-function" by B. P. Hoover, 125–138, "On the value of the Naperian base" by D. H. Lehmer, 139–143; "On differential inversive geometry" by F. Morley, 144–146. No. 3, July, 1926: "On the structure of a continuum, limited and irreducible between two points" by A. Wilson, 147–168; "The expansion problem for a certain system of ordinary linear second order differential equations" by M. G. Carman, 169–182; "The distribution of lift over thin wing sections" by C. A. Shook, 183–203; "The correspondence between the tangent plane of a surface and its point of contact" by E. P. Lane, 204–214; "On multiple iterated integrals" by H. J. Ettlinger, 215-222; "Cubic involutions and a C_3" by H. S. White, 223–224.

Annals of Mathematics, second series, volume 27, no. 3, March 1926: "Transformation of the Kummer criteria in connection with Fermat's last theorem" by H. S. Vandiver, 171–176; "Algebraic potential curves" by J. L. Coolidge, 177–186; "A method of deriving the infinite double products in the theory of elliptic functions from the multiplication theorems" by G. Mittag-Leffler, 187–194; "Miscellaneous questions in the theory of differential equations. 1. On the method of Frobenius" by E. Hille, 195–198; "Non-measurable functions connected with certain functional equations" by H. Blumberg, 199–208; "Entire functions defined by certain power series" by J. T. Colpitts, 209–223; "Approximation of curves and surfaces by algebraic curves and surfaces" by P. A. Smith, 224–244; "Note on matrices in a given field" by J. H. M. Wedderburn, 245–248; "On the arithmetical applications of the

UNDERGRADUATE MATHEMATICS CLUBS

EDITED BY H. J. ETTLINGER, 3110 Harris Park Ave., Austin, Texas

CLUB ACTIVITIES

GAMMA ETA MU, University of Redlands, Redlands, California.

At the suggestion of advanced students, an organization known as the Gamma Eta Mu Club was formed at the University of Redlands on November 13, 1924, for the purpose of promoting interest in mathematical subjects. Membership was at first limited to those majoring or minoring in mathematics, but on March 1, 1926 was extended to include those majoring in physics or engineering. The following programs have been given:

December 8, 1924. "Pre-Greek history of mathematics" by Kate Smallin; "Inscribed and circumscribed circles in a parabolic segment" by Jack Boren; "Construction of regular decagon" by Lewis Hammen.

January 19, 1925. Picnic at Camp Kill Kare.

February 16. "Greek history of mathematics" by Sarah Russell; "Ladder and post problem" by Roy Slocum.

March 16. "Hindu mathematics" by Alexis Maredudin; "Four common tangents to two circles" by Julius Williams.

April 20. "Twelve eminent mathematicians" by William Garner and Nelson Painter; "Graduate work in mathematics" by Leslie Hosegood; "Magic squares" by Doris Smith.

May 18. "Life of Newton" by Walter Burger; "Nine point circle" by Marion Scott; "Fourth dimension" by Mrs. Ethyl White.

May 25. Picnic at Plunge Creek.

October 14. Picnic in Mill Creek Canyon.

October 26. "History of π" by Helen Irwin; "Evolute of $x=(t^2+1)/4$, $y=t^3/6$" by James Wright.

November 16. "Life of Euclid" by Fred Riedman; "Maximum chord of cardioid" by Joseph Hartshorn.

December 14. "Babylonian mathematics" by Alexis Maradudin; "Locus of center of circle cutting three given circles at a constant angle" by Mrs. Ethyl White.

January 18, 1926. "Fermat's last theorem" by George Shinn; "A particle falling where resistance is proportional to the square of velocity" by Jack Boren.

February 15. "Curves satisfying condition $s^2=4ky$, where s is the length of arc" by William Richardson.

March 16. Lecture and demonstration of Willys Knight motor by the company's representative.

April 19. "San Bernardino's water system" by Leslie Hosegood; "Construction of circle tangent to three other circles" by Clara Seaton.

May 19. Picnic at Plunge Creek.

The officers were: President, William Richardson; vice-president, Clara Seaton; Secretary-treasurer, Mrs. Ethyl White; faculty advisers, Professor O. W. Albert, mathematics, and Professor H. E. Marsh, physics.

(Report by Professor Albert)

CENTENARY COLLEGE MATHEMATICS CLUB, Shreveport, La.

In the early part of October, 1925, those students of Centenary College who had a special interest in mathematics were called together and a Mathematics Club was organized. Those instrumental in organizing the club were Miss Velinsky and Dean Hardin of the mathematics department and Dr. I. Maizlish, head of the physics department. The officers elected are:

Emmet Meadows, President; Miss Emily Dean Odom, Secretary-treasurer.

The Club meets monthly. Some of the topics thus far discussed are:

"The story of Pi", Emmet Meadows;

"Discovery of Neptune", Leon Scales;

"Life and works of Willard Gibbs", W. G. Banks;

"Practical mathematics", Mr. Anderson;

"History of arithmetic", Clingman Munday;

"Value of mathematics", Dean John A. Hardin.

In his closing address Dean Hardin expressed his pleasure at the rapid progress of the Club, which had an average attendance of fifteen for the year. He stated that the prospect was very bright for an even more successful organization next year.

(Report by Emily Dean Odom, secretary-treasurer)

UNDERGRADUATE MATHEMATICS CLUB, University of Iowa, Iowa City, Iowa.

[*1925*, 432]

Altogether thirteen meetings were held during the past year. A consistent effort was made to provide interesting programs and the result was an average attendance of fifty. A new feature was a short social time at the beginning of each meeting during which tea and cakes were served. In the latter half of the year we tried particularly to interest Freshmen and the attendance at one special meeting was ninety-nine. The subjects discussed and the speakers follow: The measurement of angles, Dr. Wilson; Complex numbers, Miss Reger and Miss Peters; Computation of Pi, Mr. Stehn; Partial fractions, Mr. Wilson and Mr. Long; The why of the number *e*, Mr. Briggs; The circles of Appolonius, Mrs. Wilson; Twenty-five years in mathematics, Prof. Rietz; Trisection of angles, Miss Reger; Russian peasant method of calculation, Mr. Allen; Number systems of the ancients, Mr. Messick; Factorials and binomial coefficients, Dr. Woods. An additional feature was the presentation at each meeting of a problem for solution at the following meeting.

(Report by Miss Leona Schuster, secretary)

JOHNS HOPKINS MATHEMATICAL CLUB, Baltimore, Md.

[*1921*, 181]

The Undergraduate Mathematics Club at the Johns Hopkins University met bi-weekly during 1925–6.

The following subjects were presented:

1. "The game 'Nim'," Louis M. Volansky.
2. "Possible and impossible 'Euclidean' constructions," Dr. A. Cohen.
3. "Problems in probability," Walter S. Dawkins.
4. "History of π," Brainerd D. Wilson.
5. "Magic squares," David M. Ashkenaz.
6. "The nine-point circle," Louis M. Volansky.
7. "Non-Euclidean geometry," W. S. Dawkins.
8. "Geometrical paradoxes," Victor R. Deitz.
9. "Geometrical representation of complex numbers," Dr. A. Cohen.

(Report by Professor A. Cohen)

WHITE MATHEMATICS CLUB, University of Kentucky, Lexington, Ky.

[*1925*, 430]

Sept. 23, 1925. Election of the following officers:

President, D. E. South; Secretary, M. C. Brown. *Essentials of Mental Measurements* by Brown and Thompson was adopted as a basis for the year's club work.

Oct. 15. "Curiosities" by Professor H. H. Downing.

Oct. 27. "Suggestions for programs" by Professor J. Morton Davis.

Nov. 19. "Elementary theory of probability" by D. E. South, instr.

Dec. 3. "Psycophysical Methods" by Professor E. L. Rees.

Dec. 17. "Difference thresholds" by Dr. F. E. LeStourgeon.

Jan. 14, 1926. "Introduction to correlation" by Prof. J. Morton Davis.

Feb. 4. "Coefficient of correlation" by Mr. M. C. Brown, instructor.

Feb. 18. "Correlation ratio" by Mr. D. O. Streyffeler, instructor.

Mar. 4. "Study of grades made by freshmen in mathematics and in mental tests at U. of K. in 1925" by Mr. E. J. Canaday, instructor.

Mar. 18. "Methods of computing correlation" by Professor P. L. Boynton of the Psychology Department.

April 18. "Types of collineations in the ternary field" by Miss Vada Lee Nelson, graduate student.

April 23. "Circular inversions" by Mr. H. W. Mobley, grad. asst.

May 13. "Irrational numbers" by Miss Mary Hester Cooper, grad. asst.

(Report by M. C. Brown, secretary)

The Mathematics Club of the University of Kansas, Lawrence, Kansas

The officers of the Mathematics Club of the University of Kansas for the year 1925-26 were: President, Elizabeth Bolinger '26; Vice President, Lloyd Young '26; Secretary-Treasurer, Vera Bolton '26.

The following topics were presented at the meetings:

October 5, 1925: "Some new properties of determinants" by Professor E. B. Stouffer

October 19: "Systems of conics through four fixed points" by H. K. Hughes, Gr.

November 2: "The slide rule for complex numbers" by Professor M. E. Rice.

November 16: "Reminiscences of Benjamin Peirce" by Marjory Council '26.

December 7: Play "Flatlanders" by members of the club. "The fourth dimension" by Professor R. H. Wheeler.

January 11, 1926: "Geometrical representations of indeterminate forms" by Grace Poe '26.

February 1: "The nine point circle" by C. A. Reagan, Gr.

February 15: "1925 as a centennial year in mathematics" by Helen McFerren '26.

March 1: "Mathematical poems" by Helen Mark '25. "Perpetual calendars" by Rose Middlekauff '26.

March 15: "Some theorems of geometrical optics" by Lloyd Young '26.

April 19: "Some magic in mathematics" by Zella Colvin, Gr.

May 2: "Algebraic curves related to conics" by P. F. Wall, Gr.

May 17: Annual picnic.

(Report by Vera Bolton)

The Napierian Club of De Pauw University, Greencastle, Indiana

The officers for the year 1925-1926 were: President, James V. Brown, '26; Vice-President, Ruth Bickel, '26; Secretary, Gertrude Hendrix, '26; Treasurer, Orin Sykes, '26. Monthly meetings were held throughout the year, and the programs were as follows:

October 14, 1925: Selection of new members.

November 11, 1925: "The Life and Work of John Napier" Elizabeth Erwin, '26.

December 9, 1925: "Einstein's Theory of the Fourth Dimension" by Harold Weeks, '26.

January 13, 1926: "Mathematics and Mathematicians in Music" by Professor H. E. H. Greenleaf.

February 24, 1926: "The Slide Rule" by Professor Clark Arnold.

March 17, 1926: A debate on the question, "Resolved that ten hours of mathematics should be required for graduation at De Pauw University." Decision negative.

April 15, 1926: "Chinese Mathematics" by Horace Yu, '26.

May 27, 1926: Alumni letters. Election of officers.

(Report by Gertrude Hendrix, secretary)

The Agnesi Club of Agnes Scott College, Decatur, Ga.

The program for the year 1925-1926 was the following:

October 6, 1925. Introductory talk by the president giving organization, purpose, and plans of the Agnesi Club. The organization took place in 1921 and the purpose was to stimulate the interest in physics, mathematics and astronomy.

November 3. "Mechanisms, planimeters, and integraphs" by Professor W. W. Rankin.

December 1. Astronomical program. "Nebulae and comets" by Miss F. Swann, '26 and Miss E. Kennedy, 26, with slides by Miss Emily E. Howson, professor of physics.

February 4, 1926. Catch problems in mathematics by Miss L. S. Wallace, '26. "History of the calculus" by Miss M. E. Hammond, '26.

March 19. "Theory of equations" by Professor L. E. Dickson, University of Chicago. Social hour.

April 6. Physics program on transverse waves. "Gamma rays" by Miss K. Pitman, '26. "X-rays" by Miss E. Lynn, '27. "Ultra violet and visible waves" by Miss H. Huff, '26. "Infra-red waves" by Miss V. B. Grant, '27. These talks were accompanied by slides and experiments. Radio program.

May 4. Mathematics play "The Eternal Triangle" written by Miss S. Slaughter, '26, and Miss F. Swann, '26, read by Miss Slaughter.

(Report by Miss Lynn, Secretary)

THE MATHEMATICS CLUB of the College of the City of Detroit, Detroit, Mich.

The Mathematical Club of the College of the City of Detroit was organized January 18, 1926 with fifteen charter members. The following officers were elected:

President, Max Coral, '27; secretary, Frances Garvey, '27; program committee, Philip Gentile, '26 and Kenneth Salisbury, '28.

The papers presented during the first semester of the club's existence were:

January 18, 1926. "Non-euclidean geometry" by Professor J. W. Baldwin.

March 24. "Mathematical recreations" by David Koretz, '26.

April 14. "Graphical solution of equations" by Professor T. R. Running, University of Michigan.

May 18. "Three famous problems" by Rose Chesluk, '27.

(Report by Frances Garvey, Secretary)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All mauscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3226. Proposed by H. E. Trefethen, Colby College.

The cross section of a circular ring is crescent shaped. The outer arc is a semicircle. The inner arc has a radius of one inch and its center is at the center of the ring. Find the greatest possible volume for the ring.

3227. Proposed by H. W. Reddick, Cooper Union Institute of Technology.

Solve the differential equation

$$\frac{d^5y}{dx^5} - \frac{dy}{dx} - \frac{4y}{x} = 0.$$

3228. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The perpendiculars dropped from the vertices of a triangle upon the lines joining the mid-points of the opposite sides to the orthocenter of the triangle meet the respective sides of the triangle in three collinear points.

The line joining these three points is perpendicular to the Euler line of the triangle.

3229. Proposed by John Biggerstaff, Seattle, Washington.

Prove that the function $\{p(u)\ p'(u)+p^2(u)-1\}$ has five zeros u_1, u_2, u_3, u_4, u_5, in a period parallelogram such that

$$\sum_{n=1}^{n=5} u_n = 2\lambda\omega_1 + 2\mu\omega_2,$$

where λ and μ are integers and $p(u)$, Weierstrass's elliptic function.

Verify that if $z=p(u)$, these values of u give the five roots of the equation $4z^5-z^4-g_2z^3+(2-g_3)\,z^2-1=0$, g_2 and g_3 being constants and solve the general quintic by this method.

3230. Proposed by C. N. Schmall, New York City.

Determine the point in an ellipse at which the two focal distances include the greatest angle.

3231. Proposed by R. B. Stone, Purdue University.

The root-mean-square (R. M. S.) of n numbers $x_1, x_2, \cdots, x_n$ is defined by the formula

$$R.M.S. = \left(\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}\right)^{1/2}.$$

(*a*) For what values of n is the R. M. S. of the first n integers also an integer?

(*b*) For what values of a and n is the R. M. S. of n successive integers

$$a, \quad a+1, \cdots \ a+n-1$$

also an integer?

(*c*) Under what conditions is the R. M. S. of n integers also an integer?

SOLUTIONS

2672 [1918, 74; 1925, 385]. Proposed by E. T. Bell, Seattle, Washington.

There is an identity in z, (1) $A(z) \equiv B(z)\ C(z)$; e.g., $A(z) = 1/(1-k^2z^2)$; $B(z) = 1/(1-kz)$; $C(z) = 1/(1+kz)$; and the formal expansions $A(z) = \Sigma a(n)z^n$, $B(z) = \Sigma b(n)z^n$, $C(z) = \Sigma c(n)z^n$, $(n = 0, 1, \cdots, \infty)$, when substituted in (1), give, on equating coefficients (2):

$$a(n) = b(n)c(0) + b(n-1)c(1) + \cdots + b(0)c(n).$$

If (2) is an identity in n, justify such a use of non-convergent series to obtain it (*e.g.*, for $|k| \geqq 1$ in the above). This method of finding important identities (2) has been freely used by Hermite and many others without question of its validity, and without offering independent proofs of (2).

Solution by Otto Dunkel, Washington University, St. Louis, Mo.

By the formal expansion of $f(z)$ in a power series in z we mean ordinarily the rule given by the Maclaurin series. Hence when we say that each of the three functions $A(z)$, $B(z)$, $C(z)$ may be expanded as in the problem, it is implied that all of their derivatives exist in the neighborhood of $z=0$. Consequently, since $A(z) \equiv B(z)\ C(z)$, we must have $A^{(n)}(0) = [B(0)C(0)]^{(n)}$, where the index n indicates the nth derivative. The process for the development of the right side in terms of the derivatives of B and C is purely formal and it is unique. To obtain this formal development, we may at first suppose that each of the three functions admit formal developments which are convergent and which represent them in value in the neighborhood of $z = 0$. Then since

$$\sum \frac{A^{(i)}(0)}{i!} z^i = \sum \frac{B^{(i)}(0)}{i!} z^i \sum \frac{C^{(i)}(0)}{i!} z^i,$$

we must have

$$\frac{A^{(n)}(0)}{n!} = \sum_{i=0}^{n} \frac{B^{(i)}(0)}{i!} \frac{C^{(n-i)}(0)}{(n-i)!}.$$

This gives the form of the development desired and it must be true whether the series converge or not. This proves the theorem. The result is Leibniz' theorem, which may be established directly by other known methods, and then this theorem may be used to establish the desired result without the use of the series.

Also solved by HARRY LANGMAN.

3151 [3147; 1925, 434]. Proposed by J. A. Bullard, U. S. Naval Academy.

The cylinder $(x/a)^2 + (y/b)^2 = 1$, passes through the sphere, $x^2 + y^2 + z^2 = a^2$. If e is the eccentricity of the ellipse $(x/a)^2 + (y/b)^2 = 1$, show that

(a) the surface cut from the sphere $= 8a^2 \arccos e$,

(b) the volume cut from the sphere $= 8a^3 [e(1 - e^2)^{1/2} + \arccos e]/3$.

SOLUTION BY J. M. EARL, University of Minnesota.

(a) The width of an elementary zone cut out by the planes $x = x$ and $x = x + dx$ is

$$ds = \sqrt{1 + \left(\frac{dz}{dx}\right)^2}\, dx \text{ where } x^2 + z^2 = a^2. \text{ Hence } ds = \frac{a\, dx}{(a^2 - x^2)^{1/2}}.$$

The element of arc along this zone is

$$ds' = \sqrt{1 + \left(\frac{dz}{dy}\right)^2}\, dy \text{ where } y^2 + z^2 = (a^2 - x^2). \text{ Hence } ds' = \frac{(a^2 - x^2)^{1/2} dy}{[(a^2 - x^2) - y^2]^{1/2}}.$$

The required area, which is 8 times the area cut off in the first octant, is

$$A = 8 \iint ds\, ds' = 8a \int_{x=0}^{x=a} dx \int_{y=0}^{y=(b/a)(a^2-x^2)1/2} \frac{dy}{[(a^2-x^2)-y^2]^{1/2}} = 8a^2 \arccos e.$$

(b) In a similar manner, the required volume is

$$V = 8\int_{x=0}^{x=a} \int_{y=0}^{y=(b/a)(a^2-x^2)1/2} z\, dx\, dy = 8\int_{x=0}^{x=a} dx \int_{y=0}^{y=(b/a)(a^2-x^2)1/2} [(a^2 - x^2) - y^2]^{1/2}\, dy$$
$$= (8a^3/3)\left[(b/a^2)(a^2 - b^2)^{1/2} + \arcsin (b/a)\right]$$
$$= (8a^3/3)\left[e(1 - e^2)^{1/2} + \arccos e\right].$$

Also solved by E. T. ALLEN, MICHAEL GOLDBERG, E. C. KIEFER, J. B. REYNOLDS, H. A. SIMMONS, and R. H. SCIOBERETI.

3152 [3148; 1925, 480]. Proposed by E. M. Berry, Purdue University.

Let Δ be the determinant $|x_1 y_2 z_3| \neq 0$, and let X_i be the cofactor of x_i in Δ and Y_i the cofactor of y_i, etc. Prove that if

$$\begin{vmatrix} 1/x_1 & 1/x_2 & 1/x_3 \\ 1/y_1 & 1/y_2 & 1/y_3 \\ 1/z_1 & 1/z_2 & 1/z_3 \end{vmatrix} = 0, \text{ then } \begin{vmatrix} 1/X_1 & 1/X_2 & 1/X_3 \\ 1/Y_1 & 1/Y_2 & 1/Y_3 \\ 1/Z_1 & 1/Z_2 & 1/Z_3 \end{vmatrix} = 0$$

and conversely.

SOLUTION BY H. A. SIMMONS, Northwestern University.

Denote the second determinant by Δ' and the third by D'. Then on expanding these determinants we find

$$\begin{aligned} \Delta' &= \lambda_1 [x_2 x_3 y_1 z_1 X_1 + x_3 x_1 y_2 z_2 X_2 + x_1 x_2 y_3 z_3 X_3], \\ D' &= \Delta \lambda_2 [X_2 X_3 Y_1 Z_1 x_1 + X_3 X_1 Y_2 Z_2 x_2 + X_1 X_2 Y_3 Z_3 x_3], \end{aligned} \tag{1}$$

where λ_1 and λ_2 are two finite factors, each different from zero—these facts being implied by the forms in which the determinants Δ' and D' are given. Now since

$$\sum_{i=1}^{3} z_i X_i = 0 \text{ and } \sum_{i=1}^{3} Z_i x_i = 0,$$

by a well known, elementary property of determinants, we can write (1) in the form

$$\Delta' = -\lambda_1(x_3 z_2 X_2 Z_3 - x_2 z_3 X_3 Z_2), \qquad D' = -\Delta^2\lambda_2(x_3 z_2 X_2 Z_3 - x_2 z_3 X_3 Z_2). \tag{2}$$

Since $\Delta \neq 0$ by hypothesis and λ_1, λ_2 do not vanish in this problem, we see that $\Delta' = 0$ if, and only if, $D' = 0$, which proves the theorem and its converse.

Also solved by HARRY LANGMAN, and J. J. NASSAU.

3153 [3149; 1925, 481]. Proposed by A. A. Bennett, Lehigh University.

Given a circle, C. Inscribe in C a simple quadrilateral whose pairs of non-adjacent sides produced meet in points P and Q respectively. Show that the circles with P and Q as centers and orthogonal to C meet in a point R, where PRQ is a right angle.

SOLUTION BY NATHAN ALTSHILLER-COURT, University of Oklahoma.

The points P, Q are two diagonal points of the complete quadrangle having the same vertices as the simple quadrilateral inscribed in the given circle C, therefore the points P, Q are conjugate with respect to C, and the polar of P with respect to C is the perpendicular from Q to the line PO joining P to the center O of C, the foot S of this perpendicular being the inverse of P with respect to C. The circle (L) having PQ for diameter will pass through S, and therefore be orthogonal[1] to C.

We have thus three circles orthogonal to C: the circle (L) and the two circles (P), (Q), having for their centers the points P, Q. Since the centers of these three circles are collinear, the three circles are coaxal[2], i.e., the common points R, R' of the circles (P), (Q) lie on the circle (L), hence PRQ is a right angle.

REMARK. We have thus incidentally proved the proposition:

If two circles are orthogonal to a third circle and their centers are conjugate with respect to that circle, the two circles are orthogonal to each other.

Also solved by J. W. CLAWSON, RUFUS CRANE, W. J. PATTERSON, J. ROSENBAUM and MABEL YOUNG.

3154 [3150; 1925, 481]. Proposed by A. S. Wiener, Cornell University.

Prove

$$\begin{vmatrix} b^2+c^2 & 0 & -cd & be & 0 \\ 0 & a^2+d^2 & 0 & ab & ad \\ -cd & 0 & d^2+e^2 & 0 & ce \\ be & ab & 0 & b^2+e^2 & 0 \\ 0 & ad & ce & 0 & a^2+c^2 \end{vmatrix} \times \begin{vmatrix} b^2+c^2 & 0 & cd & be & 0 \\ 0 & a^2+d^2 & 0 & ab & ad \\ cd & 0 & d^2+e^2 & 0 & ce \\ be & ab & 0 & b^2+e^2 & 0 \\ 0 & ad & ce & 0 & a^2+c^2 \end{vmatrix}$$

$$= \begin{vmatrix} b^4+c^4 & 0 & -c^2d^2 & b^2e^2 & 0 \\ 0 & a^4+d^4 & 0 & a^2b^2 & a^2d^2 \\ -c^2d^2 & 0 & d^4+e^4 & 0 & c^2e^2 \\ b^2e^2 & a^2b^2 & 0 & b^4+e^4 & 0 \\ 0 & a^2d^2 & c^2e^2 & 0 & a^4+c^4 \end{vmatrix}.$$

[1] Nathan Altshiller-Court, *College Geometry*, p. 147, Johnson Publishing Co., Richmond, Va., 1925.

[2] *Ibid*, p. 188.

SOLUTION BY HARRY LANGMAN, Brooklyn, N. Y.

If we square the determinant

$$\begin{vmatrix} 0 & b & \mp c & 0 & 0 \\ a & 0 & 0 & d & 0 \\ 0 & 0 & d & 0 & e \\ b & e & 0 & 0 & 0 \\ 0 & 0 & 0 & a & c \end{vmatrix}$$

we obtain the first or second determinant of the problem, according as the upper or lower sign before c is taken. Expanding this according to the minors of the first two columns, we obtain at once

$$c(b^2d^2 \mp a^2e^2).$$

Replacing each letter in this equality by its square (choosing the upper sign) and squaring, we obtain the value of the third determinant. The identity is then evident.

3155 [3151; 1925, 481]. Proposed by C. C. Wylie, University of Iowa.

In a concave-convex lens the radii of curvature of the convex and concave surfaces are r_1 and $r_2 > r_1$. Determine the thickness and diameter of the lens so that the centre of mass shall be in the concave surface.

(Moulton's *Celestial Mechanics*, new edition, p. 29)

SOLUTION BY J. L. RILEY, Ouachita College, Arkadelphia, Ark.

The equations of the circles may be expressed in the forms:

$$y^2 = -x^2 + 2r_1x \quad \text{and} \quad y^2 = -x^2 - t^2 + 2r_2x + 2tx - 2r_2t$$

where t is the thickness of the lens.

The centroid of the lens is

$$\bar{x} = \frac{\int_0^\alpha (2r_1x^2 - x^3)dx - \int_t^\alpha (2r_2x^2 + 2tx^2 - x^3 - 2r_2tx - t^2x)dx}{\int_0^\alpha (2r_1x - x^2)dx - \int^\alpha (2r_2x + 2tx - x^2 - 2r_2t - t^2)dx}$$

where

$$\alpha = \frac{t(t+2r_2)}{2(r_2 - r_1 + t)}.$$

Remembering that $\bar{x}$ must equal t, we get

$$1 = \frac{\dfrac{(t+2r_2)^3}{(r_2 - r_1 + t)^2} - 2t - 8r_2}{\dfrac{6(t+2r_2)^2}{r_2 - r_1 + t} - 8t - 24r_2}.$$

Hence

$$t^3 + (-6r_1 + 4r_2)t^2 + (6r_1^2 - 20r_1r_2 + 2r_2^2)t + 16r_1^2r_2 - 8r_1r_2^2 = 0.$$

Since t is small, we neglect the terms in t^3 and t^2, and find that, approximately,

$$t = \frac{4r_1r_2^2 - 8r_1^2r_2}{3r_1^2 - 10r_1r_2 + r_2^2}.$$

3159 [3155; 1925, 520]. Proposed by R. E. Moritz, University of Washington.

Show that

$$\frac{d^n}{dx^n}[x^{m+n}(\log x)^n] = n!x^m\left[1 + {}_n^mp_1 \log x + {}_n^mp_2 \frac{(\log x)^2}{2!} + \cdots {}_n^mp_n \frac{(\log x)^n}{n!}\right],$$

where ${}_n^mp_k$ denotes the sum of the products of the numbers $m+1, m+2, \cdots, m+n$, taken k at a time.

I. SOLUTION BY M. S. KNEBELMAN, Princeton University.

Let

$$x = e^y\ ;\ \text{then}\ \frac{d}{dx} = e^{-y}\frac{d}{dy} = e^{-y}D,\ \text{where}\ D, \equiv \frac{d}{dy}\cdot$$

With this substitution, we get

$$\frac{d^n}{dx^n}\left[x^{m+n}(\log x)^n\right] = (e^{-y}D)^n\left[e^{(m+n)y}y^n\right].$$

To evaluate the last expression, we have two well known theorems on the symbolic operator D:

$$(e^{-y}D)^n = e^{-ny}\prod_{\alpha=1}^{n}(D - n + \alpha) \tag{1}$$

$$f(D)\left[e^{ay}F(y)\right] = e^{ay}f(D+a)\left[F(y)\right]. \tag{2}$$

By (1), our expression becomes

$$e^{-ny}\prod_{\alpha=1}^{n}(D - n + \alpha)\left[e^{(m+n)y}y^n\right]$$

and by (2) it further reduces to

$$e^{my}\prod_{\alpha=1}^{n}(D + m + \alpha)\left[y^n\right].$$

Now

$$\prod_{\alpha=1}^{n}(D + m + \alpha)$$

is a polynomial in D; it is of the nth degree and its coefficients are of the form ${}_n^m p_k$, $({}_n^m p_0 \equiv 1)$. We may therefore write

$$\prod_{\alpha=1}^{n}(D + m + \alpha) = \sum_{k=0}^{n} {}_n^m p_k D^{n-k}.$$

Again

$$D^{n-k}\left[y^n\right] = n(n-1)(n-2)\cdots(k+1)y^k = \frac{n!}{k!}y^k\ ;$$

therefore,

$$(e^{-y}D)^n\left[e^{(m+n)y}y^n\right] = n!e^{my}\sum_{k=0}^{n} {}_n^m p_k \frac{y^k}{k!}$$

and since $y = \log x$ we finally have

$$\frac{d^n}{dx^n}\left[x^{m+n}(\log x)^n\right] = n!x^m\sum_{k=0}^{n} {}_n^m p_k \frac{(\log x)^k}{k!}\cdot$$

II. SOLUTION BY THE PROPOSER.

By Maclaurin's theorem

$$x^{m+n+k} = x^{m+n}\left[1 + k\log x + \frac{k^2(\log x)^2}{2!} + \cdots + \frac{k^n(\log x)^n}{n!} + \cdots\ \right] \tag{1}$$

Differentiating with respect to x, we have, the series on the right being uniformly convergent for all values of $x \neq 0$,

$$\begin{aligned}(m+n+k)x^{m+n+k-1} &= (m+n+k)x^{m+n-1}\left[1 + k\log x + \cdots + \frac{k^n(\log x)^n}{n!} + \cdots\right]\\ &= \frac{d}{dx}(x^{m+n}) + k\frac{d}{dx}(x^{m+n}\log x) + \cdots + \frac{k^n d}{n!dx}\left[x^{m+n}(\log x)^n\right] + \cdots\end{aligned}$$

Similarly, on differentiating both sides of [1] n times with respect to x,

$$(m+n+k)(m+n+k-1)\cdots(m+k+1)x^m\left[1+k\log x+\cdots+\frac{k^n(\log x)^n}{n!}+\cdots\right]$$

$$=\frac{d^n}{dx^n}(x^{m+n})+k\frac{d^n}{dx^n}(x^{m+n}\log x)+\cdots+\frac{k^n d^n}{n!dx^n}[x^{m+n}(\log x)^n]+\cdots \tag{2}$$

Now

$$(m+n+k)(m+n+k-1)\cdots(m+k+1)=k^n+{}_n^mp_1k^{n-1}+{}_n^mp_2k^{n-2}+\cdots+{}_n^mp_{n-1}k+{}_n^mp_n.$$

hence, on comparing coefficients of k^n in [2] and multiplying the result by $n!$ we obtain the desired result.

Also solved by THEODORE BENNETT.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The New York Academy of Sciences offers a prize (the A. Cressy Morrison Prize) to be awarded in December, 1926, for an essay on the intra-atomic activity of the sun.

Kenyon College has conferred the honorary degree of doctor of laws on Professor M. T. PUPIN of Columbia University.

Johns Hopkins University celebrated its fiftieth anniversary on October 22–23, 1926. Among the alumni who delivered addresses on this occasion were Professor A. B. COBLE, of the University of Illinois, and L. P. EISENHART, of Princeton University.

The Mathematical Association was represented by Professor W. L. PORTER at the semi-centennial of the founding of the Agricultural and Mechanical College of Texas October 14–16, 1926; by Professor W. L. MILNE at the inauguration of Doctor Arnold Bennett Hall as president and the semi-centennial of the founding of the University of Oregon October 18–23, 1926; and by Professor ANNA H. PALMIÉ at the hundredth anniversary of Western Reserve University November 12–13, 1926.

Professor L. D. AMES, of the Texas Technological College, has been appointed professor of mathematics at the University of Southern California.

Mr. C. F. BARR, of Purdue University, has been appointed assistant professor of mathematics at the University of Wyoming.

Dr. E. M. BERRY, of Purdue University, has been appointed head of the department of mathematics at Lynchburg College.

Mr. E. C. BOWER, of the United States Naval Observatory, has been appointed assistant professor of mathematics and astronomy at Ohio Wesleyan University.

Associate Professor ABRAHAM COHEN, of Johns Hopkins University, has been promoted to a full professorship of mathematics.

Associate Professor H. B. CURTIS, of Marquette University, has been appointed head of the department of mathematics at Lake Forest College.

Assistant Professor H. G. H. GREENLEAF of De Pauw University has been appointed associate professor of mathematics.

Mr. D. C. HARKIN has been appointed lecturer in mathematics at the University of Manitoba.

At Cornell University, Professor W. A. HURWITZ has been made chairman of the department of mathematics.

Miss MYRA I. JOHNSON has been appointed professor of mathematics at Blackstone College for Women.

Mr. G. A. LYLE, of Lehigh University, has been appointed adjunct professor of mathematics at the Texas Technological College.

Associate Professor H. R. PHALEN, of the Armour Institute of Technology, has been appointed professor of mathematics at St. Stephen's College.

Assistant Professor G. C. PRIESTER of the University of Minnesota has been appointed associate professor of mathematics and mechanics and placed in charge of the materials testing laboratory in the college of engineering.

At the University of Wyoming, Professor O. H. RECHARD has been made head of the department of mathematics.

Acting Professor B. H. REDDITT of Lebanon Valley College has been appointed assistant professor of mathematics at Kenyon College.

Professor L. V. ROBINSON, of Oklahoma City University, has been appointed adjunct professor of mathematics at the Texas Technological College.

Professor S. A. ROWLAND, of Union College, has been appointed professor of mathematics at Ohio Wesleyan University.

Associate Professor MARY E. SINCLAIR, of Oberlin College, has been promoted to a full professorship of mathematics.

Assistant Professor H. L. SMITH, of the University of Minnesota, has been appointed assistant professor of mathematics at Louisiana State University.

Associate Professor F. W. SPARKS, of Louisiana State Normal College, has been appointed associate professor of mathematics at the Texas Technological College.

Assistant Professor J. S. TURNER, of Iowa State College, has been promoted to an associate professorship of mathematics.

Mr. A. H. WAIT, of the University of Wisconsin, has been appointed adjunct professor of mathematics at the Texas Technological College.

The following appointments as instructor in mathematics are announced:

Case School of Applied Science, Mr. RICHARD BURINGTON, Cornell University, Mr. A. G. MONTGOMERY and Mr. H. L. SCHUG; Hunter College, Miss LAURA GUGGENBUHL; Lehigh University, Mr. L. J. PARADISO and Mr. G. W. RIDDLE; Montana State School of Mines, Mr. ALEXANDER MASLOW; University of Nebraska, Mr. H. P. DOOLE; University of New Hampshire, Mr. M. R. SOLT; New York University (Washington Square College), Miss ELIZABETH BERGER; Princeton University, Mr. M. S. KNEBELMAN; Texas Technological College, Mr. P. K. REES.

Dr. A. HOWE, dean of the College of Liberal Arts of the University of Denver, and director of the Chamberlin Observatory of the University has died, at the age of sixty-seven.

Professor W. J. HUSSEY, director of the observatory of the University of Michigan, died in London, October 28, 1926, at the age of sixty-four. Professor Hussey was en route to Bloemfontein, South Africa, to assist in the establishment of a university observatory station there.

Dr. E. NIPHER, Professor emeritus of physics at Washington University, died October 6, 1926, at the age of seventy-eight.

Dr. C. A. WALDO, professor emeritus at Washington University, died October 1, 1926 at the age of seventy-four.

ADDENDA AND CORRIGENDA

P. 44, middle, *for* "Dension" *read* "Denison".
P. 53, line 8 from bottom, *for* "Eurpoe" *read* "Europe".
P. 56, 1.2, *for* "Schloring" *read* "Schorling".
P. 121, in the determinant, insert "j" in first row, second column.
P. 123, formula (11) should read "$\nabla\cdot\sigma\times\tau=\tau\cdot\nabla\times\sigma-\sigma\cdot\nabla\times\tau$".
P. 123, in the display between (11) and (12) delete the second dot in "$\tau\cdot\sigma\cdot\times\tau=0$".
P. 123, last line, insert "f" as the third last denominator.
P. 217, insert "IV." preceding "A CRYSTALLOGRAPHIC . . . .".
P. 218, insert "V." preceding "ON THE DUALS . . . .".
P. 236, delete fifth and sixth last lines.
P. 255, Theorem III, for "paramteers" read "parameters".
P. 389, *for* problem number **3145** *read* **3149**. (See page 483).
P. 403, 1.20, *for* "molar" *read* "molecular".
P. 430, 1.6, et seq., Correction of problem numbers **3144 – 3193**.
P. 434, 1.4, *for* "Goast" *read* "Coast".
P. 459, last line, *for* "substracts" *read* "subtracts".
P. 478, title, *for* "UNDERGRADUATES" *read* "UNDERGRADUATE".
P. 487, title, *for* "847" *read* "487".
P. 488, 1.5 from bottom, *for* "are" *read* "is".
P. 498, title, insert "I" in "STAT STICS".
P. 499, 1.5, *for* "are" *read* "is".
P. 508, 1.2 from bottom, insert "a" after "perform".

Attention is called to the corrigendum on page 430 concerning the numbering of problems proposed and solved.